Example A string is stretched and fixed between two points $(0,0)$ and $(l,0)$. Motion is initiated by displacing the string in the form $u = \lambda \sin(\pi x/l)$ and released from rest at time $t=0$. Find the displacement of any point on the string at any time t.

Soln The displacement $u(x,t)$ is governed by $u_{tt} = c^2 u_{xx}$, $0 < x < l$, $t > 0$, → (*)

BCs: $u(0,t) = u(l,t) = 0$

ICs: $u(x,0) = \lambda \sin \frac{\pi x}{l}$, $u_t(x,0) = 0$.

Taking Laplace transform, we get

$$c^2 \frac{d^2 \bar{u}}{dx^2} = s^2 \bar{u}(x,s) - s\,\underbrace{u(x,0)}_{=\lambda\sin\frac{\pi x}{l}} - \underbrace{u_t(x,0)}_{=0}$$

$$\boxed{\frac{d^2 \bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} = -\frac{\lambda s}{c^2} \sin \frac{\pi x}{l}}$$

Its general soln is

$$\bar{u}(x,s) = A e^{\frac{s}{c}x} + B e^{-\frac{s}{c}x} + \frac{\lambda s \sin(\pi/l)x}{s^2 + \pi^2 c^2/l^2} \quad \text{→ (1)}$$

Laplace transform of (*) yields

$$\bar{u}(0,s) = 0, \quad \bar{u}(l,s) = 0 \quad \text{→ (2)}$$

$$A + B = 0 \qquad (3)$$

$$A e^{sl/c} + B e^{-sl/c} = 0 \qquad (4)$$

Solving (3) and (4) we have $A = B = 0$.

Thus from (1) $\bar{u}(x,s) = \dfrac{\lambda s \sin(\pi x/l)}{s^2 + \pi^2 c^2/l^2}$.

Laplace inversion yields.

$$u(x,t) = L^{-1}[\bar{u}(x,s)]$$

$$= \lambda L^{-1}\left[\frac{s}{s^2 + \frac{\pi^2 c^2}{l^2}}\right] \sin\frac{\pi x}{l}.$$

$$= \boxed{\lambda \cos\left(\frac{\pi c}{l} t\right) \sin\frac{\pi x}{l}}.$$

which gives the displacement of the string for a given time.

Example An infinitely long string having one end at $x=0$ is initially at rest on the x-axis. The end $x=0$ undergoes a periodic transverse displacement described by $A_0 \sin\omega t, t>0$. Find the displacement at any time t of any point on the string at any time t.

Soln Let $u(x,t)$ be the transverse displacement of the string at any point x and at any time t. Then the transverse displacement of the string is described by the

PDE: $u_{tt} = c^2 u_{xx}$, $x>0$, $t>0$ — (1)

(17)

BCs: $u(0,t) = A_0 \sin \omega t, \quad t > 0 \; (A_0 = \text{constant})$. ——— (2)

ICs: $u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \geq 0.$

The Laplace transform of (1) is

$$c^2 \frac{d^2 \bar{u}}{dx^2} = s^2 \bar{u} - s\,u(x,0) - u_t(x,0).$$

using initial conditions we have.

$$\boxed{\frac{d^2 \bar{u}}{dx^2} - \frac{s^2}{c^2} \bar{u}(x,s) = 0} \quad \text{——— (3)}$$

Laplace transform of (2) is

$$\boxed{\bar{u}(0,s) = \frac{A_0 \omega}{s^2 + \omega^2}} \quad \text{——— (4)}$$

The general soln of (3) is

$$\bar{u}(x,s) = A\, e^{(s/c)x} + B\, e^{-(s/c)x} \longrightarrow (5)$$

Since the displacement $\bar{u}(x,s)$ is bounded as $x \to \infty$, we get $A = 0$ and, therefore, from (5)

$$\bar{u}(x,s) = B\, e^{-\frac{s}{c}x} \longrightarrow (6)$$

Using (4) in (6) we obtain

$$B = \bar{u}(0,s) = \frac{A_0 \omega}{s^2 + \omega^2} \longrightarrow (7)$$

(7) in (6) yields

$$\bar{u}(x,s) = \frac{A_0 \omega}{s^2 + \omega^2}\, e^{-\frac{s}{c}x}$$

Taking inverse Laplace we get

$$u(x,t) = L^{-1}[\bar{u}(x,s)] = L^{-1}\left[\frac{A_0 \omega}{s^2 + \omega^2}\, e^{-\frac{s}{c}x}\right]$$

$$= \begin{cases} A_0 \sin \omega (t - x/c). & \text{if } t > x/c \\ 0 & \text{if } t < x/c \end{cases}$$

Eg: Find the soln of

$$u_t = k\,u_{xx}, \quad 0 < x < \infty, \quad t > 0$$

$$u(x,t) \to 0 \text{ as } x \to \infty$$

$$u(0,t) = g(t)$$

$$u(x,0) = 0$$

assuming $L^{-1}\left[\exp\left(-\sqrt{\frac{s}{k}}\,x\right)\right] = \frac{x}{2\sqrt{k\pi t^3}} \exp\left(-\frac{x^2}{4kt}\right)$.

Soln Taking Laplace transform, we have

$$k\frac{d^2\bar{u}}{dx^2} = s\bar{u}(x,s) - u(x,0)$$

$u(x,0) = 0$ ∴

$$\boxed{\frac{d^2\bar{u}}{dx^2} - \frac{s}{k}\bar{u} = 0}$$

soln is

$$\bar{u}(x,s) = A\exp\left(\sqrt{\frac{s}{k}}\,x\right) + B\exp\left(-\sqrt{\frac{s}{k}}\,x\right) \quad \text{(1)}$$

The Laplace transform of the BCs give

$$\bar{u}(x,s) \to 0 \text{ as } x \to \infty \quad \text{(2)}$$

$$\bar{u}(0,s) = \bar{g}(s) \quad \text{(3)}$$

Using (2) in (1) we get $A = 0$

$$\therefore \bar{u}(x,s) = B\exp\left(-\sqrt{\frac{s}{k}}\,x\right) \quad \text{(4)}$$

From (3) & (4)

$$B = \bar{g}(s) \quad \text{(5)}$$

$$\therefore \boxed{\bar{u}(x,s) = \bar{g}(s)\exp\left(-\sqrt{\frac{s}{k}}\,x\right)}$$

Laplace inversion of ⑤ yields

$$u(x,t) = L^{-1}\left[\bar{g}(s)\, e^{-\sqrt{s/k}\,x}\right]$$

$$= L^{-1}\left[\left[L(g(t))\right] L\left[\frac{x}{2\sqrt{k\pi t^3}}\, e^{-x^2/4kt}\right]\right]$$

By convolution thm we have

$$u(x,t) = \int_0^t \frac{x \exp\left[-x^2/4k(t-u)\right]}{2\sqrt{\pi k}\,(t-u)^{3/2}}\, g(u)\, du$$

Example Solve $u_t(x,t) = u_{xx}(x,t) - h\,u(x,t)$ ——— ①

$h =$ constant, $0 < x < \pi$, $t > 0$

BCS: $u(0,t) = 0$, $t > 0$, $u(\pi,t) = 1$, $t > 0$ ——— ②

IC: $u(x,0) = 0$, $t = 0$ ——— ③

Taking Laplace transform of ① we get

$$\frac{d^2\bar{u}}{dx^2} - (h+s)\,\bar{u}(x,s) = 0 \quad \left[\text{As } s\bar{u}(x,s) - u(x,0) = \frac{d^2\bar{u}}{dx^2} - h\bar{u}\right]$$

The general soln is

$$\bar{u}(x,s) = A\cosh\left(\sqrt{h+s}\,x\right) + B\sinh\left(\sqrt{h+s}\,x\right) \longrightarrow ④$$

Taking Laplace transform of ② we have

$$\bar{u}(0,s) = 0,\quad \bar{u}(\pi,s) = 1/s \longrightarrow ⑤$$

$\bar{u}(0,s) = 0 \Rightarrow A = 0$ from ④ Hence

$$\bar{u}(x,s) = B\sinh\left(\sqrt{h+s}\,x\right) \longrightarrow ⑥$$

$\bar{u}(\pi, s) = \frac{1}{s} = B \sinh(\sqrt{h+s}\,\pi)$

$$\Rightarrow \boxed{B = \frac{1}{s \sinh(\sqrt{h+s}\,\pi)}} \longrightarrow (7)$$

Using (7) in (6) we get

$$\boxed{\bar{u}(x,s) = \frac{1}{s \sinh(\sqrt{h+s}\,\pi)} \sinh(\sqrt{h+s}\,x)}$$

By means of complex inversion formula

$$u(x,t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{st} \sinh(\sqrt{h+s}\,x)}{s \sinh(\sqrt{h+s}\,\pi)} \longrightarrow (8)$$

= sum of the contributions from all the poles of the integrand.

From (8), the poles are given by $s = 0$, and $\sinh(\sqrt{h+s}\,\pi) = 0 = \sin(i\sqrt{h+s}\,\pi)$. Therefore,

$i\sqrt{h+s}\,\pi = n\pi$, implying $h+s = -n^2$.

Thus, the integrand has poles at $s = 0$, and

$s = -n^2 - h$, $n = 1, 2, 3, \ldots$

The residue of the expression

$$\frac{e^{st}}{s} \frac{\sinh(\sqrt{h+s}\,x)}{\sinh(\sqrt{h+s}\,\pi)} \text{ at } s = 0 \text{ is } \boxed{\frac{\sinh(\sqrt{h}\,x)}{\sinh(\sqrt{h}\,\pi)}} \longrightarrow (9)$$

The residue of the integrand at $s = -n^2 - h$ is

$$\frac{e^{-(n^2+h)t} \sinh(\sqrt{-n^2}\,x)}{\sinh\sqrt{-n^2}\,\pi + \frac{(-n^2-h)\pi}{2\sqrt{-n^2}} \cosh(\sqrt{-n^2}\,\pi)}, \quad n = 1, 2, 3, \ldots$$

Using the relations $\sinh x = -i \sin ix$, and $\cosh x = \cos ix$,

the above residues become

$$\frac{[-(n^2+h)t]\, i \sin nx}{-\frac{(n^2+h)\pi}{2in}\cos n\pi} = \frac{2n \exp[-(n^2+h)t] \sin nx}{(n^2+h)\pi \cos n\pi}$$

$$= \frac{2n \exp[-(n^2+h)t] \sin nx}{(n^2+h)\,\pi\,(-1)^n} \longrightarrow (10)$$

Hence from (8) — (10) we get

$$\boxed{u(x,t) = \frac{\sinh(\sqrt{h}\,x)}{\sinh(\sqrt{h}\,\pi)} + \frac{2e^{-ht}}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^n \exp(-n^2 t) \sin nx}{(n^2+h)}}$$

Ex Find the inverse Laplace transform of $\dfrac{e^{-1/s}}{\sqrt{s}}$.

Sol

$$\frac{1}{\sqrt{s}} e^{-\frac{1}{s}} = \frac{1}{\sqrt{s}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, s^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, s^{n+1/2}}.$$

$$L^{-1}\left[\frac{e^{-1/s}}{\sqrt{s}}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L^{-1}\left[\frac{1}{s^{n+1/2}}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{t^{n-1/2}}{\Gamma(n+\frac{1}{2})} \quad (1)$$

$$\Gamma(n+1/2) = \frac{(2n)!}{2^{2n}\, n!}\sqrt{\pi}$$

using in (1) we get

$$L^{-1}\left[\frac{e^{-1/s}}{\sqrt{s}}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n}\, t^{n-1/2}}{(2n)!\sqrt{\pi}}$$

$$= \frac{t^{-1/2}}{\sqrt{\pi}} - \frac{4\, t^{1/2}}{2!\sqrt{\pi}} + \frac{2^4\, t^{3/2}}{4!\sqrt{\pi}} - \cdots$$

$$= \frac{1}{\sqrt{\pi t}}\left[1 - \frac{(2\sqrt{t})^2}{2!} + \frac{(2\sqrt{t})^4}{4!} - \cdots\right]$$

$$= \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}.$$

Ex (5.1) Find the Laplace transform of $\operatorname{erf}(1/\sqrt{t})$.

$$L[\operatorname{erf}(1/\sqrt{t})] = \int_0^\infty e^{-st}\operatorname{erf}(1/\sqrt{t})\,dt$$

$$= \int_0^\infty e^{-st}\left[\frac{2}{\sqrt{\pi}}\int_0^{1/\sqrt{t}} e^{-u^2}du\right]dt$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty\int_0^{1/\sqrt{t}} e^{-st-u^2}\,du\,dt. \longrightarrow (1)$$

Now, changing the order of integration, we get.

$$L[\operatorname{erf}(1/\sqrt{t})] = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}du\int_0^{1/u^2} e^{-st}\,dt.$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}du\left[\frac{e^{-st}}{-s}\right]_0^{1/u^2}$$

$$= -\frac{2}{s\sqrt{\pi}}\int_0^\infty\left[e^{-u^2-s/u^2} - e^{-u^2}\right]du \longrightarrow (2)$$

Consider

$$I(a,b) = \int_0^\infty e^{-a^2u^2-b^2/u^2}\,du \longrightarrow (2.a)$$

$$I(a,0) = \int_0^\infty e^{-a^2u^2}\,du = \frac{\sqrt{\pi}}{2a} \longrightarrow (3)$$

$$\left(\text{since } \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}du = 1\right)$$

Differentiating (2a) w.r.t. b partially, we have

$$\frac{\partial I}{\partial b} = -2b\int_0^\infty u^{-2}\, e^{-a^2u^2-b^2/u^2}\,du. \longrightarrow (4)$$

Let $au = x$ then $du = dx/a$ and so (2a) gives

$$I(a,b) = \frac{1}{a}\int_0^\infty \exp(-x^2 - a^2b^2/x^2)\,dx$$

$$= \frac{1}{a} I(1, ab) \longrightarrow (5)$$

(by 2a)

Let $b/u = x$ then (4) gives

$$\frac{\partial I(a,b)}{\partial b} = 2b\int_\infty^0 \frac{x^2}{b^2}\exp\left(-\frac{a^2b^2}{x^2} - x^2\right)\frac{b}{x^2}\,dx$$

$$= -2\int_0^\infty \exp\left(\frac{-x^2 - a^2b^2}{x^2}\right)dx$$

$$= -2\, I(1, ab) \longrightarrow (6)$$

Eliminating $I(1, ab)$ from (5) & (6) we get

$$\frac{\partial I}{\partial b} = -2a\, I$$

On integrating $\ln I = -2ab + \ln c$

$$\boxed{I(a,b) = c\,e^{-2ab}} \longrightarrow (7)$$

$$I(a,0) = c \longrightarrow (8)$$

(8) in (7) gives $\boxed{I(a,b) = I(a,0)\,e^{-2ab}} \longrightarrow (9)$

or using (2a) and (3)

$$\int_0^\infty e^{-a^2u^2 - b^2/u^2}\,du = \frac{\sqrt{\pi}}{2a}\,e^{-2ab} \longrightarrow (10)$$

Using (10) in (2) used to

$$L[\operatorname{erf}(1/\sqrt{t})] = \frac{-2}{s\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}e^{-2\sqrt{s}} - \frac{\sqrt{\pi}}{2}\right]$$

$$= \frac{1}{s}\left(1 - e^{-2\sqrt{s}}\right).$$

Ex Solve

$$\dot{x} + ky = a\sin kt, \qquad \dot{y} - kx = a\cos kt \quad ; \; x(0) = 0, \; y(0) = b.$$

Taking Laplace transform we have.

$$s\bar{x}(s) - x(0) + k\bar{y} = \frac{ak}{s^2+k^2} \quad \text{—— (1)}$$

$$s\bar{y}(s) - y(0) - k\bar{x} = \frac{as}{s^2+k^2} \quad \text{—— (2)}$$

using conditions we get from (1) & (2) as $skx + k^2y =$

$$s\bar{x} + k\bar{y} = ak/s^2+k^2 \quad \text{—— (3)}$$

$$s\bar{y} - k\bar{x} = \frac{as}{s^2+k^2} + b \quad \text{—— (4)}$$

Solving (3) and (4) we get

$$\bar{x} = \begin{vmatrix} \frac{ak}{s^2+k^2} & k \\ \frac{as}{s^2+k^2} + b & s \end{vmatrix} \Bigg/ \begin{vmatrix} s & k \\ -k & s \end{vmatrix} \qquad \frac{aks}{s^2+k^2} - \frac{ask}{s^2+k^2} + bk$$

$$y(s) \begin{vmatrix} \frac{ak}{s^2+k^2} & k \\ \frac{as}{s^2+k^2} + b & s \end{vmatrix}$$

$$= \frac{-bk}{s^2+k^2} \quad \text{—— (5)}$$

Similarly

$$\bar{y} = \frac{a}{s^2+k^2} + \frac{bs}{s^2+k^2} \quad \text{from (4) \& (5)}$$

Laplace inversion yields

$$\boxed{x(t) = -b\sin kt}, \quad y(t) = \frac{a}{k}\sin kt + b\cos kt.$$