KJ 45

Laplace Transform

Laplace transform is a mathematical tool which can be used to solve several problems in science and engineering. This was first introduced by a French mathematician 'Laplace' in the year 1790 in his work on probability theorem. This technique became popular when Heaviside applied to the solution of an ordinary differential equation referred hereafter as ODE, representing a problem in electrical engineering.

To the basic question as to why one should learn Laplace transform technique when other techniques are available, the answer is very simple. Transforms are used to accomplish the solution of certain problems with less effort and in a simple routine way.

To illustrate, consider the problem of finding the value of x from the equation

$$x^{1.85} = 3 \qquad (1)$$

It is extremely difficult task to this this problem algebraically. However, taking Logarithms on both sides, we have the transformed equation as

$$1.85 \ln x = \ln 3 \qquad (2)$$

In this transformed equation, the algebraic ~~equation~~ operation and exponentiation have been changed to multiplication which immediatly gives

$$\ln x = \frac{\ln 3}{1.85}.$$

To get the required result, it is enough if we take the antilogarithm on both sides of the above equation, which yields

$$x = \ln^{-1}\left(\frac{\ln 3}{1 \cdot 85}\right).$$

With the help of any ordinary calculator, we can compute x. Following this simple example, the Laplace transform method reduces the solution of an ODE to the solution of an algebraic equation. Infact, this method has a particular advantage in finding the solution of an O.DE with appropriate ICs, without first finding the general solution and then using ICs for evaluating the arbitrary constants. Also, when the Laplace transform technique is applied to a PDE, it reduces the number of independent variables by one.

Function of exponential Order

Let $f(t)$ is a piecewise continuous function and $\exists$ a real number r_0 and a positive number M such that

$$\lim_{t \to \infty} |f(t)|\, e^{-rt} \leq M \text{ for } r > r_0.$$

and the limit does not exist when $r < r_0$. Such a function is said to be of exponential order r_0 and we write

$$|f(t)| = O(e^{r_0 t}).$$

always finite, which means that $f(t)$ is bounded. Thus, for any bounded function $f(t)$, $|f(t)|\,e^{-rt} \to 0$ for all $r>0$. The order of such a function is zero. However, variables such as electrical charge and mechanical displacement may increase without limit but of course proportional to t. Such functions are also of exponential order. (3)

Examples

(i) $\lim_{t\to\infty} t\,e^{-rt} = 0$

t^n is of exponential order zero can be seen as:

$$\lim_{t\to\infty} t^n e^{-rt} = \lim_{t\to\infty}\left(\frac{t^n}{e^{rt}}\right) = \lim_{t\to\infty}\frac{nt^{n-1}}{re^{rt}}$$

(by L-Hospital's rule)

Applying the L-Hospital rule repeatedly, we get

$$\lim_{t\to\infty} t^n e^{-rt} = \lim_{t\to\infty}\left(\frac{n!}{r^n e^{rt}}\right) = 0 \qquad r>0$$

(ii) In an unstable system a function may increase as e^{at} and

$$\lim_{t\to\infty} e^{at}e^{-rt} = \lim_{t\to\infty} e^{-(r-a)t} = 0 \text{ if } r-a>0$$

Thus the function e^{at} is of exponential order a. $\Rightarrow r>a$.

(iii) $\exp(t^n)\,(n>1)$ is not of exponential order, since $\lim_{t\to\infty}\exp(t^n)\,e^{-rt} = \lim_{t\to\infty}\exp[t(t^{n-1}-r)] \to \infty$ for any finite value of r.

Def (Laplace transform)

Let $f(t)$ be a continuous and single-valued function of the real variable t defined for all t, $0 < t < \infty$ and is of exponential order. Then the Laplace transform of $f(t)$ is defined as $\bar{f}(s)$ denoted by

$$\bar{f}(s) = L[f(t); s] = \int_0^{\infty} e^{-st} f(t)\, dt \quad \text{--- (3)}$$

over that range of values of s for which the integral exists. Here, s is a parameter, real or complex. From (3)

$$f(t) = L^{-1}[\bar{f}(s)]$$

where L is the operator that transforms $f(t)$ into $\bar{f}(s)$, called Laplace transform operator and L^{-1} is the inverse Laplace transform operator.

(*) The Laplace transform belongs to the family of 'integral transforms'. An integral transform $\bar{f}(s)$ of $f(t)$ is defined by an integral of the form

$$\int_a^b k(s,t) f(t)\, dt = \bar{f}(s) \quad \text{--- (4)}$$

where $k(s,t)$, a function of two variables s and t, is called the kernel of the integral transform. The kernels and limits of integration for various integral transforms are given as: (which is not exhaustive):

Kernels and Limits for Various Integral Transforms

Name of the Transform	$k(s,t)$	a	b
Laplace transform	e^{-st}	0	∞
Fourier transform	$e^{ist}/\sqrt{2\pi}$	$-\infty$	∞
Fourier Sine transform	$\sqrt{\frac{2}{\pi}}\sin st$	0	∞
Fourier Cosine transform	$\sqrt{\frac{2}{\pi}}\cos st$	0	∞
Hankel transform	$t\,J_n(st)$	0	∞
Mellin transform	t^{s-1}	0	∞

(*) The integral transforms defined above are applicable, either for semi-infinite or infinite domains. Similarly, finite integral transforms can be defined on finite domains.

(**) If $f(t)$ is piecewise continuous in the range $t \geq 0$ and is of exponential order γ, then the Laplace transform $\bar{f}(s)$ of $f(t)$ exists for all $s > \gamma$.

Pf
$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$
$$= I_1 + I_2$$

Since $f(t)$ is piecewise continuous function on every finite interval $0 < t < T$, I_1 exists, whereas

$$|I_2| = \left| \int_{T}^{\infty} e^{-st} f(t)\, dt \right|$$

$$\leq \int_{T}^{\infty} |e^{-st} f(t)|\, dt \longrightarrow (4a)$$

But $f(t)$ is of exponential order; therefore

$$|f(t)| < M e^{rt} \qquad \text{——————} (4b)$$

Therefore (4a) after using (4b) yields

$$|I_2| \leq \int_{T}^{\infty} M e^{-(s-r)t}\, dt$$

$$= \frac{M e^{-(s-r)T}}{s-r}, \quad s > r.$$

Thus I_2 can be made as small as we like provided T is large enough and therefore, I_2, exists. Hence $\bar{f}(s)$ exists for $s > r$.

Transform of Some elementary Functions

Example Find the Laplace transform of (i) 1 (ii) 0 (iii) t, (iv) e^{at} (v) e^{-at}.

Soln

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

(i) $\bar{f}(s) = \int_0^\infty e^{-st}(1)\, dt = \left| \frac{e^{-st}}{-s} \right|_0^\infty = \frac{1}{s}$ if $s > 0$

$\boxed{(ii)}$ $\bar{f}(s) = \int_0^\infty e^{-st}(0)\, dt = 0$

$\boxed{(iii)}$ $\bar{f}(s) = \int_0^\infty e^{-st}\, t\, dt = \int_0^\infty \underset{I}{t}\, d\left(\underset{II}{\frac{e^{-st}}{-s}}\right) dt$

(using integ. by parts)

$$= t\frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt = 1/s^2$$

$\boxed{(iv)}$ $\bar{f}(s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a},\ s>a,$$

$\boxed{(v)}$ $\bar{f}(s) = \int_0^\infty e^{-st} e^{-at}\, dt = \int_0^\infty e^{-(s+a)t}\, dt$

$$= \frac{1}{s+a}.$$

Example. Find the Laplace transform of (i) cos at (ii) sin at

Soln

$\boxed{(i)}$ $\bar{f}(s) = \int_0^\infty e^{-st} \cos at\, dt = Re \int_0^\infty e^{-st} e^{iat}\, dt$

$$= Re\left[\frac{e^{-(s-ia)}}{-(s-ia)}\right]_0^\infty$$

$$= Re\left[\frac{1}{\ldots}\right]$$

$$\bar{f}(s) = Re\left[\frac{1}{s-ia} \times \frac{s+ia}{s+ia}\right]$$

$$= Re\left[\frac{s+ia}{s^2+a^2}\right]$$

$$= \boxed{\frac{s}{s^2+a^2}}$$

$\boxed{(ii)}$ $$\bar{f}(s) = \int_0^\infty e^{-st} \sin at \, dt$$

$$= Im\left[\int_0^\infty e^{-st} e^{iat} dt\right]$$

$$= Im\left[\frac{1}{s-ia}\right]$$

$$= Im\left[\frac{s+ia}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2}$$

Example. Find the Laplace transform of (i) $\cosh at$ (ii) $\sinh at$

Sol^n (i) $f(t) = \cosh at$

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) \, dt = \int_0^\infty e^{-st} \cosh at \, dt$$

$$= \int_0^\infty e^{-st}\left[\frac{e^{at}+e^{-at}}{2}\right] dt$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}$$

(ii) $$\bar{f}(s) = \int_0^\infty e^{-st} \sinh at \, dt = \int_0^\infty e^{-st}\left[\frac{e^{at}-e^{-at}}{2}\right] dt$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$$

Ex Find the Laplace transform of t^n, where n is a positive integer. (9)

Soln $f(t) = t^n$

$$\bar{f}(s) = \int_0^\infty e^{-st} t^n \, dt = \int_0^\infty \underset{I}{t^n} \, d\left(\underset{II}{\frac{e^{-st}}{-s}}\right) dt$$

(Using integ. by parts)

$$= t^n \frac{e^{-st}}{-s}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$$

$$= \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} \, dt$$

$$= \frac{n}{s} L[t^{n-1}]$$

Similarly

$$L[t^{n-1}] = \frac{n-1}{s} L[t^{n-2}]$$

$$L[t^{n-2}] = \frac{n-2}{s} L[t^{n-3}]$$

$\vdots$

$$L[t^2] = \frac{2}{s} L[t]$$

$$L[t] = \frac{1}{s^2}$$

$\therefore$

$$L[t^n] = \bar{f}(s) = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s^2}$$

$$= \frac{n!}{s^{n+1}}$$

which can also be expressed in terms of Gamma function as

$$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}.$$

when n is positive integer then $\Gamma(n+1) = n!$.