The Wave Equation

Generally speaking, the method of separation of variables is applied to IBVPs for the wave equation in much the same way as for the heat equation. Here, however, the time component satisfies a second order ODE, which introduces a second IC.

String with fixed endpoints

The corresponding IBVP is

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad 0<x<L, \quad t>0, \qquad (1)$$

$$u(0,t)=0, \quad u(L,t)=0, \quad t>0, \qquad (2)$$

$$\left.\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned}\right\} \quad 0<x<L. \qquad (3)$$

$$u(x,t) = X(x)\,T(t) \longrightarrow (4)$$

From (1) and (4)

$$X\ddot{T} = c^2 X'' T$$

or $$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda \text{ (separation constant)}$$

$$\boxed{\begin{aligned} &X'' + \lambda X = 0, \quad 0 < x < L \\ &\ddot{T} + c^2 \lambda T = 0, \quad t > 0, \end{aligned}} \longrightarrow (5)$$

$u(0,t) = 0 \Rightarrow X(0) = 0$

$u(L,t) = 0 \Rightarrow X(L) = 0$

$$\boxed{X(0) = X(L) = 0} \longrightarrow (6)$$

so From (5) $D^2 + \lambda = 0 \quad D = \pm i\sqrt{\lambda}$

$X(x) = A_1 \cos\sqrt{\lambda}x + A_2 \sin\sqrt{\lambda}x$

$X(0) = 0 \Rightarrow A_1 = 0 \longrightarrow (6a)$

$\therefore X(x) = A_2 \sin\sqrt{\lambda}\, x$

$X(L) = 0 = A_2 \sin\sqrt{\lambda} L$

$A_2 \neq 0$ for non-trivial soln so $\sin\sqrt{\lambda}L = \sin n\pi$,

$$\sqrt{\lambda_n} = \frac{n\pi}{L}, \quad n = 1, 2, \cdots$$

$$\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \cdots} \longrightarrow (7)$$

$\therefore$ from (6a)

$$\boxed{X_n(x) = A_n \sin \frac{n\pi}{L} x} \longrightarrow (8)$$

From (5) $m^2 + c^2\lambda = 0 \Rightarrow m^2 = -\lambda c^2$

$m = \pm i\sqrt{\lambda}\, c$

$$T_n(t) = B_{1n} \cos\sqrt{\lambda_n}\, ct + B_{2n} \sin\sqrt{\lambda_n}\cdot ct$$

$$\boxed{T_n(t) = B_{1n} \cos \frac{n\pi}{L} ct + B_{2n} \sin \frac{n\pi}{L} ct} \rightarrow (9)$$

From (4), (8) and (9)

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[a_{1n} \cos \frac{n\pi}{L} ct + a_{2n} \sin \frac{n\pi}{L} ct \right]} \rightarrow (10)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_{1n} \sin \frac{n\pi}{L} x \longrightarrow (11)$$

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left(\frac{n\pi}{L} c \right) \left[-a_{1n} \sin \frac{n\pi}{L} ct + a_{2n} \cos \frac{n\pi}{L} c(t) \right]$$

$$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} a_{2n} \left(\frac{n\pi}{L} c \right) \sin \frac{n\pi}{L} x \cos \frac{n\pi}{L} c(0).$$

$$\boxed{g(x) = \sum_{n=1}^{\infty} a_{2n} \left(\frac{n\pi}{L} c \right) \sin \frac{n\pi}{L} x} \longrightarrow (12)$$

From (11) and (12)

$$a_{1n} = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx, \quad n = 1, 2, \dots$$

$$a_{2n} = \frac{2}{\left(\frac{n\pi c}{\cancel{L}}\right) \cancel{L}} \int_0^L g(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

Example Solve

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0<x<L, t>0 \longrightarrow (1)$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t>0 \longrightarrow (2)$$

$$u(x,0) = \begin{cases} x, & 0<x \leq \frac{1}{2}, \\ 1-x, & \frac{1}{2} < x < 1, \end{cases} \longrightarrow (3)$$

$$u_t(x,0) = 0, \quad 0<x<1. \longrightarrow (4)$$

Soln Here $c=1$ and $L=1$ and from previous Example we can write i.e. from Eq. (10) at P-(30)

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \left[a_{1n} \cos n\pi t + a_{2n} \sin n\pi t\right] \longrightarrow (5)$$

where

$$a_{1n} = 2\int_0^1 \overset{f(x)}{u(x,0)} \sin n\pi x \, dx, \quad n=1,2,\ldots \longrightarrow (6)$$

$$a_{2n} = \frac{2}{n\pi}\int_0^1 \overset{g(x)}{u_t(x,0)} \sin n\pi x \, dx, \quad n=1,2,\ldots \longrightarrow (7)$$

From (6)

$$a_{1n} = 2\left\{\int_0^{1/2} u(x,0) \sin n\pi x \, dx + \int_{1/2}^1 u(x,0) \sin n\pi x \, dx\right\}$$

$$= 2\left\{\int_0^{1/2} x \sin n\pi x \, dx + \int_{1/2}^1 (1-x) \sin n\pi x \, dx\right\} \quad \text{by (3)}$$

Using integ. by parts we have

$$a_{1n} = 2\left\{\left[x\left(-\frac{1}{n\pi}\right)\cos n\pi x\right]_0^{1/2} + \int_0^{1/2} \frac{1}{n\pi} \cos n\pi x \, dx\right.$$

$$\left. + \left[(1-x)\left(-\frac{1}{n\pi}\right)\cos n\pi x\right]_{1/2}^1 - \int_{1/2}^1 \frac{1}{n\pi} \cos n\pi x \, dx\right\}$$

For $\lambda > 0$

$$\boxed{T_n(t) = B_{1n} \cos \frac{n\pi ct}{L} + B_{2n} \sin \frac{n\pi ct}{L}} \rightarrow (11)$$

For $\lambda = 0$ we have from (5)

$$T'' = 0$$

$$\boxed{T_0(t) = B_{10} + B_{20} t} \rightarrow (12)$$

and from (10)

$$X_0(x) = A_0.$$

Thus the soln is

$$\boxed{u(x,t) = a_{10} + a_{20} t + \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \left(a_{1n} \cos \frac{n\pi ct}{L} + a_{2n} \sin \frac{n\pi ct}{L} \right)} \rightarrow (13)$$

(where $a_{1n} = A_m B_{1n}$ etc)

Now from (13)

$$\frac{\partial u(x,t)}{\partial t} = a_{20} + \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \left[-\frac{n\pi c}{L} a_{1n} \sin \frac{n\pi ct}{L} + \frac{n\pi c}{L} a_{2n} \cos \frac{n\pi ct}{L} \right] \rightarrow (14)$$

From (13) and (14) we get.

$$u(x,0) = f(x) = a_{10} + \sum_{n=1}^{\infty} a_{1n} \cos \frac{n\pi x}{L} \quad (15)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = a_{20} + \sum_{n=1}^{\infty} \frac{n\pi c}{L} a_{2n} \cos \frac{n\pi x}{L}, \quad (16)$$

and Consequently from (15) and (16)

$$a_{10} = \frac{1}{L}\int_0^L f(x)\,dx, \quad (17)$$

$$a_{1n} = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\,dx, \; n=1,2,\cdots \quad (18)$$

$$a_{20} = \frac{1}{L}\int_0^L g(x)\,dx, \quad (19)$$

$$a_{2n} = \frac{2}{\cancel{L}\left(\frac{n\pi c}{\cancel{L}}\right)}\int_0^L g(x) \cos \frac{n\pi x}{L}\,dx$$

or

$$a_{2n} = \frac{2}{n\pi c}\int_0^L g(x) \cos \frac{n\pi x}{L}\,dx, \; n=1,2,\cdots \quad (20)$$

Thus the solution of the given problem is (13) where a_{10}, a_{20}, a_{1n} and and a_{2n} are given by Eqs. (17) to (20), respectively.

Example To find the solution of the IBVP

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0<x<1, \quad t>0, \longrightarrow (1)$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t>0 \longrightarrow (2)$$

$$\left.\begin{array}{l} u(x,0) = \cos 2\pi x, \\ u_t(x,0) = -2\pi \cos \pi x \end{array}\right\} 0<x<1. \longrightarrow (3)$$

Soln Here $c=1$ and $L=1$ so the solution (10 at P-30) is

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \left[a_{1n} \cos n\pi t + a_{2n} \sin n\pi t \right], \longrightarrow (4)$$

$n=1,2,\ldots$

where from P-30

$$\left.\begin{array}{l} a_{1n} = 2\int_0^1 \underbrace{f(x)}_{=u(x,0)} \sin n\pi x \, dx \\ a_{2n} = \dfrac{2}{n\pi}\int_0^1 \underbrace{g(x)}_{=\frac{\partial u(x,0)}{\partial t}} \sin n\pi x \, dx \end{array}\right\} \longrightarrow (5)$$

Using (3) in (5) we have , $n=1,2,\ldots$

$$\left.\begin{array}{l} a_{1n} = 2\int_0^1 \cos 2\pi x \sin n\pi x \, dx \\ a_{2n} = \dfrac{2}{n\pi}\int_0^1 -2\pi \cos \pi x \sin n\pi x \, dx \end{array}\right] \longrightarrow (6)$$

Now from (6) and (4) we get the solution.

(37)

Example

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0<x<1, \; t>0, \quad (1)$$

$$u_x(0,t)=0, \quad u_x(1,t)=0, \quad t>0, \quad \to (2)$$

$$u(x,0) = 0, \quad \to (3)$$

$$u_t(x,0) = \begin{cases} -1, & 1/4 \le x \le 3/4, \\ 0, & \text{otherwise} \end{cases} \quad \to (4)$$

Soln. Here we have $c=1$ and $L=1$ and so soln is given by Eq. 13 at P-34 as

$$u(x,t) = a_{10} + a_{20}t + \sum_{n=1}^{\infty} \cos n\pi x \left(a_{1n}\cos n\pi t + a_{2n}\sin n\pi t\right) \quad \to (5)$$

where (17) to (20) at P-35 we have

$$a_{10} = \int_0^1 f(x)\,dx, \quad (6)$$

$$a_{1n} = 2\int_0^1 \underbrace{f(x)}_{=u(x,0)} \cos n\pi x\,dx, \; n=1,2,\ldots \quad (7)$$

$$a_{20} = \int_0^1 \underbrace{g(x)}_{=\frac{\partial u(x,0)}{\partial t}} dx \quad (8)$$

$$a_{2n} = \frac{2}{n\pi}\int_0^1 g(x)\cos n\pi x\,dx \quad (9)$$

Here $f(x) = 0 = u(x,0)$ by 3 so from (6) and (7) $\boxed{a_{10} = 0, \quad a_{1n} = 0}$, $n = 1,2,\ldots \to (10)$

From (8)

$$a_{20} = \int_0^1 \frac{\partial u}{\partial t}(x,0)\, dx$$

$$= \int_0^{1/4} \frac{\partial u}{\partial t}(x,0)\, dx + \int_{1/4}^{3/4} \frac{\partial u}{\partial t}(x,0)\, dx + \int_{3/4}^{1} \frac{\partial u}{\partial t}(x,0)\, dx$$

$$= \int_0^{1/4} (\cancel{-0})\, dx + \int_{1/4}^{3/4} (\cancel{-1})\, dx + \int_{3/4}^{1} (0)\, dx$$

$$= - \left| x \right|_{1/4}^{3/4} = \frac{-3}{4} + \frac{1}{4} = \boxed{\frac{-1}{2}}$$

$$\boxed{a_{20} = \frac{-1}{2}} \longrightarrow (11)$$

From (9)

$$a_{2n} = \frac{2}{n\pi} \left\{ \int_0^{1/4} g(x) \cos n\pi x\, dx + \int_{1/4}^{3/4} g(x) \cos n\pi x\, dx + \int_{3/4}^{1} g(x) \cos n\pi x\, dx \right\}$$

using (4) we have

$$a_{2n} = \frac{2}{n\pi} \int_{1/4}^{3/4} (-1) \cos n\pi x\, dx$$

$$= \frac{2}{n\pi} \left| - \frac{\sin n\pi x}{n\pi} \right|_{1/4}^{3/4}$$

$$a_{2n} = \frac{2}{n\pi}\left(\frac{-1}{n\pi}\right)\Big[\sin n\pi x\Big]_{1/4}^{3/4}$$

$$= \frac{-2}{n^2\pi^2}\left[\sin\frac{3n\pi}{4} - \sin\frac{n\pi}{4}\right]$$

$$\boxed{a_{2n} = \frac{2}{n^2\pi^2}\left[\sin\frac{n\pi}{4} - \sin\frac{3n\pi}{4}\right]} \to (11)$$

$$\boxed{a_n = \frac{-4}{n^2\pi^2}\sin\frac{n\pi}{4}\cos\frac{n\pi}{2}, \; n=1,2,\dots} \to (12)$$

after using $\sin\alpha - \sin\beta$

Using (10) to (12) we have from (5)

$$u(x,t) = -\frac{t}{2} - \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\sin\frac{n\pi}{4}\cos\frac{n\pi}{2}\cos(n\pi x)\sin n\pi t$$

$x = a^2 - (2-c)^2 - y^2$

$2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$