Classification of 2nd order partial differential equation (PDE)

The general form of 2nd order PDE is

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

(where A, B, C are the linear functions of the independent variable x & y or constants)

Elliptic eq: $B^2 - AC < 0$ ✓

Parabolic : $B^2 - AC = 0$ ✓

Hyperbolic : $B^2 - AC > 0$ ✓

Examples

1. $u_{xx} + u_{yy} = 0$, $A = 1$, $C = 1$, $B = 0$

$B^2 - AC = -1 < 0$ (Elliptic)

2. $u_{xx} = \alpha^2 u_t$ (Diffusion Eq)

$A = 1$, $B = 0$, $C = 0$

$B^2 - AC = 0$ (Parabolic Eq)

3. $u_{xx} = \frac{1}{c^2} u_{tt}$

$u_{xx} - \frac{1}{c^2} u_{tt} = 0$

$A = 1$, $B = 0$, $C = -\frac{1}{c^2}$

$B^2 - AC = \frac{1}{c^2} > 0$ (Hyperbolic Eq).

Exercise

Classify the following eqs.

1. $u_{xx} - 2u_{xy} + 2u_{yy} + 5u_x + 6u_y + 7 = 0$

2. $u_{xx} - 3u_{xy} + \frac{1}{2}u_{yy} + 16u_y = 0$

Reduction of 2nd order PDE into Canonical Form ②

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Consider

$$A u_{xx} + 2B u_{xy} + C u_{yy} + F(x, y, u, u_x, u_y) = 0 \quad (1)$$

where A, B, C are functions of x and y and u have continuous first and 2nd order partial derivative. Also, A, B, C do not vanish at the same time.

Let us transform the independent variables by substituting ?

$$\xi = \xi(x, y)$$
$$\eta = \eta(x, y)$$

ξ and η are such that

$$\frac{D(\xi, \eta)}{D(x, y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq 0$$

In the new coordinate (ξ, η) (1) can be written as

$$\bar{A} \frac{\partial^2 u}{\partial \xi^2} + 2\bar{B} \frac{\partial^2 u}{\partial \xi \partial \eta} + \bar{C} \frac{\partial^2 u}{\partial \eta^2} + F(\xi, \eta, u, u_\xi, u_\eta) = 0,$$

where

$$\bar{A} = A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2,$$

$$\bar{C} = A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2,$$

$$\bar{B} = A \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}.$$

Aside

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\right)$$

$$= \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x}\right]\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$

$$= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$

$\Longrightarrow$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}} \quad (2)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \eta}{\partial x}\right]\frac{\partial \xi}{\partial y}$$

$$+ \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x}\right]\frac{\partial \eta}{\partial y}$$

$$+ \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}$$

$$= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y}$$

$$+ \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}$$

$$\boxed{= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}\right)\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}} \rightarrow (3)$$

Now

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y}$$

$$+ \frac{\partial u}{\partial \eta}\frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial y}\right)$$

$$= \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \eta}{\partial y}\right]\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} +$$

$$+ \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y}\right]\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2}$$
$$+ \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 u}{\partial y^2}$$

or

$$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}} \quad (4)$$

Using (2), (3) and (4) in (1) we have

$$A\left[\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}\right] + 2B\left[\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}\right)\frac{\partial^2 u}{\partial \xi \partial \eta}\right.$$
$$\left. + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 u}{\partial x \partial y}\right]$$
$$+ C\left[\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}\right] +$$
$$+ F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0$$

$$= \left[A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial \xi^2} +$$

(P.T.O)

$$+2\left[A\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\cdot\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right]\frac{\partial^2 u}{\partial\xi\partial\eta}$$

$$+\left[A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial\eta^2}+F\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)=0$$

or

$$\boxed{\bar{A}\frac{\partial^2 u}{\partial\xi^2}+2\bar{B}\frac{\partial^2 u}{\partial\xi\partial\eta}+\bar{C}\frac{\partial^2 u}{\partial\eta^2}+F\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)=0} \longrightarrow (5)$$

where

$$\bar{A}=A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2,$$

$$\bar{B}=A\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\cdot\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial x}\cdot\frac{\partial\xi}{\partial y}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y},$$

$$\bar{C}=A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2 \longrightarrow (6)$$

Note that

$$\bar{B}^2-\bar{A}\bar{C}=(B^2-AC)\left(\frac{\partial\xi}{\partial x}\cdot\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2 \longrightarrow (7)$$

where A, B, C are not zero at the same time. Thus the transformation of independent variables does not change the type of eq.

Since $\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}=\begin{vmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y}\\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y}\end{vmatrix}\neq 0$

So $\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2$ is a positive expression.

<u>Show that</u> $\bar{B}^2-\bar{A}\bar{C}=(B^2-AC)\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2$

<u>A Side</u> From (6),

$$\bar{B}^2-\bar{A}\bar{C}=\left[A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right]^2$$

$$-\left[A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2\right]\times\left[A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2\right]$$

$$\bar{B}^2-\bar{A}\bar{C}=A^2\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2+B^2\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2+C^2\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2$$

$$+2AB\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)$$

$$+2BC\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+2AC\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}$$

$$-A^2\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2-2AB\left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}-AC\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2$$

$$-2AB\left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}-4B^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}-2BC\left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}$$

$$-AC\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2-2BC\left(\frac{\partial\xi}{\partial y}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}-C^2\left(\frac{\partial\eta}{\partial y}\right)^2\left(\frac{\partial\xi}{\partial y}\right)^2$$

$$=B^2\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2+B^2\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2+2B^2\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}$$

$$+\cancel{2AB\left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}}+\cancel{2AB\left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}}+\cancel{2BC\left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}}$$

$$+\cancel{2BC\left(\frac{\partial\xi}{\partial y}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}}+2AC\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}-\cancel{2AB\left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}}$$

$$-AC\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2-\cancel{2AB\left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial x}}-4B^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}$$

$$-\cancel{2BC\left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}}-AC\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2-\cancel{2BC\left(\frac{\partial\xi}{\partial y}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}}$$

So

$$\bar{B}^2-\bar{A}\bar{C}=B^2\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2+B^2\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2$$

$$+2B^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+2AC\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}$$

$$-AC\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2-AC\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2$$

$$=B^2\left[\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2-2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\right]$$

$$-AC\left[\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2-2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\right]$$

$$=(B^2-AC)\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2$$

Example $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2 \longrightarrow (1)$

$A = 4 \ , \ B = \frac{5}{2} \ , \ C = 1$

$B^2 - AC = \frac{9}{4} > 0$, ~~exist~~ Hyperbolic eq.

$A\,dy - (B \pm \sqrt{B^2 - AC})\,dx = 0 \longrightarrow (2)$

Using values of A, B, C in (2) we get

$4dy - 4dx = 0 \Rightarrow dy - dx = 0 \longrightarrow (3)$

$4dy - dx = 0 \longrightarrow (4)$

Integ (3) & (4)

$y - x = c_1$

$4y - x = c_2$

So
$$\boxed{\begin{aligned} \xi &= y - x \\ \eta &= 4y - x \end{aligned}} \longrightarrow (5)$$

$$\left.\begin{array}{l} \xi_x = -1 \ , \ \xi_{xx} = 0 \ , \ \xi_y = 1 \\ \xi_{yy} = 0 \ , \ \xi_{xy} = 0 \\ \eta_x = -1 \ , \ \eta_{xx} = 0 \ , \ \eta_y = 4 \ , \ \eta_{yy} = 0 \ , \ \eta_{xy} = 0 \end{array}\right\} \longrightarrow (6)$$

Example $4u_{xx}+5u_{xy}+u_{yy}+u_x+u_y=2 \longrightarrow (1)$

$A=4, \quad B=\frac{5}{2}, \quad C=1$

$B^2-AC=\frac{9}{4}>0$, ~~~~ Hyperbolic eq.

$A\,dy-(B\pm\sqrt{B^2-AC})\,dx=0 \longrightarrow (2)$

Using values of A, B, C in (2) we get

$4dy-4dx=0 \Rightarrow dy-dx=0 \longrightarrow (3)$

$4dy-dx=0 \longrightarrow (4)$

Integ (3), & (4),

$y-x=c_1$

$4y-x=c_2$

So $\boxed{\begin{aligned}\xi&=y-x\\ \eta&=4y-x\end{aligned}} \longrightarrow (5)$

$$\left.\begin{aligned}&\xi_x=-1, \quad \xi_{xx}=0, \quad \xi_y=1\\ &\xi_{yy}=0, \quad \xi_{xy}=0\\ &\eta_x=-1, \quad \eta_{xx}=0, \quad \eta_y=4, \quad \eta_{yy}=0, \quad \eta_{xy}=0\end{aligned}\right\} \longrightarrow (6)$$

using (6) we get

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$\boxed{u_x = -u_\xi - u_\eta} \longrightarrow (7)$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$\boxed{u_y = u_\xi + 4u_\eta} \longrightarrow (8)$$

and

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2$$

$$+ u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

using (6) we get

$$u_{xx} = u_{\xi\xi}(-1)^2 + 2u_{\xi\eta}(-1)(-1) + u_{\eta\eta}(-1)^2$$

$$+ u_\xi(0) + u_\eta(0)$$

or

$$\boxed{u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}} \longrightarrow (9)$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + (\xi_x\eta_y + \eta_x\xi_y)u_{\xi\eta}$$

$$+ u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}$$

$$= u_{\xi\xi}(-1)(1) + (-5)u_{\xi\eta} + u_{\eta\eta}(-1)(4) + u_\xi(0) + u_\eta(0)$$

or

Now Consider

$$\boxed{A\left(\frac{\partial \varphi}{\partial x}\right)^2 + 2B\left(\frac{\partial \varphi}{\partial x}\right)\left(\frac{\partial \varphi}{\partial y}\right) + C\left(\frac{\partial \varphi}{\partial y}\right)^2 = 0} \rightarrow (8)$$

Case I Hyperbolic Equation.

$B^2 - AC > 0$

Consider that either $A \neq 0$ or $C \neq 0$, Eq (8) can be written as

$$(\varphi_x)^2 + \frac{2B}{A}\left(\frac{\partial \varphi}{\partial x}\right)\left(\frac{\partial \varphi}{\partial y}\right) + \frac{C}{A}\left(\frac{\partial \varphi}{\partial y}\right)^2 = 0$$

This is quadratic in φ_x & φ_y. Consider it is quadratic in φ_x then ~~we~~ we have by formula

$$\varphi_x = -\frac{B\varphi_y \pm \varphi_y\sqrt{B^2-AC}}{A}$$

or

$$A\varphi_x = (-B \pm \sqrt{B^2-AC})\varphi_y$$

or

$$\boxed{A\frac{\partial \varphi}{\partial x} + (B \pm \sqrt{B^2-AC})\frac{\partial \varphi}{\partial y} = 0} \rightarrow (9)$$

Thence $\left[A\frac{\partial \varphi}{\partial x} + (B + \sqrt{B^2-AC})\frac{\partial \varphi}{\partial y}\right]$ and $\left[A\frac{\partial \varphi}{\partial x} + (B - \sqrt{B^2-AC})\frac{\partial \varphi}{\partial y}\right]$ are factors of (8) So (8) can be written as

$$\left[A\frac{\partial \varphi}{\partial x} + (B + \sqrt{B^2-AC})\frac{\partial \varphi}{\partial y}\right]\left[A\frac{\partial \varphi}{\partial x} + (B - \sqrt{B^2-AC})\frac{\partial \varphi}{\partial y}\right] \rightarrow (10)$$

From above eq

$$A\frac{\partial \varphi}{\partial x} + (B + \sqrt{B^2-AC})\frac{\partial \varphi}{\partial y} = 0 \rightarrow (11)$$

$$A\frac{\partial \varphi}{\partial x} + (B - \sqrt{B^2-AC})\frac{\partial \varphi}{\partial y} = 0 \rightarrow (12)$$

Note that the solution of (11) & (12) will be the solution of (8)

To solve (11) & (12) the corresponding auxiliary eqⁿ are

$$\frac{dx}{A} = \frac{dy}{B + \sqrt{B^2 - AC}} \longrightarrow (13)$$

$$\frac{dx}{A} = \frac{dy}{B - \sqrt{B^2 - AC}} \longrightarrow (14)$$

From (13) & (14), it follows that

$$\boxed{\begin{array}{l} A\,dy - (B + \sqrt{B^2 - AC})\,dx = 0 \\ A\,dy - (B - \sqrt{B^2 - AC})\,dx = 0 \end{array}} \longrightarrow (15)$$

Solutions of (15) are

$$\boxed{\begin{array}{l} \phi_1(x,y) = \text{constt} \\ \phi_2(x,y) = \text{constt} \end{array}} \longrightarrow (16)$$

Let $\xi = \xi(x,y) = \phi_1(x,y)$

$\eta = \eta(x,y) = \phi_2(x,y)$

[ϕ_1 & ϕ_2 are the sol. of (8) so ξ & η are also solutions of (8)] so

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + C\left(\frac{\partial \xi}{\partial y}\right)^2 = 0$$

$\Rightarrow \bar{A} = 0$ using (6)

&

$$A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + C\left(\frac{\partial \eta}{\partial y}\right)^2 = 0$$

i.e. $\boxed{\bar{C} = 0}$ using (6)

So the transformed eq (5) becomes