## Example

Consider the partial differential equation as

$$u_x - u_y = 0 \quad \text{—— (i)}$$ Partial

Using the transformation

$$\xi = x+y \quad \text{—— (ii)}$$

$$\eta = x-y$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = -u_\eta$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \eta} = u_\eta$$

(1) gives

$$2u_\eta = 0$$

$2 \neq 0$ so $\Rightarrow$ $u_\eta = 0$

Integ $u = f(\xi) = f(x+y)$ by (ii)

General solution

$$\boxed{u(x,y) = f(x+y)}$$

where $f(x+y)$ is an arbitrary function. Thus we see that each of the functions

$$(x+y)^n$$

$$\sin n(x+y)$$

$$\cos n(x+y)$$

$$\exp n(x+y), \quad n = 1,2,3,\ldots$$

is a solution of Eq.-(i). The fact that a simple equation such as (i) yields infinitely many solutions is an indication of an added difficulty which must be overcome in the study of partial differential equations. Thus, we generally prefer to directly determine the particular solution of a partial differential equation satisfying prescribed supplementary conditions.

## Superposition principle

If $u_1, u_2$ are solutions of a linear homogeneous partial differential equation then $w = c_1 u_1 + c_2 u_2$ ($c_1$ and $c_2$ are constants) is also a solution of the equation

~~[illegible]~~

**Example.** Consider the one dimensional heat equation

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

If $u_1$ and $u_2$ are two solutions of above equation then

$$\frac{1}{k}\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2} \longrightarrow (2)$$

$$\frac{1}{k}\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} \longrightarrow (3)$$

We have to show that $c_1 u_1 + c_2 u_2$ is also a solution of Eq. (1).

i.e. $$\frac{1}{k}\frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) = c_1\left(\frac{1}{k}\frac{\partial u_1}{\partial t}\right) + c_2\left(\frac{1}{k}\frac{\partial u_2}{\partial t}\right)$$

$$= c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2} \quad \text{(by using (2) and}$$

$$= \frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2)$$

Proving that $c_1u_1 + c_2u_2$ is also a solution of (1).

**Note** Although the superposition principle is stated for two solutions only, it is true for any finite linear combination of solutions. Furthermore, under proper restrictions it is also true for any infinite number of solutions. If $u_i, i = 1, 2, \ldots$ are solutions of a homogeneous linear partial differential equation, then

$$w = \sum_{i=1}^{\infty} c_i u_i$$

is also a solution of the equation.

For Linear inhomogeneous equations we have the following result.

**Result** If $u_1, u_2$ are solutions of a linear inhomogeneous equation then $u_1 - u_2$ is a solution of the corresponding homogeneous equation.

Consider the general form of a ~~second~~ partial differential equation

$$au + \sum_{i}^{n} b_i (u)_{x_i} + \cdots = f(x)$$

If $u_1$ and $u_2$ are two solutions of above equation then

$$au_1 + \sum_{i}^{n} b_i (u_1)_{x_i} + \cdots = f(x)$$

$$au_2 + \sum_{i}^{n} b_i (u_2)_{x_i} + \cdots = f(x)$$

subtracting

$$a(u_1 - u_2) + \sum_{i=1}^{n} b_i (u_1 - u_2)_{x_i} = 0$$

as desired

## Three important Partial differential Equations

(a) The heat equation

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(b) The wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

(c) The Laplace equation or the potential equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

## Boundary and Initial Conditions

- For ordinary differential equation, it is easy to solve the problem since the adjustment of the constants $c_1, \dots, c_n$ in the general solution

$$y = c_1 u_1 + \dots + c_n u_n$$

(where $u_1, u_2, \dots u_n$ is a fundamental set of solutions to a given linear equation)

to the boundary conditions is reducible to a finite set of algebraic equations that can be solved with little effort.

The corresponding problem for partial differential equations is far more complicated, since, in general, the number of independent solutions for such equations is infinite. Moreover, the boundaries of the regions of the independent variables over which we desire to solve the equations are not discrete points as in the one-dimensional case but are continuous curves or surfaces. Thus, the complete formulation of a

...ical system in terms of partial differential ...ations requires careful attention not only to the equations that govern the system but also to the correct formulation of the boundary conditions. Furthermore, most differential equations that we encounter in applications are mathematical expressions of physical laws (e.g. the heat equation is an expression of the law of energy conservation). Therefore, in order to obtain a unique solution, we must specify the initial conditions in addition to the boundary conditions.

## Solution of a boundary value problem

By a solution to a boundary value problem on an open region D, we mean a function u that satisfies the differential equation on D and is continuous on $D \cup \partial D$, and satisfies the specified boundary conditions on $\partial D$.

**Def.**: The boundary conditions are linear if they express a linear relationship between u and its partial derivatives (up to the appropriate order) on $\partial D$. (In other words, a boundary condition is linear if it is expressed as a linear equation between u and it's derivatives on $\partial D$.)

For second-order partial differential equations ($n=2$), linear boundary conditions can take one of the following three forms:

The boundary conditions specify the values of the unknown function $u$ on the boundary. This type of boundary condition is called the Dirichlet condition.

. The boundary conditions specify the derivative of $u$ in the direction normal to the boundary, which is written as $\partial u / \partial \bar{n}$. This type of boundary condition is called the Neumann condition.

. The boundary conditions specify a linear relationship between $u$ and its normal derivative on the boundary. These are referred to as mixed boundary conditions or Robin's boundary conditions.

The general form of such a boundary condition is

$$\left[\alpha u + \beta \frac{\partial u}{\partial \bar{n}}\right]\Bigg|_{\partial D} = f(x)\Bigg|_{\partial D} \qquad \alpha, \beta \text{ constants.}$$

Remark

The normal derivative on the boundary $\partial u / \partial \bar{n}$ is defined as

$$\frac{\partial u}{\partial \bar{n}} = \operatorname{grad} u \cdot \bar{n} = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) \cdot \bar{n}$$

where $\bar{n}$ is the outward normal ...

If $u(x,t)$ is the displacement of a vibrating string and its ends are fixed at $x=0$ and $x=L$, then the conditions $u(0,t)=0$ and $u(L,t)=0$ are Dirichlet conditions.

Example 2: Suppose $u(x,t)$ is the temperature in a rod of length a. If the rod is perfectly insulated at $x=0$ and $x=L$, the heat flux at these points is zero. The appropriate boundary conditions are

$$\frac{\partial u}{\partial x}(0,t)=0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t)=0.$$

These are Neumann boundary conditions.

Example 3: Suppose in Example 2 that poor insulation is used at the ends of the rod. Then the boundary conditions then take the form

$$u(0,t)+\frac{\partial u}{\partial x}(0,t)=0 \quad \text{and} \quad u(L,t)+\frac{\partial u}{\partial x}(L,t)=u_0$$

This is an example of Robin's condition.

Note Similar to Superposition principle for the solutions of linear partial differential equations, we have the superposition principle for linear boundary conditions i.e

(E1)

**Result:** If $u_1, u_2$ are solutions of a linear homogeneous partial differential equation with linear boundary conditions

$$\left[\alpha u_1(\bar{x}) + \beta \frac{\partial u_1(\bar{x})}{\partial \bar{n}}\right]\Bigg|_{\partial D} = f(\bar{x})\Big|_{\partial D}$$

$$\left[\alpha u_2(\bar{x}) + \beta \frac{\partial u_2(\bar{x})}{\partial \bar{n}}\right]\Bigg|_{\partial D} = g(\bar{x})\Big|_{\partial D}$$

where $\alpha, \beta$ are constants, then $\omega = u_1 + u_2$ is a solution of the partial differential equation that satisfies the boundary conditions

$$\left[\alpha \omega(\bar{x}) + \beta \frac{\partial \omega(\bar{x})}{\partial \bar{n}}\right]\Bigg|_{\partial D} = (f+g)(\bar{x})\Big|_{\partial D}.$$

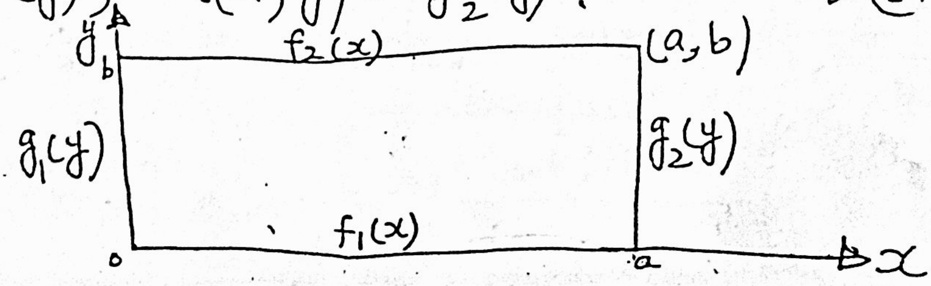
The above result is particularly useful in applications in which the boundary conditions are complex.

**Example**

Consider the Laplace equation $\nabla^2 u = 0$ ⟶ (1) in rectangle with the following linear boundary conditions

$$u(x,0) = f_1(x), \quad u(x,b) = f_2(x), \longrightarrow (2)$$

$$u(0,y) = g_1(y), \quad u(a,y) = g_2(y). \longrightarrow (3)$$



(ii) (a) We split the problem into two parts:

| | |
|---|---|
| $\nabla^2 u_1 = 0$ | $\nabla^2 u_2 = 0$ |
| $u_1(x,0) = f_1(x)$ | $u_2(x,0) = 0,$ |
| $u_1(x,b) = f_2(x)$ | $u_2(x,b) = 0$ |
| $u_1(0,y) = 0$ | $u_2(0,y) = g_1(y)$ |
| $u_1(a,y) = 0$ | $u_2(a,y) = g_2(y)$ |

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Obviously, if we solve $u_1, u_2$ then $u_1 + u_2$ is a solution of Laplace equation, which satisfies all the boundary conditions [(2) and (3)].

Note: that Neumann boundary conditions usually do not specify a unique solution of a BVP.

Example Consider the solution

$$\nabla^2 u = 0 \quad \text{——— (1)}^*$$

with the Neumann boundary conditions

$$\frac{\partial u}{\partial y}(x,0) = f_1(x), \quad \frac{\partial u}{\partial y}(x,b) = f_2(x), \longrightarrow (2)^*$$

$$\frac{\partial u}{\partial x}(0,y) = g_1(y), \quad \frac{\partial u}{\partial x}(a,y) = g_2(y) \longrightarrow (3)^*$$

It is obvious that if $u$ is a solution of this BVP, then $w = u + c$ ($c$ is a constant) is also a soln of (1) with the b.cs $(2)^*$ and $(3)^*$. Thus Neumann b.cs determine the solution of this BVP only up to a constant.