Unit Transform Methods

1320

(4)

Introduction

Joesph Fourier, a French mathematician, had invented a method called Fourier transform in 1801, to explain the flow of heat around an anchor ring. Since then, it has become a powerful tool in diverse fields of science and engineering. It can provide a means of solving unwieldy equation that describe dynamic responses to electricity, heat or light. In some cases, it can also identify the regular contributions to a fluctuating signal, thereby helping to make sense of observations in Astronomy, Medicine and Chemistry. Fourier transform has become indispensable in the numerical calculations needed to design electrical circuits, to analyze the mechanical vibrations, and to study wave propagation.

Fourier transform techniques have been widely used to solve problems involving semi-infinite or totally infinite range of the variables or unbounded regions.

In order to deal with such pro...
it is necessary to generalise Fourier
series to include infinite intervals and
to introduce the concept of Fourier
integral. In the present analysis,
we deal with Fourier integral
representations and Fourier transform
together with some applications to
Diffusion, Wave and Laplace equation.

Fourier Integral Representations

Def (Dirichlet's conditions)

A function $f(x)$ is said to have satisfied Dirichlet's conditions in the interval $(-p, p)$ provided $f(x)$ is periodic, piecewise continuous and has a finite number of relative maxima and minima in $(-p, p)$.

Let a ftn $f(x)$ be periodic with period $2p$, i.e., $f(x+2p) = f(x)$, and satisfy Dirichlet's conditions in the interval $(-p, p)$. Then $f(x)$ has a Fourier series representation for every x in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right) \quad (1)$$

where

(3)

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(t)\,dt \quad (1)$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi}{p}t\,dt, \quad n = 1, 2, \cdots \quad (2)$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi}{p}t\,dt, \quad n = 1, 2, \cdots \quad (3)$$

ere a_n, b_n are called Fourier coefficients.

Fourier series representation, however, an be extended to some non-eriodic functions also provided the ntegral of the modulus of such a (tn $f(t)$) satisfies the condition

$$\int_{-\infty}^{\infty} |f(t)|\,dt \text{ is finite.}$$

($f(x)$ is absolutely integrable)

Fourier integral representation of $f(x)$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\{s(t-x)\}\,dt\,ds.$$

) Fourier Integral Thm

If $f(x)$ satisfies Dirichlet's conditions for $-\infty < x < \infty$ and if the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely cgt, then

$$\frac{1}{\pi}\int_{0}^{\infty} d\alpha \int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt = \frac{1}{2}\left[f(x+0) + f(x-0)\right]$$

To establish the Fourier integral we should be aware of the Riemann-Lebesgue lemma and the Riemann Localisation lemma.

Riemann-Lebesgue lemma

If $f(x)$ satisfies Dirichlet's conditions in the interval (a,b), then each of the integrals

$$\int_a^b f(x)\sin Nx\,dx, \quad \int_a^b f(x)\cos Nx\,dx$$

tends to zero as $N \to \infty$.

Riemann localization lemma

If $f(t)$ satisfies Dirichlet's conditions in the interval $(0, a)$, where a is finite, then

$$\int_0^a f(t)\frac{\sin Nt}{t}\,dt \to \frac{\pi}{2} f(0+) \text{ as } N \to \infty.$$

Fourier Transform Pairs

Def. Let $f(x)$ be a function defined on $(-\infty,\infty)$ and is piecewise continuous, differentiable in each finite interval and is absolutely integrable on $(-\infty,\infty)$. Then the Fourier transform of $f(x)$ is

$$\mathcal{F}[f(x)] = \bar{F}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{+i\alpha x}\, dx$$

and inverse Fourier transform is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{F}(\alpha)\, e^{-i\alpha x}\, d\alpha .$$

(α is Fourier transform parameter).

Fourier Sine transform pair

$$\bar{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \alpha x\, dx = \mathcal{F}_s(f(x))$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_s(\alpha) \sin \alpha x\, d\alpha = \mathcal{F}_s^{-1}[\bar{f}_s(\alpha)]$$

Fourier Cosine transform pair

$$\bar{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \alpha x\, dx = \mathcal{F}_c[f(x)],$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_c(\alpha) \cos \alpha x\, d\alpha = \mathcal{F}_c^{-1}[\bar{f}_c(\alpha)]$$

Transform of Some Elementary functions

Ex Find the Fourier transform of $f(x) = e^{-x^2/2}$

Soln

$$\bar{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$-\frac{x^2}{2} + i\alpha x$

$-\left(\frac{x^2}{2} - i\alpha x + \left(\frac{i\alpha}{2}\right)^2 - \left(\frac{i\alpha}{2}\right)^2\right)$

$= \frac{(x - i\alpha)^2}{2} + \frac{i^2\alpha^2}{2}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{i\alpha x}\, dx$$

completing squares and adding and subtracting

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-i\alpha)^2/2}\, e^{-\alpha^2/2}\, dx$$

(completing the Square)

$$= \frac{e^{-\alpha^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-i\alpha)^2/2}\, dx \longrightarrow (1)$$

Put $\frac{x - i\alpha}{\sqrt{2}} = t$

$\frac{dx}{\sqrt{2}} = dt.$

$$\bar{f}(\alpha) = \frac{e^{-\alpha^2/2}}{\sqrt{\cancel{2}\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, \cancel{\sqrt{2}}\, dt.$$

$$= \frac{e^{-\alpha^2/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{\sqrt{a}}$$

$$= \frac{e^{-\alpha^2/2}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{1}$$

$$= e^{-\alpha^2/2}$$

(*) Gaussian is a self Fourier function.

Belsian

Ex Find the Fourier transform of

$$f(x) = e^{-a|x|}, \quad -\infty < x < \infty$$

Soln

$$|x| = \begin{cases} -x, & x<0 \\ x, & x \geq 0 \end{cases}$$

$$\bar{F}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{ax} e^{i\alpha x}\, dx + \int_{0}^{\infty} e^{-ax} e^{i\alpha x}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a+i\alpha)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{(-a+i\alpha)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+i\alpha} - \frac{1}{-a+i\alpha} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+\alpha^2} \right].$$

Ex Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

$\pm x > a$

$x > a$

$-a$ $\quad -x > a$

$x < -a$

and hence evaluate

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha}\, d\alpha, \quad \int_{0}^{\infty} \frac{\sin \alpha a}{\alpha}\, d\alpha.$$

Soln By def.

$$\bar{F}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)\, e^{i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha x}}{i\alpha} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i\alpha a}}{i\alpha} - \frac{e^{-i\alpha a}}{i\alpha} \right) = \frac{2}{\alpha\sqrt{2\pi}} \left(\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i} \right)$$

Therefore

$\alpha > 0$? ?

$$\bar{f}(\alpha) = \begin{cases} \dfrac{2\sin\alpha a}{\alpha\sqrt{2\pi}}, & \\ \lim_{\alpha \to 0} \dfrac{2a}{\sqrt{2\pi}} \dfrac{\sin\alpha a}{\alpha a} = \dfrac{2a}{\sqrt{2\pi}}, & \alpha = 0 \end{cases}$$

Now
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} \bar{f}(\alpha)\, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2\sin\alpha a}{\alpha\sqrt{2\pi}} e^{-i\alpha x}\, d\alpha$$

$$= \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

i.e. $$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha a(\cos\alpha x - i\sin\alpha x)}{\alpha} d\alpha = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Hence, $$\int_{-\infty}^{\infty} \frac{\sin\alpha a \cos\alpha x}{\alpha} d\alpha = \begin{cases} \pi, & |x| \le a \\ 0, & |x| > a. \end{cases}$$

Also, by setting $x = 0$ in above equation we obtain $$\int_{-\infty}^{\infty} \frac{\sin\alpha a}{\alpha} d\alpha = \pi.$$

Since the integrand is even, we can have $$\int_{0}^{\infty} \frac{\sin\alpha a}{\alpha} d\alpha = \pi/2.$$

Ex Find the Fourier cosine and sine transforms of e^{-bx} and evaluate the integrals

(i) $\int_0^\infty \frac{\cos\alpha x}{\alpha^2+b^2}\,d\alpha$ (ii) $\int_0^\infty \frac{\alpha\sin\alpha x}{\alpha^2+b^2}\,d\alpha$.

Soln Given $f(x) = e^{-bx}$

$$\bar{F}_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\alpha x\,dx$$

$$\bar{F}_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\alpha x\,dx$$

using f(x) we get.

$$\bar{F}_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\cos\alpha x\,dx \quad \text{——— (1)}$$

$$\bar{F}_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\sin\alpha x\,dx \quad \text{——— (2)}$$

Let $I_1 = \int_0^\infty e^{-bx}\cos\alpha x\,dx$ (integ. by parts)

$I_2 = \int_0^\infty e^{-bx}\sin\alpha x\,dx$

we have

$$I_1 = \left(-\frac{1}{b}e^{-bx}\cos\alpha x\right)_0^\infty - \frac{\alpha}{b}\int_0^\infty e^{-bx}\sin\alpha x\,dx = \frac{1}{b} - \frac{\alpha}{b}I_2 \quad \text{(3)}$$

$$I_2 = \left(-\frac{1}{b}e^{-bx}\sin\alpha x\right)_0^\infty + \frac{\alpha}{b}\int_0^\infty e^{-bx}\cos\alpha x\,dx = \frac{\alpha}{b}I_1 \quad \text{(4)}$$

Solving (3) and (4), we obtain

$$I_1 = \frac{b}{\alpha^2+b^2}, \quad I_2 = \frac{\alpha}{\alpha^2+b^2} \quad \text{——— (5)}$$

Hence using (5) in (1) & (2) we have

$$\bar{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{b}{\alpha^2+b^2},$$

$$\bar{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2+b^2}.$$

then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_c(\alpha) \cos\alpha x \, d\alpha$$

$$e^{-bx} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{b}{\alpha^2+b^2}\right) \cos\alpha x \, d\alpha$$

$$\Rightarrow \boxed{\int_0^{\infty} \frac{\cos\alpha x}{\alpha^2+b^2} d\alpha = \frac{\pi}{2b} e^{-bx}.}$$

Similarly

Similarly,

$$\int_0^{\infty} \frac{\alpha \sin\alpha x}{\alpha^2+b^2} = \frac{\pi}{2} e^{-bx}.$$

Ex Find the Fourier Sine transform f(x), if $f(x) = \begin{cases} 0, & 0 < x < a, \\ x, & a \le x \le b, \\ 0, & x > b. \end{cases}$

soln

$$\bar{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin\alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_a^{b} x \sin\alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(-\frac{x\cos\alpha x}{\alpha}\right)_a^b + \frac{1}{\alpha} \int_a^b \cos\alpha x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a\cos\alpha a - b\cos\alpha b}{\alpha} + \frac{\sin\alpha b - \sin\alpha a}{\alpha^2} \right].$$