Properties of Laplace transform

We present a few important properties of the Laplace transform in the following results which will enable us to find the Laplace transform of a combination of functions whose transforms are known.

Linearity Property

If c_1 and c_2 are any two constants and $\bar{f_1}(s)$ and $\bar{f_2}(s)$ are the Laplace transforms, respectively of $f_1(t)$ and $f_2(t)$, then

$$L\left[c_1 f_1(t) + c_2 f_2(t)\right] = c_1 L\left[f_1(t)\right] + c_2 L\left[f_2(t)\right]$$

$$= c_1 \bar{f_1}(s) + c_2 \bar{f_2}(s).$$

Pf

$$L\left[c_1 f_1(t) + c_2 f_2(t)\right] = \int_0^\infty e^{-st}\left[c_1 f_1(t) + c_2 f_2(t)\right] dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt$$

$$= c_1 \bar{f_1}(s) + c_2 \bar{f_2}(s)$$

$$= c_1 L[f_1(t)] + c_2 L[f_2(t)].$$

Shifting Property

If a function is multiplied by e^{at}, the transform of the resultant is obtained by replacing 's' by 's-a' in the transform of the original function. That is, if

$$L[f(t)] = \bar{f}(s)$$

then

$$L[e^{at} f(t)] = \bar{f}(s-a).$$

Pf $L[e^{at}f(t)] = \int_0^\infty e^{-st}[e^{at}f(t)]dt$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = \bar{f}(s-a).$$

Similarly $L[e^{-at}f(t)] = \bar{f}(s+a)$.

(x) Multiplication by power of t

If $L[f(t)] = \bar{f}(s)$ then $L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}\bar{f}(s)$

$$= (-1)^n \bar{f}^{(n)}(s)$$

$n = 1, 2, 3, \ldots$

Pf By def. $\bar{f}(s) = \int_0^\infty e^{-st}f(t)\,dt$

$$\Rightarrow \frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st}f(t)\,dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}\{e^{-st}f(t)\}\,dt$$

$$= -\int_0^\infty t\,e^{-st}f(t)\,dt$$

$$= -L[tf(t)]$$

(Interchange the operations of differentiation and integration for which we assume that the necessary conditions are satisfied and since there are two variables s and t, we use the notation of partial derivative)

$$\Rightarrow L[tf(t)] = -\frac{d}{ds}\bar{f}(s)$$

Repeating the application of above result

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\bar{f}(s).$$

(iii) Differentiation Property

If $L[f(t)] = \bar{f}(s)$

then
$$L\left[f^{(n)}(t)\right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{n-2}(0) - f^{n-1}(0).$$

Pf By def

$$L[f'(t)] = \int_0^\infty \underset{I}{e^{-st}}\, \underset{II}{f'(t)}\, dt$$

$$= \left| e^{-st} f(t) \right|_0^\infty + s \int_0^\infty e^{-st} f(t)\, dt$$

$$= -f(0) + s L[f(t)]$$

$$= s\bar{f}(s) - f(0). \quad \text{--- (1)}$$

Similarly

$$L[f''(t)] = sL[f'(t)] - f'(0) \quad \text{--- (2)}$$

$$= s[s\bar{f}(s) - f(0)] - f'(0)$$

by (1) in (2)

$$= s^2 \bar{f}(s) - s f(0) - f'(0).$$

$$L[f'''(t)] = s^3 \bar{f}(s) - s^2 f(0) - s f'(0) - f''(0)$$

In general

$$L\left[f^{(n)}(t)\right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

This property is very useful for solving differential equations.

$\boxed{\text{Example}}$ Find the Laplace transform of
(i) $e^{at}\cos bt$ (ii) $e^{at}\sin bt$ (iii) $e^{at}\cosh bt$
(iv) $e^{at}t^n$ (v) $\cos at \cosh bt$.

$\boxed{\text{Solu}}$ Using the shifting property.

$\boxed{(i)}$ $L\left[e^{at}\cos bt\right] = \left.\frac{s}{s^2+b^2}\right|_{s\to s-a}$

$$= \frac{s-a}{(s-a)^2+b^2}$$

Here when compare
$L\left[e^{at}f(t)\right] = \bar{f}(s-a)$
$f(t) = \cos bt$
$\bar{f}(s) = \frac{s}{s^2+b^2}$
$\bar{f}(s-a) = \frac{s-a}{(s-a)^2+b^2}$

$\boxed{(ii)}$ $L\left[e^{at}\underbrace{\sin bt}_{f(t)}\right] = \left.\frac{b}{s^2+b^2}\right|_{s\to s-a}$

$$= \frac{b}{(s-a)^2+b^2}.$$

$\boxed{(iii)}$ $L\left[e^{at}\underbrace{\cosh bt}_{f(t)}\right] = \left.\frac{s}{s^2-b^2}\right|_{s\to s-a} = \frac{s-a}{(s-a)^2\underbrace{-}_{\text{minus}}b^2}$

$\boxed{(iv)}$ $L\left[e^{at}\underbrace{t^n}_{f(t)}\right] = \left.\frac{n!}{s^{n+1}}\right|_{s\to s-a} = \frac{n!}{(s-a)^{n+1}}.$

$\boxed{(v)}$ $L[\cos at \cosh bt] = L\left[\cos at\left(\frac{e^{bt} - e^{-bt}}{2}\right)\right]$.

$= \frac{1}{2}\left\{L[e^{bt}\cos at] - L[e^{-bt}\cos at]\right\}$

$= \frac{1}{2}\left\{\left.\frac{s}{s^2 + a^2}\right|_{s \to s-b} - \left.\frac{s}{s^2 + a^2}\right|_{s \to s+b}\right\}$

$= \frac{1}{2}\left\{\frac{s-b}{(s-b)^2 + a^2} - \left(\frac{s+b}{(s+b)^2 + a^2}\right)\right\}$.

$\boxed{\text{Example}}$ Find the Laplace transform of the following:

(i) $t^2 e^{at}$ (ii) $t \sin at$ (iii) $t^2 \cos at$ (iv) $t^n e^{-at}$.

$\boxed{\text{Soln}}$ $\boxed{(i)}$ $L[t^2 e^{at}] = (-1)^2 \frac{d^2}{ds^2}\left(L[e^{at}]\right)$.

$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-a}\right)$

$= \frac{d}{ds}\left[\frac{-1}{(s-a)^2}\right]$

$= \frac{2}{(s-a)^3}$.

As $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \bar{f}(s)$

$= (-1)^n \frac{d^n}{ds^n} L(f(t))$

$\boxed{\text{Alternatively}}$

By shifting property

$L[e^{at} t^2] = \left.\frac{2!}{s^3}\right|_{s \to s-a} = \frac{2}{(s-a)^3}$

$\boxed{(ii)}$ $L[t \sin at] = (-1)^1 \frac{d}{ds} L[\sin at] = -\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right)$

$= \frac{2as}{(s^2 + a^2)^2}$.

S (iii) $L\left[t^2 \cos at\right] = (-1)^2 \frac{d^2}{ds^2} L[\cos at]$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{2s^3-6sa^2}{(s^2+a^2)^3}.$$

(iv) $L\left[e^{-at} t^n\right] = \left.\frac{n!}{s^{n+1}}\right|_{s \to s+a}$ (By Shifting property)

$$= \frac{n!}{(s+a)^{n+1}}$$

Example Find the Laplace transform of
(i) $te^{-4t}\sin 3t$ (ii) $\sin 2t \sin 3t$ (iii) $\sin^3 2t$.

Soln (i) $L\left[te^{-4t}\sin 3t\right] = L\left[e^{-4t}(t\sin 3t)\right]$ —— (1)

As $L[t\sin 3t] = -\frac{d}{ds}\left(L(\sin 3t)\right) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$.

Now using shifting property on (1) we have

$L\left[e^{-4t}(t\sin 3t)\right] = \left.\frac{6s}{(s^2+9)^2}\right|_{s\to(s+4)} = \frac{6(s+4)}{[(s+4)^2+9]^2}$

$$= \frac{6(s+4)}{(s^2+8s+25)^2}.$$

(ii) Since $\sin 2t \sin 3t = \frac{1}{2}\left[\cos(2t-3t) - \cos(2t+3t)\right]$

$= \frac{1}{2}\left[\cos t - \cos 5t\right]$. as $\cos(-t) = \cos t$

$$L[\sin 2t \sin 3t] = \frac{1}{2}\left[L(\cos t) - L(\cos 5t)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right] = \frac{12s}{(s^2+1)(s^2+25)}$$

$\boxed{\text{(iii)}}$ $\sin 6t = \sin 3(2t) = 3\sin 2t - 4\sin^3 2t$

$$\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$$

$$L[\sin^3 2t] = \frac{3}{4}L[\sin 2t] - \frac{1}{4}L[\sin 6t]$$

$$= \frac{3}{4}\left[\frac{2}{s^2+4}\right] - \frac{1}{4}\left[\frac{6}{s^2+36}\right]$$

$$= \frac{3}{2(s^2+4)} - \frac{3}{2(s^2+36)}$$

$$= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right] = \frac{48}{(s^2+4)(s^2+36)}$$

$\boxed{\text{Ex}}$ Find the Laplace transform of $f(t)$ defined as

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi, \\ 0, & t > \pi \end{cases}$$

$\boxed{\text{Soln}}$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st}\sin t\,dt + \int_\pi^\infty e^{-st}(0)\,dt$$

$$= \operatorname{Im}\int_0^\pi e^{-st+it}\,dt = \operatorname{Im}\int_0^\pi e^{(i-s)t}\,dt$$

$$= \operatorname{Im}\left[\frac{e^{(i-s)t}}{i-s}\right]_0^\pi = \operatorname{Im}\left[\frac{e^{(i-s)\pi} - 1}{i-s}\right] = \operatorname{Im}\left(\frac{i+s}{s^2+1}\right)\left[1 - e^{-s\pi}(\cos\pi + i\sin\pi)\right]$$

$$= \operatorname{Im}\frac{(i+s)}{s^2+1}\left[1 + e^{-s\pi}\right] = \frac{1+e^{-s\pi}}{1+s^2}$$

$\operatorname{Im}\dfrac{1-e^{(i-s)\pi}}{s-i}$

$= \operatorname{Im}\left[\dfrac{(1-e^{i\pi}e^{-s\pi})(s}{(s-i)(s+i)}\right.$

$= \operatorname{Im}\left[\dfrac{s+i}{s^2+1}(1-e^{i\pi}e\right.$

$= \operatorname{Im}\left(\dfrac{s+i}{s^2+1}\right)(1-e^{-s\pi}($

$\operatorname{Im}\dfrac{i+s}{s^2+1}(1+e$

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(v) Initial Value Theorem

If $f(t)$ and $f'(t)$ are Laplace transformable and $\bar{f}(s)$ is the Laplace transform of $f(t)$, then the behavior of $f(t)$ in the nbhd. of $t=0$ corresponds to the behavior of $s\bar{f}(s)$ in the nbhd of $s=\infty$. Mathematically

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\bar{f}(s).$$

Pf Since

$$L[f'(t)] = s\bar{f}(s) - f(0) \longrightarrow (1)$$

Taking $\lim_{s\to\infty}$ we have from (1)

$$\lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to\infty}[s\bar{f}(s) - f(0)] \longrightarrow (2)$$

Since s is independent of t so

$$\lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty\left[\lim_{s\to\infty} e^{-st} f'(t)\right]dt = 0 \longrightarrow (3)$$

Using (3) in (2) we obtain

$$\lim_{s\to\infty}[s\bar{f}(s) - f(0)] = 0$$

$$\Rightarrow \boxed{\lim_{s\to\infty} s\bar{f}(s) = f(0) = \lim_{t\to 0} f(t).}$$

Example Let $f(t)$ be a polynomial of degree n then

$$f(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n. \longrightarrow (1)$$

$$\bar{f}(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \dots + \frac{n!\,a_n}{s^{n+1}} \longrightarrow (2)$$

Taking $\lim_{s\to\infty}$ we have from (1) & (2)

$$\boxed{\lim_{s\to\infty} s\bar{f}(s) = a_0 = \lim_{t\to 0} f(t)}$$

$\boxed{Ex}$ Verify the initial value thm for the function (18)

$$f(t) = 1 + e^{-t}(\sin t + \cos t) \longrightarrow (1)$$

$\boxed{Soln}$

$$f(t) = 1 + e^{-t}(\sin t + \cos t)$$

$$\bar{F}(s) = \frac{1}{s} + \left(\frac{1}{s^2+1} + \frac{s}{s^2+1}\right)_{s \to s+1},$$

$$= \frac{1}{s} + \left(\frac{2+s}{s^2+2s+1}\right).$$

$$s\bar{f}(s) = 1 + \frac{2s+s^2}{s^2+2s+2}$$

$$\lim_{s\to\infty} s\bar{f}(s) = \lim_{s\to\infty}\left[1 + \frac{2/s + 1}{1 + 2/s + 2/s^2}\right] = 1+1 = 2 \longrightarrow (2)$$

Also from (1)

$$f(0) = 1+1 = 2 \longrightarrow (3)$$

From (2) and (3)

$$\lim_{s\to\infty} s\bar{f}(s) = f(0).$$

$\boxed{(*) \text{ (Final Value thm)}}$

If $f(t)$ and $f'(t)$ are Laplace transformable and $\bar{f}(s)$ is the Laplace transform of $f(t)$, then the behavior of $f(t)$ in the nbhd of $t = \infty$ corresponds to the behavior of $s\bar{f}(s)$ in the nbhd of $s = 0$. Mathematically

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s)$$

Pf

$$L[f'(t)] = s\bar{F}(s) - f(0)$$

$$\int_0^\infty e^{-st} f'(t)\,dt = s\bar{F}(s) - f(0)$$

$$\lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to 0} s\bar{f}(s) - \lim_{s\to 0} f(0) \longrightarrow (1)$$

(19)

or $\int_0^\infty \underset{s \to 0}{Lt}\ e^{-st} f'(t)\,dt = \int_0^\infty f'(t)\,dt = [f(t)]_0^\infty$

$= \underset{t \to \infty}{Lt}\ f(t) - f(0)$ ——— (2)

using (2) in (1) we get

$\underset{t\to\infty}{Lt}\ f(t) - \cancel{f(0)} = \underset{s\to 0}{Lt}\ s\bar{f}(s) - \cancel{\underset{s\to 0}{Lt}\ f(0)}$

$$\Rightarrow \boxed{\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s\bar{f}(s)}$$

(*) (Division by t)

If $L[f(t)] = \bar{f}(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds.$

$R.H.S = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$

$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt$ (by changing the order of integration)

$= \int_0^\infty f(t)\left(\frac{e^{-st}}{-t}\right)_s^\infty dt = \int_0^\infty \frac{f(t)}{t} e^{-st}\,dt$

$= L\left[\frac{f(t)}{t}\right] = R.H.S.$

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Note In applying this rule, one should be careful. Since $f(t)/t$ may have an infinite discontinuity at $t=0$, it may not be integrable. If $f(t)/t$ is not integrable, then its Laplace transform does not exist.

For example, at $t=0$, the function $\sin t/t$ does not have an infinite discontinuity, while the function $\cos t/t$ has an infinite discontinuity.