CHAPTER TWO

Mathematical Models

2.1 THE CLASSICAL EQUATIONS

The three basic types of second-order partial differential equations are:

a. The wave equation

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0 \qquad (2.1.1)$$

b. The heat equation

$$u_t - k(u_{xx} + u_{yy} + u_{zz}) = 0 \qquad (2.1.2)$$

c. The Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0 \qquad (2.1.3)$$

In this section, we list a few more common linear partial differential equations of importance in applied mathematics, mathematical physics, and engineering science. Such a list naturally cannot ever be complete. Included are only equations of most common interest:

d. The Poisson equation

$$\nabla^2 u = f(x, y, z) \qquad (2.1.4)$$

e. The Helmholtz equation

$$\nabla^2 u + \lambda u = 0 \qquad (2.1.5)$$

f. The biharmonic equation

$$\nabla^4 u = \nabla^2(\nabla^2 u) = 0 \qquad (2.1.6)$$

g. The biharmonic wave equation

$$u_{tt} + c^2 \nabla^4 u = 0 \qquad (2.1.7)$$

h. The telegraph equation

$$u_{tt} + au_t + bu = c^2 u_{xx} \tag{2.1.8}$$

i. The Schrödinger equations in Quantum Physics

$$i\hbar\psi_t = \left[\left(-\frac{\hbar^2}{2m}\right)\nabla^2 + V(x, y, z)\right]\psi \tag{2.1.9}$$

$$\nabla^2\Psi + \frac{2m}{\hbar^2}[E - V(x, y, z)]\Psi = 0 \tag{2.1.10}$$

j. The Klein–Gordon equation

$$\Box u + \lambda^2 u = 0 \tag{2.1.11}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.1.12}$$

is the Laplace operator in rectangular Cartesian coordinates (x, y, z),

$$\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{2.1.13}$$

is the D'Alembertian, and in all equations λ, a, b, c, m, E are constants and $h = 2\pi\hbar$ is the Planck constant.

Many problems in mathematical physics reduce to the solving of partial differential equations, in particular, the partial differential equations listed above. We will begin our study of these equations by first examining in detail the mathematical models representing physical problems.

2.2 THE VIBRATING STRING

One of the most important problems in mathematical physics is the vibration of a stretched string. Simplicity and frequent occurrence in many branches of mathematical physics make it a classic example in the theory of partial differential equations.

Let us consider a stretched string of length l fixed at the end points. The problem here is to determine the equation of motion which characterizes the position $u(x, t)$ of the string at time t after an initial disturbance is given.

In order to obtain a simple equation we make the following assumptions:

1. The string is flexible and elastic, that is, the string cannot resist bending moment and thus the tension in the string is always in the direction of the tangent to the existing profile of the string.

2. There is no elongation of a single segment of the string and hence, by Hooke's law, the tension is constant.
3. The weight of the string is small compared with the tension in the string.
4. The deflection is small compared with the length of the string.
5. The slope of the displaced string at any point is small compared with unity.
6. There is only pure transverse vibration.

We consider a differential element of the string. Let T be the tension at the end points as shown in Fig. 2.2.1. The forces acting on the element of the string in the vertical direction are

$$T \sin \beta - T \sin \alpha$$

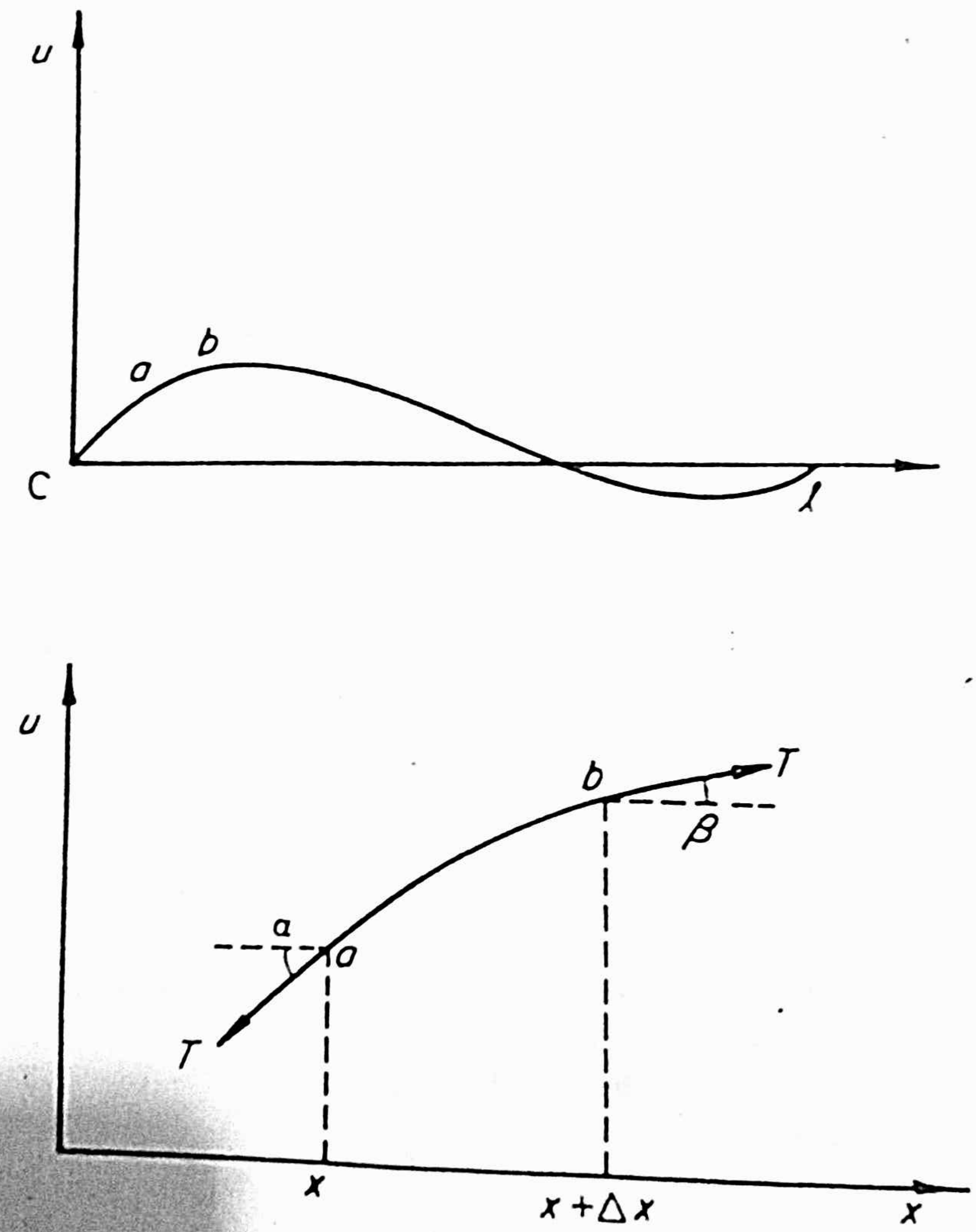

Figure 2.2.1

By Newton's second law of motion, the resultant force is equal to the mass times the acceleration. Hence,

$$T \sin \beta - T \sin \alpha = \rho \, \Delta s u_{tt} \tag{2.2.1}$$

where ρ is the line density and Δs is the small arc length of the string. Since the slope of the displaced string is small, we have

$$\Delta s \simeq \Delta x$$

Since the angles α and β are small

$$\sin \alpha \simeq \tan \alpha \qquad \sin \beta \simeq \tan \beta$$

Thus Eq. (2.2.1) becomes

$$\tan \beta - \tan \alpha = \frac{\rho \, \Delta x}{T} u_{tt} \tag{2.2.2}$$

But, from calculus we know that

$$\tan \alpha = (u_x)_x$$

and

$$\tan \beta = (u_x)_{x+\Delta x}$$

at time t. Equation (2.2.2) may thus be written as

$$\frac{1}{\Delta x}[(u_x)_{x+\Delta x} - (u_x)_x] = \frac{\rho}{T} u_{tt}$$

In the limit as Δx approaches zero, we find

$$u_{tt} = c^2 u_{xx} \tag{2.2.3}$$

where $c^2 = T/\rho$. This is called the *one-dimensional wave equation.*

If there is an external force f per unit length acting on the string, Eq. (2.2.3) assumes the form

$$u_{tt} = c^2 u_{xx} + f^*, \qquad f^* = f/\rho \tag{2.2.4}$$

where f may be pressure, gravitation, resistance, and so on.

2.3 THE VIBRATING MEMBRANE

The equation of the vibrating membrane occurs in a great number of problems in applied mathematics and mathematical physics. Before we derive the equation for the vibrating membrane we make certain simplifying assumptions as in the case of the vibrating string:

1. The membrane is flexible and elastic, that is, the membrane cannot resist bending moment and the tension in the membrane is always in the direction of the tangent to the existing profile of the membrane.

2. There is no elongation of a single element of the membrane and hence, by Hooke's law, the tension is constant.
3. The weight of the membrane is small compared with the tension in the membrane.
4. The deflection is small compared with the minimal diameter of the membrane.
5. The slope of the displayed membrane at any point is small compared with unity.
6. There is only pure transverse vibration.

We consider a small element of the membrane. Since the deflection and slope are small, the area of the element is approximately equal to $\Delta x \Delta y$. If T is the tensile force per unit length, then the forces acting on the sides of the element are $T\,\Delta x$ and $T\,\Delta y$, as shown in Fig. 2.3.1.

The forces acting on the element of the membrane in the vertical direction are

$$T\,\Delta x \sin\beta - T\,\Delta x \sin\alpha + T\,\Delta y \sin\delta - T\,\Delta y \sin\gamma$$

Since the slopes are small, sines of the angles are approximately equal to their tangents. Thus the resultant force becomes

$$T\,\Delta x(\tan\beta - \tan\alpha) + T\,\Delta y(\tan\delta - \tan\gamma)$$

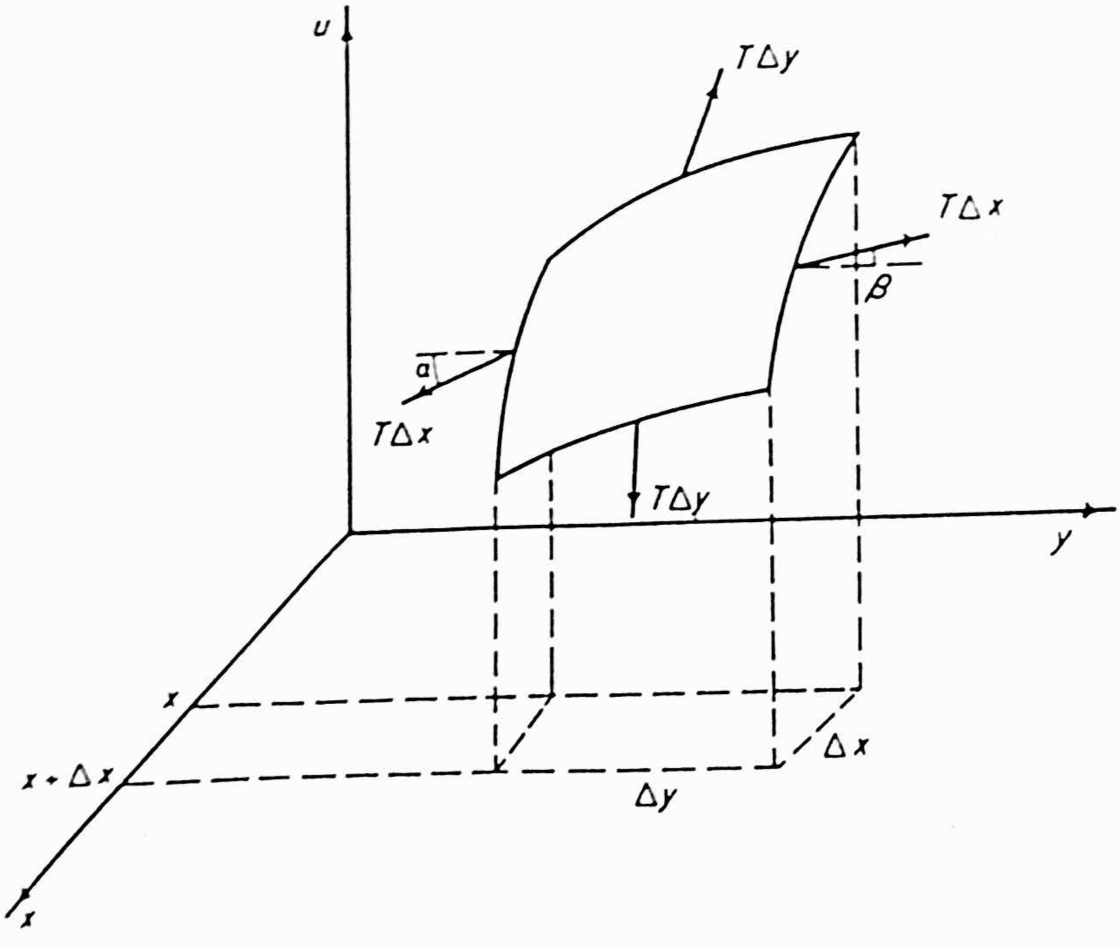

Figure 2.3.1

By Newton's second law of motion, the resultant force is equal to the mass times the acceleration. Hence

$$T\,\Delta x(\tan\beta - \tan\alpha) + T\,\Delta y(\tan\delta - \tan\gamma) = \rho\,\Delta A u_{tt} \qquad (2.3.1)$$

where ρ is the mass per unit area, $\Delta A \simeq \Delta x \Delta y$ is the area of this element, and u_{tt} is computed at some point in the region under consideration. But from calculus, we have

$$\tan\alpha = u_y(x_1, y)$$
$$\tan\beta = u_y(x_2, y + \Delta y)$$
$$\tan\gamma = u_x(x, y_1)$$
$$\tan\delta = u_x(x + \Delta x, y_2)$$

where x_1 and x_2 are the values of x between x and $x + \Delta x$, and y_1 and y_2 are the values of y between y and $y + \Delta y$. Substituting these values in (2.3.1), we obtain

$$T\,\Delta x[u_y(x_2, y + \Delta y) - u_y(x_1, y)] + T\,\Delta y[u_x(x + \Delta x, y_2) - u_x(x, y_1)] = \rho\,\Delta x\,\Delta y u_{tt}$$

Division by $\rho\,\Delta x\,\Delta y$ yields

$$\frac{T}{\rho}\left[\frac{u_y(x_2, y + \Delta y) - u_y(x_1, y)}{\Delta y} + \frac{u_x(x + \Delta x, y_2) - u_x(x, y_1)}{\Delta x}\right] = u_{tt} \qquad (2.3.2)$$

In the limit as Δx approaches zero and Δy approaches zero, we obtain

$$u_{tt} = c^2(u_{xx} + u_{yy}) \qquad (2.3.3)$$

where $c^2 = T/\rho$. This equation is called the *two-dimensional wave equation*.

If there is an external force f per unit area acting on the membrane, Eq. (2.3.3) takes the form

$$u_{tt} = c^2(u_{xx} + u_{yy}) + f^* \qquad (2.3.4)$$

where $f^* = f/\rho$

2.4 WAVES IN AN ELASTIC MEDIUM

If a small disturbance is originated at a point in an elastic medium, neighboring particles are set into motion, and the medium is put under a state of strain. We consider such states of motion to extend in all directions. We assume that the displacements of the medium are small and that we are not concerned with translation or rotation of the medium as a whole.

Let the body under investigation be homogeneous and isotropic. Let

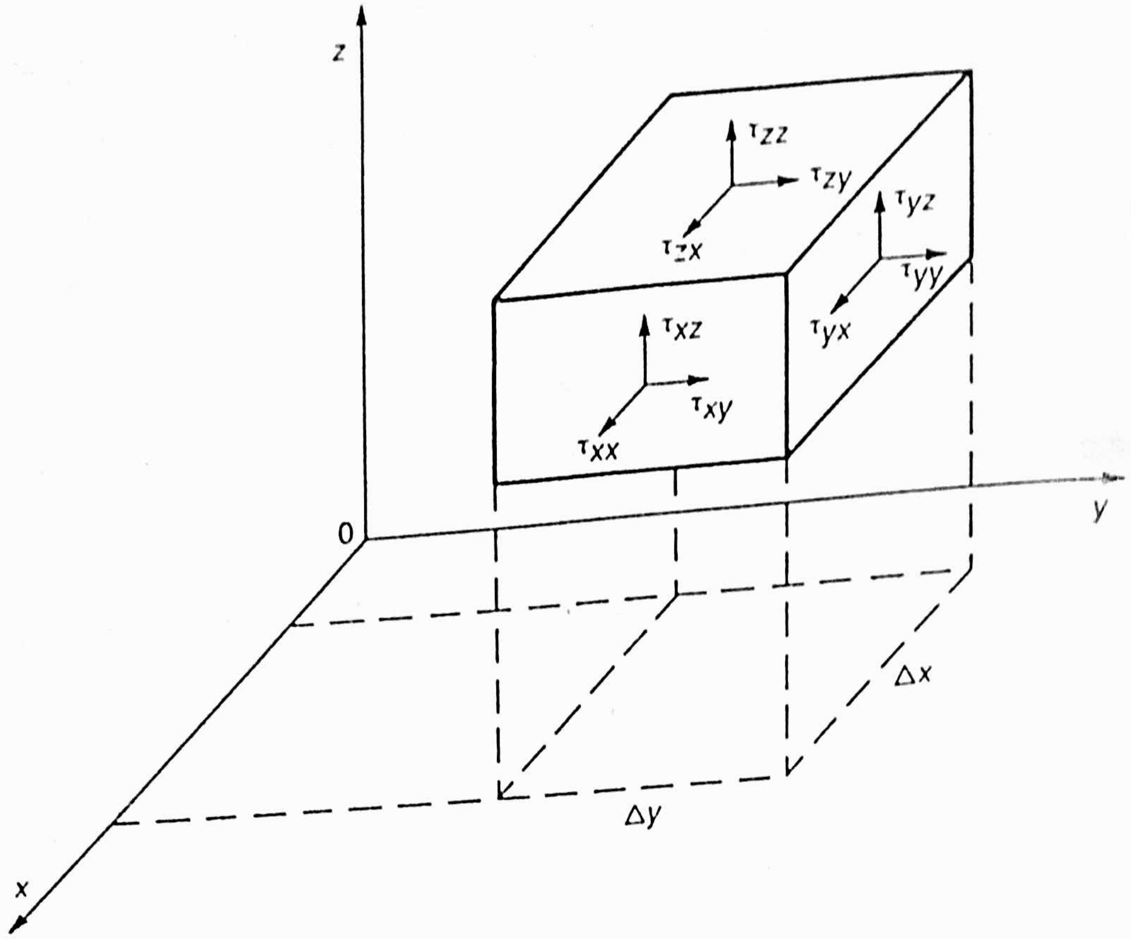

Figure 2.4.1

ΔV be a differential volume of the body, and let the stresses acting on the faces of the volume be $\tau_{xx}, \tau_{yy}, \tau_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}$. The first three stresses are called the *normal stresses* and the rest are called the *shear stresses*. (See Fig. 2.4.1)

We shall assume that the stress tensor τ_{ij} is symmetric,[3] that is,

$$\tau_{ij} = \tau_{ji} \quad i \neq j \quad i, j = x, y, z \tag{2.4.1}$$

Neglecting the body forces, the sum of all the forces acting on the volume element in the x-direction is

$$[(\tau_{xx})_{x+\Delta x} - (\tau_{xx})_x]\,\Delta y\,\Delta z + [(\tau_{xy})_{y+\Delta y} - (\tau_{xy})_y]\,\Delta z\,\Delta x + [(\tau_{xz})_{z+\Delta z} - (\tau_{xz})_z]\,\Delta x\,\Delta y$$

By Newton's law of motion this resultant force is equal to the mass times the acceleration. Thus we obtain

$$[(\tau_{xx})_{x+\Delta x} - (\tau_{xx})_x]\,\Delta y\,\Delta z + [(\tau_{xy})_{y+\Delta y} - (\tau_{xy})_y]\,\Delta z\,\Delta x + [(\tau_{xz})_{z+\Delta z} - (\tau_{xz})_z]\,\Delta x\,\Delta y = \rho\,\Delta x\,\Delta y\,\Delta z\,u_{tt} \tag{2.4.2}$$

[3] The condition of the rotational equilibrium of the volume element.

where ρ is the density of the body and u is the displacement component in the x-direction. Hence, in the limit as ΔV approaches zero, we obtain

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} \tag{2.4.3}$$

Similarly, the following two equations corresponding to y and z directions are obtained:

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho \frac{\partial^2 v}{\partial t^2} \tag{2.4.4}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{2.4.5}$$

where v and w are the displacement components in the y and z directions respectively.

We may now define linear strains [40] as

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} & \varepsilon_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} & \varepsilon_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} & \varepsilon_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{aligned} \tag{2.4.6}$$

in which ε_{xx}, ε_{yy}, ε_{zz} represent unit elongations and ε_{yz}, ε_{zx}, ε_{xy} represent unit shearing strains.

In the case of an isotropic body, generalized Hooke's law takes the form

$$\begin{aligned} \tau_{xx} &= \lambda\theta + 2\mu\varepsilon_{xx} & \tau_{yz} &= \mu\varepsilon_{yz} \\ \tau_{yy} &= \lambda\theta + 2\mu\varepsilon_{yy} & \tau_{zx} &= \mu\varepsilon_{zx} \\ \tau_{zz} &= \lambda\theta + 2\mu\varepsilon_{zz} & \tau_{xy} &= \mu\varepsilon_{xy} \end{aligned} \tag{2.4.7}$$

where $\theta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ is called the dilatation and λ and μ are Lame's constants.

Expressing stresses in terms of displacements we have

$$\begin{aligned} \tau_{xx} &= \lambda\theta + 2\mu \frac{\partial u}{\partial x} \\ \tau_{xy} &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \tau_{xz} &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned} \tag{2.4.8}$$

By differentiating Eqs. (2.4.8), we obtain

$$\begin{aligned}
\frac{\partial \tau_{xx}}{\partial x} &= \lambda \frac{\partial \theta}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} \\
\frac{\partial \tau_{xy}}{\partial y} &= \mu \frac{\partial^2 v}{\partial x\, \partial y} + \mu \frac{\partial^2 u}{\partial y^2} \qquad (2.4.9) \\
\frac{\partial \tau_{xz}}{\partial z} &= \mu \frac{\partial^2 w}{\partial x\, \partial z} + \mu \frac{\partial^2 u}{\partial z^2}
\end{aligned}$$

Substitution of Eqs. (2.4.9) into Eq. (2.4.3) yields

$$\lambda \frac{\partial \theta}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 w}{\partial x\, \partial z}\right) + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho \frac{\partial^2 u}{\partial t^2} \qquad (2.4.10)$$

We note that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 w}{\partial x\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial \theta}{\partial x}$$

and introduce the notation

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The symbol Δ or ∇^2 is called the *Laplace operator*. Hence, Eq. (2.4.10) becomes

$$(\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu \nabla^2 u = \rho \frac{\partial^2 u}{\partial t^2} \qquad (2.4.11)$$

In a similar manner, we obtain the other two equations which are

$$(\lambda + \mu)\frac{\partial \theta}{\partial y} + \mu \nabla^2 v = \rho \frac{\partial^2 v}{\partial t^2} \qquad (2.4.12)$$

$$(\lambda + \mu)\frac{\partial \theta}{\partial z} + \mu \nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2} \qquad (2.4.13)$$

In vector form, the equations of motion assume the form

$$(\lambda + \mu)\, \text{grad div}\, \mathbf{u} + \mu \nabla^2 \mathbf{u} = \rho \mathbf{u}_{tt} \qquad (2.4.14)$$

where $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and $\theta = \text{div}\, \mathbf{u}$.

(i) If div $\mathbf{u} = 0$, the general equation becomes

$$\mu \nabla^2 \mathbf{u} = \rho \mathbf{u}_{tt}$$

or

$$\mathbf{u}_{tt} = c^2 \nabla^2 \mathbf{u} \qquad (2.4.15)$$

where the velocity c of a propagated wave is

$$c = \sqrt{\mu/\rho}$$

This is the case of an equivoluminal wave propagation, since the volume expansion θ is zero for waves moving with this velocity. Sometimes these waves are called *waves of distortion* because the velocity of propagation depends on μ and ρ; the shear modulus μ characterizes the distortion and rotation of the volume element.

(ii) When curl $\mathbf{u} = \mathbf{0}$, the identity

$$\text{curl curl } \mathbf{u} = \text{grad div } \mathbf{u} - \nabla^2 \mathbf{u}$$

gives

$$\text{grad div } \mathbf{u} = \nabla^2 \mathbf{u}$$

Then the general equation becomes

$$(\lambda + 2\mu)\nabla^2 \mathbf{u} = \rho \mathbf{u}_{tt}$$

or

$$\mathbf{u}_{tt} = c^2 \nabla^2 \mathbf{u} \tag{2.4.16}$$

where the velocity of propagation is

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

This is the case of an *irrotational* or *dilatational* wave propagation, since curl $\mathbf{u} = \mathbf{0}$ describes irrotational motion. Equations (2.4.15) and (2.4.16) are called the *three-dimensional wave equations.*

In general the wave equation may be written as

$$u_{tt} = c^2 \nabla^2 u \tag{2.4.17}$$

where the Laplace operator may be one, two, or three dimensional. The importance of the wave equation stems from the fact that this type of equation arises in many physical problems; for example, sound waves in space, electrical vibration in a conductor, torsional oscillation of a rod, shallow water waves, linearized supersonic flow in a gas, waves in an electric transmission line, waves in magnetohydrodynamics, and longitudinal vibrations of a bar.

2.5 CONDUCTION OF HEAT IN SOLIDS

We consider a domain D^* bounded by a closed surface B^*. Let $u(x, y, z, t)$ be the temperature at a point (x, y, z) at time t. If the temperature is not constant, heat flows from places of higher temperature to places of lower temperature. Fourier's law states that the rate of flow is proportional to the gradient of the temperature. Thus the velocity of the heat flow in an isotropic body is

$$\mathbf{v} = -K \operatorname{grad} u \tag{2.5.1}$$

where K is a constant, called the thermal conductivity of the body.

Let D be an arbitrary domain bounded by a closed surface B in D^*. Then the amount of heat leaving D per unit time is

$$\iint_B v_n \, ds$$

where $v_n = \mathbf{v} \cdot \mathbf{n}$ is the component of $\mathbf{v}$ in the direction of the outer unit normal $\mathbf{n}$ of B. Thus by Gauss' theorem (Divergence theorem)

$$\iint_B v_n \, ds = \iiint_D \operatorname{div}(-K \operatorname{grad} u) \, dx \, dy \, dz$$

$$= -K \iiint_D \nabla^2 u \, dx \, dy \, dz \tag{2.5.2}$$

But the amount of heat in D is given by

$$\iiint_D \sigma \rho u \, dx \, dy \, dz \tag{2.5.3}$$

where ρ is the density of the material of the body and σ is its specific heat. Assuming that integration and differentiation are interchangeable, the rate of decrease of heat in D is

$$-\iiint_D \sigma \rho \frac{\partial u}{\partial t} \, dx \, dy \, dz \tag{2.5.4}$$

Since the rate of decrease of heat in D must be equal to the amount of heat leaving D per unit time, we have

$$-\iiint_D \sigma \rho u_t \, dx \, dy \, dz = -K \iiint_D \nabla^2 u \, dx \, dy \, dz$$

or

$$\iiint_D [\sigma \rho u_t - K \nabla^2 u] \, dx \, dy \, dz = 0 \tag{2.5.5}$$

for an arbitrary D in D^*. We assume that the integrand is continuous. If we suppose that the integrand is not zero at a point (x_0, y_0, z_0) in D, then by continuity, the integrand is not zero in the small region surrounding the point (x_0, y_0, z_0). Continuing in this fashion we extend the region encompassing D. Hence the integral must be nonzero. This contradicts (2.5.5). Thus, the integrand is zero everywhere, that is,

$$u_t = k\nabla^2 u \tag{2.5.6}$$

where $k = K/\sigma\rho$. This is known as the *heat equation*.

This type of equation appears in a great variety of problems in mathematical physics, for example the concentration of diffusing material, the motion of a tidal wave in a long channel, transmission in electrical cables, and unsteady boundary layers in viscous fluid flows.