Now Consider

$$\boxed{A\left(\frac{\partial\phi}{\partial x}\right)^2 + 2B\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right) + C\left(\frac{\partial\phi}{\partial y}\right)^2 = 0} \longrightarrow (8)$$

Case I   Hyperbolic Equation.

$B^2 - AC > 0$

Consider that either $A \neq 0$ or $C \neq 0$, Eq (8), can be written as

$$(\phi_x)^2 + \frac{2B}{A}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right) + \frac{C}{A}\left(\frac{\partial\phi}{\partial y}\right)^2 = 0$$

This is quadratic in ϕ_x & ϕ_y Consider it is quadratic in ϕ_x then ~~we~~ we have by formula

$$\phi_x = -\frac{B\phi_y \pm \phi_y\sqrt{B^2-AC}}{A}$$

or

$$A\phi_x = (-B \pm \sqrt{B^2-AC})\,\phi_y$$

or

$$\boxed{A\frac{\partial\phi}{\partial x} + (B \pm \sqrt{B^2-AC})\frac{\partial\phi}{\partial y} = 0} \longrightarrow (9)$$

Hence $\left[A\frac{\partial\phi}{\partial x} + (B+\sqrt{B^2-AC})\frac{\partial\phi}{\partial y}\right]$ and

$\left[A\frac{\partial\phi}{\partial x} + (B - \sqrt{B^2-AC})\frac{\partial\phi}{\partial y}\right]$ are factors of (8) So (8) can be written as

$$\left[A\frac{\partial\phi}{\partial x} + (B+\sqrt{B^2-AC})\frac{\partial\phi}{\partial y}\right]\left[A\frac{\partial\phi}{\partial x} + (B-\sqrt{B^2-AC})\frac{\partial\phi}{\partial y}\right] = 0 \longrightarrow (10)$$

From above eq

$$A\frac{\partial\phi}{\partial x} + (B+\sqrt{B^2-AC})\frac{\partial\phi}{\partial y} = 0 \longrightarrow (11)$$

$$A\frac{\partial\phi}{\partial x} + (B-\sqrt{B^2-AC})\frac{\partial\phi}{\partial y} = 0 \longrightarrow (12)$$

Note that the solution of (11) & (12) will be the solution of (8)

To solve (11) & (12) the corresponding auxillary eqs are

$$\frac{dx}{A} = \frac{dy}{B+\sqrt{B^2-AC}} \longrightarrow (13)$$

$$\frac{dx}{A} = \frac{dy}{B-\sqrt{B^2-AC}} \longrightarrow (14)$$

From (13) & (14), it follows that

$$\boxed{\begin{aligned} A\,dy - (B+\sqrt{B^2-AC})\,dx = 0 \\ A\,dy - (B-\sqrt{B^2-AC})\,dx = 0 \end{aligned}} \longrightarrow (15)$$

Solutions of (15) are

$$\boxed{\begin{aligned} \phi_1(x,y) = \text{constt} \\ \phi_2(x,y) = \text{constt} \end{aligned}} \longrightarrow (16)$$

Let $\xi = \xi(x,y) = \phi_1(x,y)$

$\eta = \eta(x,y) = \phi_2(x,y)$

[ϕ_1 & ϕ_2 are the sol. of (8) so ξ & η are also solutions of (8)] so

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + C\left(\frac{\partial \xi}{\partial y}\right)^2 = 0$$

$\Rightarrow \bar{A} = 0$ using (6)

& $$A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + C\left(\frac{\partial \eta}{\partial y}\right)^2 = 0$$

i.e $\boxed{\bar{C} = 0}$ using (6)

So the transformed eq. (5) becomes

$$\bar{B}u_{\xi\eta} + \bar{F} = 0$$

Example $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2 \longrightarrow (1)$

$A = 4, \quad B = \frac{5}{2}, \quad C = 1$

$B^2 - AC = \frac{9}{4} > 0$, ~~are~~ Hyperbolic eq.

$A\,dy - (B \pm \sqrt{B^2 - AC})\,dx = 0 \longrightarrow (2)$

Using values of A, B, C in (2) we get

$4dy - 4dx = 0 \Rightarrow dy - dx = 0 \longrightarrow (3)$

$4dy - dx = 0 \longrightarrow (4)$

Integ (3) & (4)

$y - x = c_1$

$4y - x = c_2$

So $\boxed{\begin{aligned} \xi &= y - x \\ \eta &= 4y - x \end{aligned}} \longrightarrow (5)$

$$\left.\begin{aligned} &\xi_x = -1, \quad \xi_{xx} = 0, \quad \xi_y = 1 \\ &\xi_{yy} = 0, \quad \xi_{xy} = 0 \\ &\eta_x = -1, \quad \eta_{xx} = 0, \quad \eta_y = 4, \quad \eta_{yy} = 0, \quad \eta_{xy} = 0 \end{aligned}\right\} \longrightarrow (6)$$

using (6) we get

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$\boxed{u_x = -u_\xi - u_\eta} \longrightarrow (7)$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$\boxed{u_y = u_\xi + 4u_\eta} \longrightarrow (8)$$

and

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2$$

$$+ u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

using (6) we get

$$u_{xx} = u_{\xi\xi}(-1)^2 + 2u_{\xi\eta}(-1)(-1) + u_{\eta\eta}(-1)^2$$

$$+ u_\xi(0) + u_\eta(0)$$

or

$$\boxed{u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}} \longrightarrow (9)$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + (\xi_x\eta_y + \eta_x\xi_y)u_{\xi\eta}$$

$$+ u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}$$

$$= u_{\xi\xi}(-1)(1) + (-5)u_{\xi\eta} + u_{\eta\eta}(-1)(4) + u_\xi(0) + u_\eta$$

or

$$\boxed{u_{xy} = -u_{\xi\xi} - 5u_{\xi\eta} - 4u_{\eta\eta}} \longrightarrow (10)$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\,\xi_y\,\eta_y + u_{\eta\eta}(\eta_y)^2 + u_\xi\,\xi_{yy} + u_\eta\,\eta_{yy}$$

$$= u_{\xi\xi}(1)^2 + 2u_{\xi\eta}(1)(4) + u_{\eta\eta}(4)^2 + u_\xi(0) + u_\eta(0)$$

$$= \boxed{u_{\xi\xi} + 8u_{\xi\eta} + 16u_{\eta\eta}} \longrightarrow (11)$$

Using (7) to (11) in (1) we get.

$$4[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}] + 5[-u_{\xi\xi} - 5u_{\xi\eta} - 4u_{\eta\eta}] + u_{\xi\xi} + 8u_{\xi\eta} + 16u_{\eta\eta} - u_\xi - u_\eta + u_\xi + 4u_\eta = 2$$

$$\Rightarrow 4u_{\xi\xi} + 8u_{\xi\eta} + 4u_{\eta\eta} - 5u_{\xi\xi} - 25u_{\xi\eta} - 20u_{\eta\eta} + u_{\xi\xi} + 8u_{\xi\eta} + 16u_{\eta\eta} - u_\xi - u_\eta + u_\xi + 4u_\eta = 2$$

$$\Rightarrow -9u_{\xi\eta} + 3u_\eta = 2$$

$$3u_{\xi\eta} - u_\eta = -\frac{2}{3} \Rightarrow \boxed{u_{\xi\eta} = \frac{1}{3}u_\eta - \frac{2}{9}}$$

- Elliptic Equation $(B^2 - AC < 0)$.

$$A\,dy - (B + \sqrt{B^2 - AC})\,dx = 0 \quad \text{———} \quad (1)$$

$\phi(x, y) = c$ is a soln of (1)

$$\Rightarrow \phi_1(x, y) + i\,\phi_2(x, y) = c$$

($\sqrt{B^2 - AC}$ is imaginary for this case)

$$\xi = \xi(x, y) = \phi_1(x, y)$$
$$\eta = \eta(x, y) = \phi_2(x, y)$$
$$\phi = \phi_1 + i\phi_2 = \xi + i\eta \longrightarrow (2)$$

Now as ξ and η are the solutions of

$$A\left(\frac{\partial \phi}{\partial x}\right)^2 + 2B\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right) + C\left(\frac{\partial \phi}{\partial y}\right)^2 = 0 \longrightarrow (3)$$

Using (2) in (3) we get

$$A\left[\frac{\partial}{\partial x}(\xi + i\eta)\right]^2 + 2B\left[\frac{\partial}{\partial x}(\xi + i\eta)\frac{\partial}{\partial y}(\xi + i\eta)\right] + C\left[\frac{\partial}{\partial y}(\xi + i\eta)\right]^2 = 0$$

$$A\left[\frac{\partial \xi}{\partial x} + i\frac{\partial \eta}{\partial x}\right]^2 + 2B\left(\frac{\partial \xi}{\partial x} + i\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial y} + i\frac{\partial \eta}{\partial y}\right) + C\left[\frac{\partial \xi}{\partial y} + i\frac{\partial \eta}{\partial y}\right]^2 = 0.$$

or

$$A\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + i^2\left(\frac{\partial \eta}{\partial x}\right)^2 + 2i\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}\right]$$
$$+2B\left[\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + i\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + i\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} + i^2\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\right]$$
$$+C\left[\left(\frac{\partial \xi}{\partial y}\right)^2 + i^2\left(\frac{\partial \eta}{\partial y}\right)^2 + 2i\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\right] = 0.$$

or.

$$A\left(\frac{\partial\xi}{\partial x}\right)^2 - A\left(\frac{\partial\eta}{\partial x}\right)^2 + 2iA\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + 2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}$$

$$+2iB\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + 2iB\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y} - 2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}$$

$$+C\left(\frac{\partial\xi}{\partial y}\right)^2 - C\left(\frac{\partial\eta}{\partial y}\right)^2 + 2iC\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} = 0 + i0$$

Equating real and imaginary parts we have.

(i.e. real part gives)

$$A\left(\frac{\partial\xi}{\partial x}\right)^2 - A\left(\frac{\partial\eta}{\partial x}\right)^2 + 2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} - 2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}$$

$$+C\left(\frac{\partial\xi}{\partial y}\right)^2 - C\left(\frac{\partial\eta}{\partial y}\right)^2 = 0$$

or

$$A\left(\frac{\partial\xi}{\partial x}\right)^2 + 2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + C\left(\frac{\partial\xi}{\partial y}\right)^2 = A\left(\frac{\partial\eta}{\partial x}\right)^2 + 2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} + C\left(\frac{\partial\eta}{\partial y}\right)^2$$

i.e. $\boxed{\bar{A} = \bar{C}}$

(imaginary part gives)

$$2A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + 2B\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + 2B\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y}$$

$$+2C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} = 0$$

$$A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right) + C\left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\eta}{\partial y}\right) = 0$$

$\Rightarrow \boxed{\bar{B} = 0}$

i.e. the transformed equation is reduced to

$$\boxed{\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} = F_2\left(\xi, \eta; u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right)}$$

Ex

$$u_{xx} + x^2 u_{yy} = 0 \quad \text{—— (1)}$$

$A = 1,\ B = 0,\ C = x^2$

$B^2 - AC = -x^2 < 0$

Elliptic eq.

Now ξ and η are respectively the real and imaginary parts of the solution of ~~the~~ eq.

$$A\,dy - (B \mp \sqrt{B^2 - AC})\,dx = 0$$

Using values of A, B and C we have

$$dy - \sqrt{-x^2}\,dx = 0$$

$$dy \pm i x\,dx = 0$$

Integ $\quad y \pm i\frac{x^2}{2} = c \quad$ as $c = \phi_1 + i\phi_2$

so $\xi = \phi_1,\ \eta = \phi_2$

$$\Rightarrow \boxed{\xi = y,\ \eta = \frac{x^2}{2}} \quad \text{—— (2)}$$

$\xi_x = 0,\quad \xi_{xx} = 0,\ \xi_y = 1,\ \xi_{yy} = 0$

$\eta_x = x,\ \eta_{xx} = 1,\ \eta_y = 0,\ \eta_{yy} = 0$ —— (3)

As

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}.$$

Using (3) we obtain

$$u_{xx} = x^2 u_{\eta\eta} + u_\eta \longrightarrow (4)$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\,\xi_y\,\eta_y + u_{\eta\eta}(\eta_y)^2 + u_\xi\,\xi_{yy} + u_\eta\,\eta_{yy}$$

$$\Rightarrow \boxed{u_{yy} = u_{\xi\xi}} \longrightarrow (5)$$

Using (4) and (5) in (1) we get

$$x^2 u_{\eta\eta} + u_\eta + x^2 u_{\xi\xi} = 0 \longrightarrow (6)$$

From (2)

$$x^2 = 2\eta$$

Using in (6) we obtain

$$2\eta u_{\eta\eta} + u_\eta + 2\eta u_{\xi\xi} = 0$$

$$2\eta(u_{\eta\eta} + u_{\xi\xi}) = -u_\eta \qquad -2\eta u_{\eta\eta} - u_\eta + 2\eta u_{\xi\xi} = 0$$

or $$\boxed{u_{\eta\eta} + u_{\xi\xi} = -\frac{1}{2\eta} u_\eta} \qquad u_{\xi\xi} - u_{\eta\eta}$$

Parabolic Equation

$$B^2 - AC = 0 \longrightarrow (1)$$

$$A\,dy - (B \pm \sqrt{B^2 - AC})\,dx = 0 \longrightarrow (2)$$

With (1), (2) becomes

$$A\,dy - B\,dx = 0$$

which has the solution $\phi(x, y) = c$

we choose $$\xi = \phi(x, y) \longrightarrow (3)$$

and η arbitrary.

Now ξ is a solution of

$$A\left(\frac{\partial \phi}{\partial x}\right)^2 + 2B\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right) + C\left(\frac{\partial \phi}{\partial y}\right)^2 = 0 \longrightarrow (4)$$

With (3) in (4) to arrive at

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + C\left(\frac{\partial \xi}{\partial y}\right)^2 = 0$$

or

$$\boxed{\bar{A} = 0} \longrightarrow (5)$$

$\mathbb{Z}_4$	0	1	2	3
0				
1		1	2	3
2		2	0	2
3				

As we know

$$A\frac{\partial \phi}{\partial x} + \left(B + \sqrt{B^2 - AC}\right)\frac{\partial \phi}{\partial y} = 0 \longrightarrow (5a)$$

$B^2 - AC = 0$

$$A\frac{\partial \xi}{\partial x} + B\frac{\partial \xi}{\partial y} = 0 \longrightarrow (6)$$

(With (3) in (5a)).

$\Rightarrow$

$$AB\frac{\partial \xi}{\partial x} + B^2\frac{\partial \xi}{\partial y} = 0$$

$$AB\frac{\partial \xi}{\partial x} + AC\frac{\partial \xi}{\partial y} = 0$$

$(A \neq 0)$.

$$\boxed{B\frac{\partial \xi}{\partial x} + C\frac{\partial \xi}{\partial y} = 0} \longrightarrow (7)$$

As

$$\bar{B} = A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial \eta}{\partial x}\underbrace{\left(A\frac{\partial \xi}{\partial x} + B\frac{\partial \xi}{\partial y}\right)}_{\to 0} + \frac{\partial \eta}{\partial y}\underbrace{\left(B\frac{\partial \xi}{\partial x} + C\frac{\partial \xi}{\partial y}\right)}_{\to 0}$$

(With (7)).

$$\Rightarrow \boxed{\bar{B} = 0} \longrightarrow (8)$$

So we get the canonical form as

$$\boxed{\frac{\partial^2 u}{\partial \eta^2} = F_3\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)}$$

Example

$$\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial y} = 0 \longrightarrow (1)$$

$A = 1$, $B = -x$, $C = x^2$

$B^2 - AC = 0$ (Parabolic Eq.)

$A\,dy - \left(B \pm \sqrt{B^2 - AC}\right)dx = 0$

$dy - (-x)\,dx = 0$

$dy + x\,dx = 0$

$A\,dy - B\,dx = 0$

using values, to obtain

$dy + x\,dx = 0$

$\frac{B +}{2A}$

Integrating

$$y + \frac{x^2}{2} = c$$

So $\boxed{\xi = y + \frac{x^2}{2}}$

and $\eta = x$ arbitrary.

$\xi_x = x$, $\xi_{xx} = 1$, $\xi_{xy} = 0$

$\xi_y = 1$, $\xi_{yy} = 0$, $\xi_{xy} = 0$

$\eta_x = 1$, $\eta_{xx} = 0$, $\eta_{xy} = 0$

$\eta_y = 0$, $\eta_{yy} = 0$, $\eta_{xy} = 0$

so we have

$u_y = u_\xi \xi_y + u_\eta . \eta_y$

$= u_\xi + u_\eta (0)$

$\Rightarrow \boxed{u_y = u_\xi} \longrightarrow (2)$

and

(23)

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$\Rightarrow$

$$\boxed{u_{xx} = x^2 u_{\xi\xi} + 2x u_{\xi\eta} + u_{\eta\eta} + u_\xi} \rightarrow (3)$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}(\eta_y)^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}.$$

$\Rightarrow$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta}(0)$$

$$\Rightarrow \boxed{u_{yy} = u_{\xi\xi}} \rightarrow (4)$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + (\xi_x\eta_y + \eta_x\xi_y)u_{\xi\eta} + \eta_x\eta_y u_{\eta\eta} + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

or

$$\boxed{u_{xy} = x u_{\xi\xi} + u_{\xi\eta}} \rightarrow (5)$$

Using all these in (1) to obtain

$$x^2 u_{\xi\xi} + 2x u_{\xi\eta} + u_{\eta\eta} + u_\xi - 2x(x u_{\xi\xi} + u_{\xi\eta}) + x^2(u_{\xi\xi}) - 2u_\xi = 0$$

$$\cancel{x^2 u_{\xi\xi}} + \cancel{2x u_{\xi\eta}} + u_{\eta\eta} + u_\xi - \cancel{2x^2 u_{\xi\xi}} - \cancel{2x u_{\xi\eta}} + \cancel{x^2 u_{\xi\xi}} - 2u_\xi = 0$$

$$\Rightarrow \boxed{u_{\eta\eta} - u_\xi = 0}$$

(2)

$$x^2 u_{xx} + 2u_{xy}(xy) + y^2 u_{yy} = 0 \longrightarrow (1)$$

$A = x^2$, $B = xy$, $c = y^2$

$B^2 - AC = 0$ (Parabolic Eq.)

$A\,dy - B\,dx = 0$

$x^2\,dy - xy\,dx = 0$

$$\frac{dy}{y} = \frac{dx}{x} \quad \text{Integ.}$$

$$\ln\frac{y}{x} = \ln c_1$$

$$\Rightarrow \frac{y}{x} = c_1 \quad , \quad \boxed{\begin{array}{l}\xi(x,y) = y/x \\ \eta = y\end{array}} \longrightarrow (2)$$

Also

$$\xi_x = \frac{-y}{x^2}, \quad \xi_{xx} = \frac{2y}{x^3}, \quad \xi_y = \frac{1}{x}, \quad \xi_{yy} = 0,$$

$$\xi_{xy} = -\frac{1}{x^2}, \quad \eta_x = 0, \quad \eta_{xx} = 0, \quad \eta_y = 1, \quad \eta_{yy} = 0$$

$$\eta_{xy} = 0$$

Now

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_\xi\xi_{xx} + u_{\eta\eta}(\eta_x) + u_\eta\eta_{xx}$$

Using values we get

$$\boxed{u_{xx} = \frac{y^2}{x^4}u_{\xi\xi} + \frac{2y}{x^3}u_\xi} \longrightarrow (3)$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_\xi\xi_{yy} + u_{\eta\eta}(\eta_y)^2 + u_\eta\eta_{yy}$$

or

$$\boxed{u_{yy} = \frac{u_{\xi\xi}}{x^2} + \frac{2}{x^2}u_{\xi\eta} + u_{\eta\eta}} \longrightarrow (4)$$

And

$$u_{xy} = u_{\xi\xi}\,\xi_x\,\xi_y + (\xi_x\eta_y + \eta_x\xi_y)\,u_{\xi\eta} + u_{\eta\eta}\,\eta_x\,\eta_y + u_\xi\,\xi_{xy} + u_\eta\,\eta_{xy}$$

or

$$\boxed{u_{xy} = -\frac{y}{x^3}u_{\xi\xi} - \frac{y}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\xi} \longrightarrow (5)$$

Using the values in (1), we have

$$x^2\left(\frac{y^2}{x^4}u_{\xi\xi} + \frac{2y}{x^3}u_\xi\right) + 2xy\left(-\frac{y}{x^3}u_{\xi\xi} - \frac{y}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\xi\right)$$

$$+ y^2\left(\frac{1}{x^2}u_{\xi\xi} + \frac{2}{x}u_{\xi\eta} + u_{\eta\eta}\right) = 0$$

$\Longrightarrow$

$$\cancel{\frac{y^2}{x^2}u_{\xi\xi}} + \cancel{\frac{2y}{x}u_\xi} - \cancel{\frac{2y^2}{x^2}u_{\xi\xi}} - \cancel{\frac{2y^2}{x}u_{\xi\eta}} - \cancel{\frac{2y}{x}u_\xi}$$

$$+ \cancel{\frac{y^2}{x^2}u_{\xi\xi}} + \cancel{\frac{2y^2}{x}u_{\xi\eta}} + y^2u_{\eta\eta} = 0$$

$$\Longrightarrow y^2\,u_{\eta\eta} = 0$$

$$\Longrightarrow \boxed{u_{\eta\eta} = 0}$$

Integ.

$$u_\eta = f(\xi)$$

Again integ.

$$u = \int f(\xi)\,d\eta + g(\xi)$$

$$= \eta f(\xi) + g(\xi)$$

$$\boxed{u = y\,f(y/x) + g(y/x)}$$

Exercise 3

Classify the following equations:

1. $u_{xx} - 2u_{xy} + 2u_{yy} + 5u_x + 6y + 7 = 0$
2. $8u_{xx} - 8u_{xy} + 2u_{yy} + 17u_x - 13u = 0$
3. $u_{xx} - 3u_{xy} + \frac{1}{2}u_{yy} + 16u_y = 0.$

Reduce to canonical form:

1. $y\frac{\partial^2 z}{\partial x^2} + (x+y)\frac{\partial^2 z}{\partial x \partial y} + x\frac{\partial^2 z}{\partial y^2} = 0,$
2. $y^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + x^2\frac{\partial^2 z}{\partial y^2} - \frac{y^2}{x}\frac{\partial z}{\partial x} - \frac{x^2}{y}\frac{\partial z}{\partial y} = 0$
3. $\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial y^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 0$
4. $\frac{\partial^2 z}{\partial x^2} - x^2\frac{\partial^2 z}{\partial y^2} = 0$
5. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$
6. $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 4x^2,$
7. $u_{xx} - 2\sin x\, u_{xy} - \cos^2 x\, u_{yy} - \cos x\, u_y = 0$
8. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
9. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
10. $u_{xx} - 3u_{xy} + \frac{1}{2}u_{yy} + 16u_y = 0.$

Ex Reduce into the canonical form and hence find the general solution

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

with Cauchy's data

$$u(x,0) = f(x),$$

$$u_t(x,0) = g(x).$$

soln $u_{xx} - \frac{1}{c^2} u_{tt} = 0 \longrightarrow (1)$

$A = 1, \; B = 0, \; C = -\frac{1}{c^2}$

$B^2 - AC = \frac{1}{c^2} > 0$ (Hyperbolic Eq).

$A\,dt - \left(B \pm \sqrt{B^2 - AC} \right) dx = 0 \longrightarrow (2)$

With A, B and C, the above equations yield

$dt - \frac{1}{c} dx = 0 \longrightarrow (3)$

$dt + \frac{1}{c} dx = 0 \longrightarrow (4)$

Integ (4) we get

$t - \frac{x}{c} = k_1^*$

$ct - x = c k_1^*$

$x - ct = k_1, \quad k_1 = -ck_1^*$

$\longrightarrow (5)$

Similarly, integ (4) we get

$x + ct = k_2, \quad k_2 = -ck_2^*$

$\longrightarrow (6)$

so

$$\boxed{\begin{aligned} \xi(x,t) &= x + ct \\ \eta(x,t) &= x - ct \end{aligned}} \longrightarrow (7)$$

$\xi_x = 1, \; \xi_{xx} = 0, \; \xi_t = c, \; \xi_{tt} = 0, \; \xi_{xt} = 0$

$\eta_x = 1, \; \eta_{xx} = 0, \; \eta_t = -c, \; \eta_{tt} = 0, \; \eta_{xt} = 0$

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx} + u_{\eta\eta}(\eta_x)^2 + u_\eta \eta_{xx}$$

$$= u_{\xi\xi}(1)^2 + 2u_{\xi\eta}(1)(1) + u_\xi(0) + u_{\eta\eta}(1)^2 + u_\eta(0)$$

$\Rightarrow$

$$\boxed{u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}} \longrightarrow (8)$$

$$u_{tt} = u_{\xi\xi}(\xi_t)^2 + 2u_{\xi\eta}\xi_t\eta_t + u_\xi \xi_{tt} + u_{\eta\eta}(\eta_t)^2 + u_\eta \eta_{tt}$$

$$= u_{\xi\xi}c^2 + 2u_{\xi\eta}(c)(-c) + u_\xi(0) + u_{\eta\eta}(-c)^2 + u_\eta(0)$$

or

$$\boxed{u_{tt} = c^2u_{\xi\xi} - 2c^2u_{\xi\eta} + c^2u_{\eta\eta}} \longrightarrow (9)$$

Using (8) and (9) in (1) we have

$$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - \frac{1}{c^2}\left(c^2u_{\xi\xi} - 2c^2u_{\xi\eta} + c^2u_{\eta\eta}\right) = 0$$

or

$$\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}} - \cancel{u_{\xi\xi}} + 2u_{\xi\eta} - \cancel{u_{\eta\eta}} = 0$$

$\Rightarrow 4u_{\xi\eta} = 0$

$4 \neq 0$ so $\boxed{u_{\xi\eta} = 0} \longrightarrow (10)$

(Required canonical form)

Integ. (10) w.r.t η we get

$$u_\xi = f(\xi)$$

Integ w.r.t ξ we have

$$u = \int f(\xi)\, d\xi + f(\eta)$$

or $$u = \phi(\xi) + \psi(\eta)$$

$$\boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)} \longrightarrow (11)$$

(General soln of (1))

<u>bcs</u>

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\}$$

From (11)

$$u(x,0) = \phi(x) + \psi(x) = f(x) \longrightarrow (12)$$

$$u_t(x,t) = c\,\phi'(x+ct) - c\,\psi'(x-ct)$$

$$u_t(x,0) = c\,\phi'(x) - c\,\psi'(x) = g(x) \longrightarrow (13)$$

From (12)

$$\phi'(x) + \psi'(x) = f'(x) \longrightarrow (14)$$

Xing (14) by c we get

$$c\,\phi'(x) + c\,\psi'(x) = c\,f'(x) \longrightarrow (15)$$

Adding (13) and (15) we obtain

$$g(x) + c\, f'(x) = 2c\, \phi'(x).$$

or $$\boxed{\phi'(x) = \frac{1}{2c} g(x) + \frac{1}{2} f'(x)} \rightarrow (16.)$$

Subtracting (13) from (15) to get

$$c\, f'(x) - g(x) = 2c\, \psi'(x)$$

$$\Rightarrow \psi'(x) = \frac{1}{2c} c\, f'(x) - \frac{1}{2c} g(x)$$

or $$\boxed{\psi'(x) = \frac{1}{2} f'(x) - \frac{1}{2c} g(x)} \rightarrow (17)$$

Integrating (16) and (17) we have

$$\phi(x) = \frac{1}{2c}\int g(x)\,dx + \frac{1}{2} f(x) + k_3 \rightarrow (18)$$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int g(x)\,dx + k_4 \rightarrow (19)$$

+ing (18) and (19)

$$\phi(x) + \psi(x) = f(x) + k_3 + k_4 \rightarrow (20)$$

From (12)

$$\phi(x) + \psi(x) = f(x)$$

so (20) yields

$$f(x) = f(x) + k_3 + k_4 \Rightarrow k_3 + k_4 = 0$$
$$\Rightarrow k_3 = -k_4 = k \text{ (say)}$$

||ing in (18) and (19) we have

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int g(x)\,dx + k$$

or

$$\boxed{\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)\,dz + k} \longrightarrow (21)$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int g(x)\,dx - k$$

$$\boxed{\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\,dz - k} \longrightarrow (22)$$

From (21) and (22)

$$\boxed{\begin{aligned} \phi(x+ct) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz + k \\ \psi(x-ct) &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(z)\,dz - k \end{aligned}} \longrightarrow (23)$$

Using (23) in (11) to obtain

$$\begin{aligned} u(x,t) &= \phi(x+ct) + \psi(x-ct) \\ &= \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) \\ &\quad + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz + \cancel{k} - \frac{1}{2c}\int_0^{x-ct} g(z)\,dz - \cancel{k} \end{aligned}$$

or

$$\begin{aligned} u(x,t) &= \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) \\ &\quad + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz - \frac{1}{2c}\int^{x-ct} g(z)\,dz \end{aligned}$$