

## The Method of Separation of Variables

Separation of variables is one of the oldest and most efficient solution techniques for a certain class of partial differential equations.

The general idea is the one of the oldest techniques learned when studying first order ODE's: when applied to PDEs, this method is similar in some respects to ODEs but eventually takes on a whole new form. The theme here is

"to reduce the given PDE into number of ODEs and then construct the solutions of the governing ODEs."

### The Heat Equation

#### Rod with zero temperature at the endpoints

Consider the IBVP

$$u_t(x,t) = k u_{xx}(x,t), \quad 0 < x < L, t > 0 \quad (PDE)$$

$$\left. \begin{array}{l} u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \\ u(x,0) = f(x), \quad 0 < x < L \end{array} \right\} \begin{array}{l} (BCS) \\ (IC) \end{array} \quad (2)$$

Here we note that the PDE and BCS are linear and homogeneous.

We seek a solution of the form

$$u(x,t) = X(x)T(t) \longrightarrow (3)$$

Using (3) in (1) and (2), i.e. we have from (1) as

$$X(x)T(t) = k X''(x)T(t)$$

$$\frac{1}{k} \frac{T'}{T} = \frac{X''}{X} \longrightarrow (4)$$

Since the left hand side above is a function of  $t$  and the right hand side is a function of  $x$  alone, this equality is possible only if both sides are equal to one and the same constant, say,  $-1$ . In other words, we must have

$$\frac{1}{k} \frac{T''}{T} = \frac{X''}{X} = -1 \longrightarrow (5)$$

where  $\lambda$  is called the separation constant. This leads to separate equations for the functions  $X$  and  $T$ :

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < L, \quad (6)$$

$$T''(t) + \lambda k T(t) = 0, \quad t > 0, \quad (7)$$

From (2) and (3),

$$U(0,t) = X(0) T(t) = 0 \text{ for all } t > 0$$

We cannot have  $T(t) = 0$  since this would imply  $U(x,t) = 0$ , which does not necessarily satisfy the IC; hence it follows that

$$\boxed{X(0) = 0} \longrightarrow (8)$$

Similarly from the second BC in (2),

$$\text{i.e. } U(L,t) = 0 \Rightarrow X(L) = 0 \longrightarrow (9)$$

We are now ready to find  $X$  and  $T$ .

The function  $X$  is the solution of the Sturm-Liouville problem (6), (8) and (9).

Clearly  $X(x) = 0$  is a solution of this problem,

but it produces  $U(x,t) = 0$ , which as mentioned earlier, is not acceptable. Consequently we want non-zero solutions  $X$ . These solutions are the eigen functions  $X_n$  corresponding to the eigen values  $\lambda_n$ , both computed as

Case 1  $\lambda > 0$

$$\text{From (6), } D^2 + \lambda = 0 \Rightarrow D^2 = -\lambda \Rightarrow D = \pm i\sqrt{\lambda}$$

$$\therefore D = \frac{d}{dx}$$

The solution is

$$X(x) = A_1 \cos \sqrt{\lambda} x + A_2 \sin \sqrt{\lambda} x \quad \rightarrow (10)$$

From (10) and (8)

$$X(0) = A_1(1) + A_2(0) = 0 \Rightarrow A_1 = 0$$

$$\therefore X(x) = A_2 \sin \sqrt{\lambda} x \quad \rightarrow (11)$$

From (9) and (11),

$$X(L) = A_2 \sin \sqrt{\lambda} L = 0$$

For non-zero solution  $A_2 \neq 0$  so

$$\sin \sqrt{\lambda} L = 0 = \sin n\pi, \quad n=1, 2, \dots$$

$$\Rightarrow \sqrt{\lambda_n} L = n\pi \Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{L}$$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \rightarrow (12)$$

From (11) and (12),

$$X_n(x) = A_n \sin \frac{n\pi}{L} x \quad \rightarrow (13)$$

$$\text{From (7), } m + \lambda K = 0, \quad m = \frac{d}{dt}$$

$$m = -\lambda K$$

$$\text{So solution is } T_n(t) = C_n e^{-\lambda_n K t} = C_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \quad \rightarrow (14)$$

From (13) & (14)

$$u_n(x,t) = X_n(x) T_n(t)$$

$$= E_n \sin \frac{n\pi}{L} x e^{-(\frac{n\pi}{L})^2 kt}, \quad n=1, 2, \dots \rightarrow (15)$$

By superposition principle, we have  $E_n = A_n C_n$

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi}{L} x e^{-(\frac{n\pi}{L})^2 kt} \quad n=1, 2, \dots \rightarrow (16)$$

$$(u(x,t)) = \sum_{n=1}^{\infty} u_n(x,t)$$

From the given initial condition in (2) and eq. (16)

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi}{L} x e^0 = f(x)$$

$$\Rightarrow E_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx \rightarrow (17)$$

Thus solution is (16)  
with  $E_n$  in (17),

(as  $f$  is piecewise smooth and can be written as an infinite linear combination of the eigen functions)

### Case 2

When  $\lambda = 0$  Then from (16) and (17),

$$\begin{cases} X''(x) = 0 \\ T'(t) = 0 \\ D = 0, 0 \end{cases}$$

$$X(x) = A_1 + A_2 x \quad (A_1 \text{ and } A_2 \text{ are constants})$$

$$X(0) = 0 \Rightarrow A_1 = 0$$

$$X(L) = 0 \Rightarrow A_2 L = 0, A_2 = 0, L \neq 0$$

$$\therefore x(x) = 0$$

and hence  $\boxed{u(x,t) = 0}$  (trivial solution)

Case 3 When  $\lambda < 0$  Then  $\mu = -\lambda > 0$  and hence (6) gives

$$x''(x) - ux = 0$$

$$D^2 - \mu = 0 \quad , \quad D = \pm \sqrt{\mu}$$

$$X(x) = A_1 \cosh(\sqrt{\mu}x) + A_2 \sinh(\sqrt{\mu}x)$$

$$X(0) = 0 \Rightarrow A_1 = 0$$

$$\therefore x(x) = A_2 \sin h (\sqrt{\mu} x)$$

$$x(l) = A_2 \sin l (\sqrt{\mu} l) = 0 \longrightarrow 18$$

Since  $\sin h \sqrt{\mu} L = 0$  only when  $L=0$  which is not the case here so from (18) we get  $A_2 = 0$  and hence  $X(x) = 0$  and

$U(x,t) = 0$  (which is again unacceptable solution)

To find the solution of the IBVP

$$u_x(x,t) = u_{xx}(x,t) \quad , \quad 0 < x < L, \quad 0 < t < T, \quad t > 0,$$

$$u(x,t) = 0, \quad u(l,t) = 0, \quad t > 0, \quad \text{---} \rightarrow ②$$

$$u(x, 0) = x, \quad 0 < x < 1 \quad \longrightarrow \quad (3)$$

Sols

Here  $L = 1$ ,  $K = 1$ ,  $f(x) = x$  so

Sohn. is

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi}{l} x e^{-\left(\frac{n\pi}{l}\right)^2 t} \quad \rightarrow ④$$

$$E_n = \frac{2}{l} \int_0^l x \sin \frac{n\pi}{l} x dx \quad \begin{matrix} & \text{(calculated earlier} \\ & \text{in previous Ex.)} \end{matrix}$$

I : II      (Using integration by parts)

$$= 2 \left[ \left\{ x \left( -\frac{1}{n\pi} \right) \cos n\pi x \right\}_0^1 + \int_0^1 \frac{1}{n\pi} \cos n\pi x dx \right]$$

$$\begin{aligned}
 &= 2 \left[ \left\{ -\frac{1}{n\pi} \cos n\pi + 0 \right\} + \frac{1}{(n\pi)^2} \left\{ \sin n\pi x \right\}^2 \right] \\
 &= (-1)^{n+1} \frac{2}{n\pi} + \frac{2}{(n\pi)^2} \left[ \sin n\pi - \sin 0 \right] \\
 &= (-1)^{n+1} \frac{2}{n\pi}, \quad n = 1, 2, \dots \quad \xrightarrow{\text{5}}
 \end{aligned}$$

With (5), (4), yields

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi x) e^{-n^2\pi^2 t} \quad n = 1, 2, \dots}$$

Ex Consider the IBVP

$$U_t(x,t) = U_{xx}(x,t), \quad 0 < x < 1, \quad t > 0,$$

$$U(0,t) = 0, \quad U(1,t) = 0, \quad t > 0,$$

$$U(x,0) = \sin 3\pi x - 2 \sin 5\pi x, \quad 0 < x < 1$$

Soln

Here  $L = 1$  and  $f(x) = \sin 3\pi x - 2 \sin 5\pi x$   
 $K = 1$

Soln. is

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi}{L}\right)x e^{-(\frac{n\pi}{L})^2 t}}$$

$$E_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$$

Here the soln is

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} E_n \sin n\pi x e^{-(n\pi)^2 t}}$$

①

$$\begin{aligned}
 E_n &= 2 \int_0^1 [\sin 3\pi x - 2 \sin 5\pi x] \sin n\pi x dx \\
 &= 2 \int_0^1 \sin 3\pi x \sin n\pi x dx - 4 \int_0^1 \sin 5\pi x \sin n\pi x dx \\
 &= E_3 + E_5.
 \end{aligned}$$

The solution ① can be written as

(The integrals on the right hand side are non-zero only when  $n = 3$  and  $n = 5$ )

$$\begin{aligned}
 u(x, t) &= E_3 \sin 3\pi x e^{-9\pi^2 k t} \\
 &\quad + E_5 \sin 5\pi x e^{-25\pi^2 k t}
 \end{aligned}$$

K = 1 → Sohm

where

$$\begin{aligned}
 E_3 &= 2 \int_0^1 \sin 3\pi x \sin n\pi x dx \\
 &= 2 \int_0^1 \sin^2 3\pi x dx \quad \because \text{integral is zero for } n \neq 3 \\
 &= 2 \int_0^1 \left( \frac{1 - \cos 6\pi x}{2} \right) dx \\
 &= \left[ x - \frac{\sin 6\pi x}{6\pi} \right]_0^1 = 1 - 0 = 1
 \end{aligned}$$

Similarly

$$\begin{aligned}
 E_5 &= -4 \int_0^1 \sin 5\pi x \sin n\pi x dx \\
 &= -4 \int_0^1 \sin^2 5\pi x dx = -4 \int_0^1 \left( \frac{1 - \cos 10\pi x}{2} \right) dx \\
 &= -\frac{4}{2} [1 - 0] = -2
 \end{aligned}$$

Hence soln. is

$$u(x, t) = \sin 3\pi x e^{-9\pi^2 t} - 2 \sin 5\pi x e^{-25\pi^2 t}$$

## Rod with insulated end points

The heat conduction problem for a uniform rod with insulated end points is described by the IBVP

$$u_t(x,t) = k u_{xx}(x,t), \quad 0 < x < L, \quad t > 0, \quad (\text{PDE}) \quad \rightarrow 1$$

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \quad \rightarrow 2$$

$$u(x,0) = f(x), \quad 0 < x < L \quad \rightarrow 3$$

Soln

$$\text{Let } u(x,t) = X(x) T(t) \quad \rightarrow 4$$

With (4) and (1) we get

$$XT' = kX''T$$

$$\Rightarrow \frac{1}{k} \frac{T'}{T} = \frac{X''}{X} = -\lambda \quad (\text{a separation constant})$$

$$\Rightarrow X'' + \lambda X = 0, \quad 0 < x < L \quad \rightarrow 5$$

$$T' + k\lambda T = 0, \quad t > 0 \quad \rightarrow 6$$

From (2) and (4)

$$X'(0) T(t) = u_x(0,t) = 0 \Rightarrow X'(0) = 0$$

$$X'(L) T(t) = u_x(L,t) = 0 \Rightarrow X'(L) = 0$$

Thus

$$\boxed{X'(0) = X'(L) = 0} \quad \rightarrow 7$$

Case 1 When  $\lambda > 0$  Then from (5),

$$D^2 + \lambda = 0$$

$$X(x) = A_1 \cos \sqrt{\lambda} x + A_2 \sin \sqrt{\lambda} x \quad D^2 = -\lambda$$

$$X'(x) = -\sqrt{\lambda} A_1 \sin \sqrt{\lambda} x + \sqrt{\lambda} A_2 \cos \sqrt{\lambda} x \quad D = \pm i\sqrt{\lambda}$$

$$X'(0) = 0 \Rightarrow \sqrt{\lambda} A_2 = 0, \quad \sqrt{\lambda} \neq 0 \text{ as } A_2 = 0$$

Thus from (8),

$$X(x) = A_1 \cos \sqrt{\lambda} x \quad \rightarrow 9$$

$$X'(x) = -A_1 \sqrt{\lambda} \sin \sqrt{\lambda} x$$

$$X'(L) = 0 = -A_1 \sqrt{\lambda} \sin \sqrt{\lambda} L$$

$A_1 \neq 0$  get non-trivial soln,  $\sqrt{\lambda} \neq 0$

$$\text{So } \sin \sqrt{\lambda} L = 0 = \sin n\pi, n=1, 2, \dots$$

$$\sqrt{\lambda_n} L = n\pi$$

$$\Rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2, n=1, 2, \dots}$$

From (9) and (10) we get

$$X_n(x) = A_n \cos\left(\frac{n\pi}{L}x\right), n=0, 1, 2, \dots \quad \rightarrow (11)$$

From (6)

$$\begin{aligned} T_n(t) &= C_n e^{-k\lambda_n t} \\ &= C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned} \quad \rightarrow (12)$$

From (9), (11) and (12)

$$\begin{aligned} U(x, t) &= \sum_{n=0}^{\infty} A_n C_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ &= \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

$$\boxed{U(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}} \quad \begin{array}{l} a_n = A_n C_n \\ 0 < x < L \end{array} \quad \rightarrow (12a)$$

$$U(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) \quad \rightarrow (13)$$

From (13) and (13) we obtain

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

where  $a_0$  and  $a_n$  are Fourier cosine series coefficients of  $f$  and are given by

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, n=1, 2, \dots$$

(14)

Thus (129) is the required solution where  $a_0$  and  $a_n$  are given by Eq. (14).

Case 2 When  $\lambda = 0$  Then from (5), and (6),

$$X'' = 0 \Rightarrow X(x) = C_1 + C_2 x$$

$$X'(x) = C_2$$

$$X'(L) = 0 \Rightarrow C_2 = 0, L \neq 0 \Rightarrow C_2 = 0$$

$$\therefore X(x) = C_1$$

$$X'(0) = 0$$

$$\therefore X(x) = 0 \quad \text{Thus } U(x, t) = 0 \quad (\text{Trivial soln})$$

Case 3

When  $\lambda < 0$  Then  $\lambda = -\mu$  ( $\mu > 0$ )

$$\therefore \text{From (5)} \quad X'' - \mu X = 0 \Rightarrow D^2 - \mu = 0$$

$$\Rightarrow D = \pm \sqrt{\mu}$$

$$X(x) = C_1 \cosh \sqrt{\mu} x + C_2 \sinh \sqrt{\mu} x \quad (15)$$

$$X'(x) = \sqrt{\mu} C_1 \sinh \sqrt{\mu} x + C_2 \sqrt{\mu} \cosh \sqrt{\mu} x$$

$$X'(0) = 0 \Rightarrow C_2 = 0 \quad C_2 \sqrt{\mu} = 0, \sqrt{\mu} \neq 0$$

$\therefore$  From (15)

$$X(x) = C_1 \cosh \sqrt{\mu} x \quad (16)$$

$$X'(x) = \sqrt{\mu} C_1 \sinh \sqrt{\mu} x$$

$$X'(L) = \sqrt{\mu} C_1 \sinh \sqrt{\mu} L = 0$$

$\sqrt{\mu} \neq 0$  and  $\sinh \sqrt{\mu} L \neq 0$  (as this is possible only when  $L=0$  which is not the case here)

$$\therefore C_1 = 0$$

Thus from (16) we have

$U(x) = 0$  and consequently from Eq.(4)

$$\boxed{U(x,t) = 0}$$

Example To find the solution of the IVP

$$U_t(x,t) = k U_{xx}(x,t), \quad 0 < x < 1, \quad t > 0,$$

$$U_x(0,t) = 0, \quad U_x(1,t) = 0, \quad t > 0,$$

$$U(x,0) = x, \quad 0 < x < 1$$

Soln

Here  $L = 1$  so solution is i.e from (12a)

$$\boxed{U(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x e^{-kn^2\pi^2 t}} \rightarrow (17)$$

where from (14)

$$a_0 = \int_0^1 x dx = \left[ \frac{x^2}{2} \right]_0^1 = \frac{1}{2} \rightarrow (17a)$$

$$\begin{aligned} a_n &= 2 \int_0^1 x \cos n\pi x dx \\ &= 2 \left\{ \left[ x \frac{\sin n\pi x}{n\pi} \right]_0^1 - \frac{1}{n\pi} \int_0^1 \sin n\pi x dx \right\} \\ &= 2 \left\{ -\frac{1}{n\pi} \left[ -\frac{\cos n\pi x}{n\pi} \right]_0^1 \right\} \\ &= \frac{2}{n^2\pi^2} [ \cos n\pi - 1 ] \end{aligned}$$

$$\boxed{a_n = \frac{2}{n^2\pi^2} [ (-1)^n - 1 ]} \rightarrow (15)$$

From (17), (17a) and (18), we obtain

$$U(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [(-1)^n - 1] \cos n\pi x e^{-kn^2 \pi^2 t}$$

### Rod with mixed homogeneous boundary conditions

The method of separation of variables can also be used in the case where endpoint is held at zero temperature while the other one is insulated.

Example Consider the IBVP

$$U_t(x,t) = U_{xx}, \quad 0 < x < 1, \quad t > 0 \rightarrow ①$$

$$U(0,t) = 0, \quad U_x(1,t) = 0, \quad t > 0 \rightarrow ②$$

$$U(x,0) = 1, \quad 0 < x < 1 \rightarrow ③$$

Soln We seek a solution of the form

$$U(x,t) = X(x) T(t) \rightarrow ④$$

From ① and ④

$$T'X = X''T$$

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$$\Rightarrow X'' + \lambda X = 0 \rightarrow ⑤$$

$$\frac{T'}{T} + \lambda T = 0 \rightarrow ⑥$$

Also From ④ and ②

$$U(0,t) = 0 = X(0)T(t) \Rightarrow \boxed{X(0)=0} \rightarrow ⑦$$

$$U_x(1,t) = X'(1)T(t) = 0 \Rightarrow \boxed{X'(1)=0} \rightarrow ⑧$$

Case I

The soln. of (5) is

$$D^2 + 1 = 0$$

$$X(x) = A_1 \cos \sqrt{\lambda} x + A_2 \sin \sqrt{\lambda} x$$

$$X(0) = 0 \Rightarrow A_1 = 0$$

$$\therefore X(x) = A_2 \sin \sqrt{\lambda} x \quad \text{---} \quad (8a)$$

$$X'(x) = \sqrt{\lambda} A_2 \cos \sqrt{\lambda} x \quad \sqrt{\lambda} A_2 \neq 0$$

$$X'(1) = \sqrt{\lambda} A_2 \cos \sqrt{\lambda} = 0 \Rightarrow \cos \left( \frac{(2n-1)\pi}{2} \right) = \cos \sqrt{\lambda}$$

$$\boxed{\sqrt{\lambda_n} = \frac{(2n-1)\pi}{2}} ; \quad n = 1, 2, \dots \quad (9)$$

From (8a) and (9)

$$X_n(x) = A_n \sin \sqrt{\lambda_n} x$$

$$\boxed{X_n(x) = A_n \sin \left( \frac{(2n-1)\pi x}{2} \right)} \quad \text{---} \quad (10)$$

$\lambda_n$  are eigen values and  $X_n$  are eigenfunctions

$$\begin{aligned} \text{From (6), } T_n(t) &= C_n e^{-\lambda_n t} \\ &= C_n e^{-(2n-1)^2 \frac{\pi^2}{4} t} \end{aligned} \quad \text{---} \quad (11)$$

From (4), (10) and (11)

$$\begin{aligned} U_n(x, t) &= X_n(x) T_n(t) \\ &= A_n \sin \left( \frac{(2n-1)\pi x}{2} \right) C_n e^{-(2n-1)^2 \frac{\pi^2}{4} t}, \end{aligned}$$

or by superposition principle

$$\boxed{U(x, t) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{(2n-1)\pi x}{2} \right) e^{-(2n-1)^2 \frac{\pi^2}{4} t}} \quad \text{---} \quad (12)$$

where

$$U(x, 0) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{(2n-1)\pi x}{2} \right)$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{(2n-1)\pi x}{2} \right)$$

$$\Rightarrow a_n = 2 \int_0^1 f(x) \sin(2n-1) \frac{\pi x}{2} dx \rightarrow (13)$$

From (3),  $f(x) = 1$  so

$$a_n = 2 \int_0^1 \sin(2n-1) \frac{\pi x}{2} dx$$

$$\begin{aligned} a_n &= 2 \left| \frac{-\cos(2n-1) \frac{\pi x}{2}}{(2n-1) \frac{\pi}{2}} \right|_0^1 \\ &= \frac{-4}{(2n-1)\pi} \left[ \cos(2n-1) \frac{\pi}{2} - 1 \right] \end{aligned}$$

$$a_n = \frac{4}{(2n-1)\pi} ; n = 1, 2, \dots$$

Thus from (12)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin(2n-1) \frac{\pi x}{2} e^{-(2n-1)^2 \pi^2 t / 4} \rightarrow (14)$$

Example The solution of the IVP

$$u_t(x,t) = u_{xx}(x,t) , 0 < x < 1 , t > 0,$$

$$u(0,t) = 0, u_x(1,t) = 0 - t > 0,$$

$$u(x,0) = x , 0 < x < 1$$

Soln.

From previous example

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(2n-1) \frac{\pi x}{2} e^{-(2n-1)^2 \pi^2 t / 4}$$

$$u(x,0) = x = \sum_{n=1}^{\infty} a_n \sin(2n-1) \frac{\pi x}{2}$$

$$a_n = 2 \int_0^1 x \sin(2n-1) \frac{\pi x}{2} dx$$

$$a_n = 2 \left\{ \int_0^1 \frac{x (-c_{2n-1}) \pi x / 2}{(2n-1) \pi / 2} dx \right\} + \int_0^1 \frac{c_{2n-1} \pi x / 2}{(2n-1) \pi / 2} dx$$

$$= 2 \left| \frac{\sin((2n-1)\pi x / 2)}{(2n-1)^2 \pi^2 / 4} \right|_0^1 = \frac{2 \sin((2n-1)\pi / 2)}{(2n-1)^2 \pi^2 / 4}$$

$\therefore$  from (2), we have

$$U(x, t) = 2 \sum_{n=1}^{\infty} \left[ \frac{\sin((2n-1)\pi / 2)}{(2n-1)^2 \pi^2 / 4} \right] \sin((2n-1) \frac{\pi}{2} x) e^{-(2n-1)^2 \pi^2 t / 4}$$

Example The solution of the IVP

$$U_t(x, t) = U_{xx}(x, t), \quad 0 < x < 1, \quad t > 0 \longrightarrow ①$$

$$U_x(0, t) = 0, \quad U(1, t) = 0, \quad t > 0 \longrightarrow ②$$

$$U(x, 0) = x, \quad 0 < x < 1, \longrightarrow ③$$

Sohm

$$U(x, t) = X(x)T(t) \longrightarrow ④$$

Then from ①

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$$\Rightarrow \boxed{\begin{aligned} X'' + \lambda X &= 0 \\ T' + \lambda T &= 0 \end{aligned}} \longrightarrow ⑤$$

$$U_x(0, t) = 0 \Rightarrow X'(0)T(t) = 0$$

$$\Rightarrow X'(0) = 0, \quad U(1, t) = 0 \Rightarrow X(1)T(t) = 0$$

$$\text{Thus } X'(0) = 0, \quad X(1) = 0 \Rightarrow X(1) = 0 \longrightarrow ⑥$$

$$\text{From } ⑤, \quad (D^2 + \lambda)X = 0 \Rightarrow D^2 = -\lambda, \\ D = \pm i\sqrt{\lambda}$$

$$X(x) = A_1 \cos \sqrt{A} x + A_2 \sin \sqrt{A} x$$

$$X'(x) = -\sqrt{A} A_1 \sin \sqrt{A} x + A_2 \sqrt{A} \cos \sqrt{A} x$$

$$X'(0) = 0 \Rightarrow A_2 \sqrt{A} = 0 \Rightarrow A_2 = 0 \quad (\text{as } \sqrt{A} \neq 0)$$

$$\therefore X(x) = A_1 \cos \sqrt{A} x$$

$$X(1) = A_1 \cos \sqrt{A} = 0$$

$$\Rightarrow \cos \sqrt{A} = 0 \quad (A_1 \neq 0 \text{ give non-trivial}$$

or  $\cos \sqrt{A} = \cos \left(\frac{(2n-1)}{2}\pi\right) \pi \quad \text{solution})$

$$\Rightarrow \sqrt{A_n} = \frac{(2n-1)}{2}\pi$$

$$\Rightarrow \boxed{A_n = \frac{(2n-1)^2 \pi^2}{4}, \quad n=1, 2, \dots}$$

$$\boxed{X_n(x) = A_n \cos \left(\frac{(2n-1)\pi}{2} x\right)} \rightarrow (7)$$

From (5)  $T(t) = C_n e^{-\lambda t}$

Using (7) and (8), in (4), we get  $\rightarrow (8)$

$$U(x,t) = \sum_{n=1}^{\infty} a_n \cos \left(\frac{(2n-1)\pi}{2} x\right) e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 \pi t}$$

$$U(x,0) = \sum_{n=1}^{\infty} a_n \cos \left(\frac{(2n-1)\pi}{2} x\right) \quad \boxed{a_n = A_n C_n}$$

Using (3), we get

$$x = \sum_{n=1}^{\infty} a_n \cos \left(\frac{(2n-1)\pi}{2} x\right)$$

$$\Rightarrow a_n = \frac{2}{1} \int_0^1 x \cos \left(\frac{(2n-1)\pi}{2} x\right) dx$$

$$\begin{aligned}
 &= 2 \left\{ \int_0^1 x \frac{2}{(2n-1)\pi} \sin((2n-1)\frac{\pi x}{2}) dx \right. \\
 &\quad \left. - \int_0^1 \frac{2}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{2} dx \right\} \\
 &= 2 \left\{ (-1)^{n+1} \frac{2}{(2n-1)\pi} + \frac{4}{(2n-1)^2\pi^2} \left[ \cos \frac{(2n-1)\pi x}{2} \right]_0^1 \right\} \\
 &= (-1)^{n+1} \frac{4}{(2n-1)\pi} - \frac{8}{(2n-1)^2\pi^2}, \quad n=1, 2, \dots
 \end{aligned}$$

Consequently the soln. is

$$U(x, t) = \sum_{n=1}^{\infty} \left[ (-1)^{n+1} \frac{4}{(2n-1)\pi} - \frac{8}{(2n-1)^2\pi^2} \right] \cos \frac{(2n-1)\pi x}{2} e^{-(2n-1)^2\pi^2 t/4}$$

Rod with endpoint in a zero temperature medium.

Consider the problem of heat flow in a uniform rod with out internal sources, when the near end point is kept at zero temperature and the far point end point is kept in open air of zero temperature.

Corresponding IBVP is

$$U_t(x, t) = k U_{xx}(x, t), \quad 0 < x < L, \quad t > 0,$$

$$U(0, t) = 0, \quad U_x(L, t) + h U(L, t) = 0, \quad t > 0, \quad \rightarrow ①$$

$$U(x, 0) = f(x), \quad 0 < x < L \quad \rightarrow ②$$

where  $h = \text{constant} > 0$ .

Since the PDE and BCS are linear and homogeneous, we use separation of variables and seeks a soln. of the form

$$U(x, t) = X(x) T(t) \quad \rightarrow ④$$

Using ④, ①, yields

$$\frac{T'}{T} = k \frac{X''}{X}$$

$$\Rightarrow \frac{T'}{RT} = \frac{x''}{x} = -\lambda$$

$$\Rightarrow \boxed{x'' + \lambda x = 0 \longrightarrow ⑤}$$

$$T' + \lambda k T = 0 \longrightarrow ⑥$$

From ② and ④ we get

$$u(0,t) = x(0)T(t) = 0 \Rightarrow \boxed{x(0) = 0}$$

$$u_x(L,t) + h u(L,t) = 0 \Rightarrow [x'(L) + h x(L)]T(t) = 0$$

$$\Rightarrow x'(L) + h x(L) = 0$$

$$\Rightarrow \boxed{x(0) = 0, x'(L) + h x(L) = 0} \longrightarrow ⑦$$

From ⑤ we have  $D^2 = -\lambda \Rightarrow D = \pm i\sqrt{\lambda}$

$$x(x) = A_1 \cos \sqrt{\lambda} x + A_2 \sin \sqrt{\lambda} x$$

$$x(0) = 0 \Rightarrow \boxed{A_1 = 0}$$

$$\boxed{x(x) = A_2 \sin \sqrt{\lambda} x} \longrightarrow ⑧$$

$$x'(x) = \sqrt{\lambda} A_2 \cos \sqrt{\lambda} x \longrightarrow ⑨$$

From ⑦ to ⑨ we have

$$\sqrt{\lambda} A_2 \cos \sqrt{\lambda} L + h A_2 \sin \sqrt{\lambda} L = 0$$

$$A_2 [\sqrt{\lambda} + h \tan \sqrt{\lambda} L] = 0$$

$A_2 \neq 0$  give non-trivial solution, so

$$\sqrt{\lambda} + h \tan \sqrt{\lambda} L = 0$$

$$\boxed{\tan \sqrt{\lambda} L = -\frac{\sqrt{\lambda}}{h}} \longrightarrow ⑩$$

$$\text{Let } \xi = \sqrt{\lambda} L$$

$$\sqrt{\lambda} = \frac{\xi}{L}$$

$$\therefore \boxed{\tan \xi = -\frac{\xi}{hL}} \longrightarrow ⑪$$

Let  $\xi_n$  are the roots of (11), Then

$$\begin{aligned}\xi_n &= \sqrt{\lambda_n} L \Rightarrow \sqrt{\lambda_n} = \frac{\xi_n}{L} \\ \Rightarrow \boxed{\lambda_n &= \left(\frac{\xi_n}{L}\right)^2}\end{aligned}\quad \rightarrow (12)$$

$\therefore$  From (8),  $n = 1, 2, \dots$

$$X_n(x) = A_n \sin \frac{\xi_n}{L} x \longrightarrow (13)$$

$$\begin{aligned}\text{From (6), } T(t) &= C_n e^{-\lambda_n k t} \\ &= C_n e^{-(\frac{\xi_n}{L})^2 k t}\end{aligned}\quad \rightarrow (14)$$

From (4), (13) and (14) we have

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{\xi_n}{L} x e^{-(\frac{\xi_n}{L})^2 k t}}\quad \rightarrow (15)$$

$$U(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{\xi_n}{L} x$$

or using (3),

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{\xi_n}{L} x$$

$$\Rightarrow a_n = \int_0^L f(x) \sin(\xi_n x/L) dx / \int_0^L \sin(\xi_n x/L) dx$$

$n = 1, 2, \dots$

$$\rightarrow (16)$$

Thus the solution of the IVP is (15), with  $a_n$  given by (16).