**Ex** Find the Laplace transform of

(i) $\frac{1-\cos t}{t}$ (ii) $\frac{\cos 2t - \cos 3t}{t}$.

**Soln**

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$$

(i) $\bar{f}(s) = L[1-\cos t] = \frac{1}{s} - \frac{s}{s^2+1}$

$\therefore$

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\ln s - \frac{1}{2}\ln(s^2+1)\right]_s^\infty = \left[\ln \frac{s}{(s^2+1)^{1/2}}\right]_s^\infty$$

$$= \left[\ln \frac{1}{(1+1/s^2)^{1/2}}\right]_s^\infty = 0 - \ln \frac{s}{(s^2+1)^{1/2}}$$

$$= \ln \frac{(s^2+1)^{1/2}}{s}$$

(ii) $$L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \bar{f}(s)\,ds \quad \text{——— (1)}$$

$$\bar{f}(s) = L[\cos 2t - \cos 3t] = \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

Using in (1) we get

$$L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] ds$$

$$= \frac{1}{2}\left[\ln(s^2+4) - \ln(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln\left(\frac{s^2+4}{s^2+9}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln \frac{1+4/s^2}{1+9/s^2}\right]_s^\infty = \frac{1}{2}\ln\left(\frac{s^2+9}{s^2+4}\right)$$

## Transform of a periodic function

A function $f(t)$ is called periodic with period $T$, if $f(t+T) = f(t)$ for all values of $t$ and $T>0$.

If $f(t)$ is a periodic function with period $T$, then

$$L[f(t)] = \int_0^T e^{-st} f(t)\,dt \Big/ (1 - e^{-sT})$$

**Pf** By def.

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt \quad \text{—— (1)}$$

Substitute $u = t - T$ in the second integral of (1) i.e.

$$L[f(t)] = \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+T)} f(u+T)\,du$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \int_0^\infty e^{-su} f(u)\,du$$

as $f(u+T) = f(u)$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} L[f(t)]$$

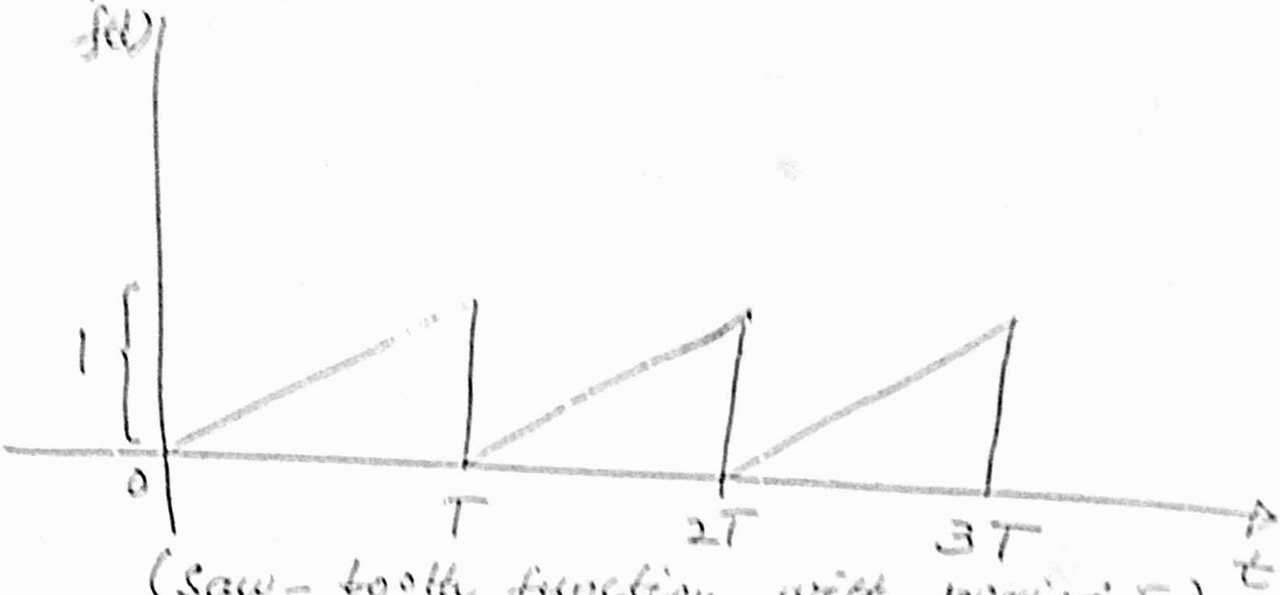
$$\Rightarrow (1 - e^{-sT}) L(f(t)) = \int_0^T e^{-st} f(t)\,dt.$$

$$\Rightarrow \boxed{L(f(t)) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt}$$

**Ex** Obtain the Laplace transform of the periodic saw-tooth wave function given by

$$f(t) = \frac{t}{T} \text{ of period } T,\ 0<t<T.$$

**Soln** The graph of the periodic saw-tooth fun is given as



(Saw-tooth function with period T)

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \frac{t}{T}\, dt$$

$$= \frac{1}{T(1-e^{-sT})} \int_0^T e^{-st}\, t\, dt$$

$$= \frac{1}{T(1-e^{-sT})} \int_0^T t\, d\left(\frac{e^{-st}}{-s}\right)$$

$$= \frac{1}{T(1-e^{-sT})} \left[ \left(\frac{t e^{-st}}{-s}\right)_0^T + \frac{1}{s}\int_0^T e^{-st} dt \right]$$

$$= \frac{1}{T(1-e^{-sT})} \left[ \frac{T e^{-sT}}{-s} - \frac{1}{s^2}(e^{-sT}-1) \right]$$

$$= \frac{1}{s^2 T} - \frac{e^{-sT}}{s(1-e^{-sT})}$$

$\boxed{Ex}$ Find the Laplace transform of the following full wave rectifier function:

$$f(t)=\begin{cases} E\sin\omega t, & 0<t<\lambda/\omega \\ 0, & \lambda/\omega<t<2\lambda/\omega \end{cases}$$

given that $f(t+\frac{2\lambda}{\omega})=f(t)$.

$\boxed{Soln}$ $f(t)$ is periodic with period $2\lambda/\omega$ so

$$L(f(t))=\frac{1}{1-e^{-2\lambda s/\omega}}\int_0^{2\lambda/\omega} e^{-st} f(t)\,dt$$

$$=\frac{1}{1-e^{-2\lambda s/\omega}}\left[\int_0^{\lambda/\omega} e^{-st} E\sin\omega t\,dt \text{ (integ. by parts)} + \cancel{\int_{\lambda/\omega}^{2\lambda/\omega} e^{-st}(0)\,dt}^{\,0}\right]$$

$$=\frac{E}{1-e^{-2\lambda s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}\left(s\sin\omega t+\omega\cos\omega t\right)\Big|_0^{\lambda/\omega}\right]$$

$$\therefore\ L[f(t)]=\frac{E}{1-e^{-2\lambda s/\omega}}\left[\frac{e^{-s\lambda/\omega}}{s^2+\omega^2}(s\sin\lambda+\omega\cos\lambda)-\frac{\omega}{s^2+\omega^2}\right].$$

$\boxed{Ex}$ Find the Laplace transform of

$$f(t)=\begin{cases} 1, & 0\le t<2 \\ -1, & 2\le t<4. \end{cases}$$

$\boxed{Soln}$ $f(t)$ is a periodic ftn with period 4 so

$$L[f(t)]=\frac{1}{1-e^{-4s}}\int_0^4 e^{-st}f(t)\,dt$$

$$=\frac{1}{1-e^{-4s}}\left[\int_0^2 e^{-st}(1)\,dt+\int_2^4 e^{-st}(-1)\,dt\right]$$

$$=\frac{1}{1-e^{-4s}}\left[\frac{-2e^{-2s}}{s}+\frac{e^{-4s}}{s}+\frac{1}{s}\right].$$

# Transform of Error function

It is defined as:

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, du \quad \rightarrow (1)$$

In terms of power series

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} u^{2n}\, du$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{t^{2n+1}}{(2n+1)}.$$

(This ftn occurs in many branches of science and engineering, for example, in probability theory, the theory of heat conduction and so on.)

$\boxed{\text{Note that}}$ the above series converge everywhere and therefore, erf(t) is an entire ftn

From (1)

$$\operatorname{erf}(0) = 0$$

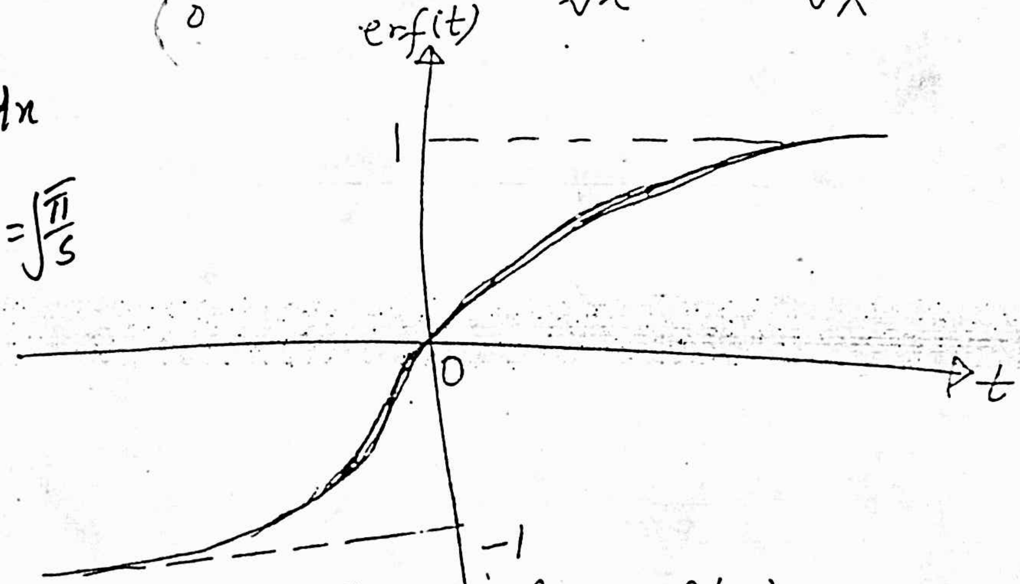
$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2}\, du = \frac{\Gamma(1/2)}{\sqrt{\pi}} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

$\frac{\sqrt{\pi}}{2}$

$$\mathcal{L}[t^{\alpha}] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx$$

$$\mathcal{L}[t^{-1/2}] = \frac{1}{\sqrt{s}} \Gamma(1/2) = \sqrt{\frac{\pi}{s}}$$




(Error ftn)

## Complementary Error ftn

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2}\, du = \frac{2}{\sqrt{\pi}} \left[ \int_0^\infty e^{-u^2}\, du - \int_0^t e^{-u^2}\, du \right]$$

$$= 1 - \operatorname{erf}(t).$$

Now

$$L[\operatorname{erf}(t)] = \int_0^\infty e^{-st} \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du\, dt.$$ (25)

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \int_u^\infty e^{-st} dt\, du$$

$$= \frac{2}{s\sqrt{\pi}} \int_0^\infty e^{-(u^2+su)} du$$ (changing order of integ)

$$= \frac{2}{s\sqrt{\pi}} e^{s^2/4} \int_0^\infty e^{-(u+s/2)^2} du$$ ✓

$0 < u < \infty$
$0 < t < \infty$
put $u = t$
$0 < u < \infty$
as $u < t$.
$\Rightarrow 0 < u < t < \infty$
$\Rightarrow u < t < \infty$

Setting $x = u + s/2$
$dx = du$ we get

$$L[\operatorname{erf}(t)] = \frac{2}{s\sqrt{\pi}} e^{s^2/4} \int_{s/2}^\infty e^{-x^2} dx$$ ✓

$$= \boxed{\frac{1}{s} e^{s^2/4} \operatorname{erfc}(s/2)}.$$

**Ex** Find the Laplace transform of $\operatorname{erf}(\sqrt{t})$.

Soln

$$L[\operatorname{erf}(\sqrt{t})] = \int_0^\infty e^{-st} \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du\, dt.$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du \int_{u^2}^\infty e^{-st} dt$$ (Changing the order of integration)

$$= \frac{2}{s\sqrt{\pi}} \int_0^\infty e^{-u^2 - su^2} du$$ (solving the 2nd integral)

$$= \frac{2}{s\sqrt{\pi}} \int_0^\infty e^{-(1+s)u^2} du \quad \text{--- (1)}$$

Setting $(1+s)u^2 = x^2$ or $\sqrt{1+s}\, u = x$, we have

$$du = dx/\sqrt{1+s}$$

Then

$$L\left[\operatorname{erf}(\sqrt{t})\right] = \frac{2}{s\sqrt{\pi}\sqrt{1+s}} \int_0^\infty e^{-x^2}\,dx$$

$$= \frac{1}{s\sqrt{1+s}}\left[\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx\right]$$

$$= \frac{1}{s\sqrt{1+s}}\left[\frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}\right]$$

$$= \frac{1}{s\sqrt{1+s}}.$$

**Ex** Find the Laplace transform of $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$.

**Soln** Let $f(t) = \sin\sqrt{t}$, then

$$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

As $L[f'(t)] = s\bar{f}(s) - f(0)$.

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = s\,L[\sin\sqrt{t}] - 0 \quad \text{—— (1)}$$

$$L[\sin\sqrt{t}] = L\left[\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right]$$

$$= L\left[t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\right]$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\cdot\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \cdots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{120}\cdot\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right] \quad \text{as } \Gamma(1/2) = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-1/4s}$$

using in (1)

$\Gamma(n+1) = n\Gamma(n)$.

$\Gamma(3/2) = \Gamma(1/2+1) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$

$\Gamma(5/2) = \Gamma(\frac{3}{2}+1) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2}\Gamma(\frac{1}{2}+1) = \frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

$L[t^\alpha] = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\,dx$, $\Gamma(1/2) = \sqrt{\pi}$