PARTIAL DIFFERENTIAL EQUATIONS

INTRODUCTION

BASIC CONCEPTS AND DEFINITIONS

A differential equation that contains, in addition to the dependent variable and the independent variables, one or more partial derivatives of the dependent variable is called a <u>partial differential equation</u> (PDE)

In general, it may be written in the form

$$f(x, y, \ldots u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots) = 0$$

where $x, y, \ldots$ are independent variables. u is dependent variable and $u_x, u_y, \ldots$ $u_{xx}, u_{xy}, \ldots$ are partial derivatives of the dependent variable (unknown function).

<u>Examples</u>

$$u u_x + u_x = y$$

$$u_{xx} + 2y u_{xy} + 3x u_{yy} = 4 \sin x$$

$$(u_x)^2 + (u_y)^2 = 1$$

$$u_{xx} - u_{yy} = 0$$

ORDER OF A PARTIAL DIFFERENTIAL EQUATION

The order of a partial differential equation is the order of the highest ordered partial derivative appearing in the equation. e.g

$$u_{xx} + 2x u_{xy} + u_{yy} = e^{y}$$

is a second order partial differential equation, and

$$u_{xxy} + x u_{yy} + 8u = 7y$$

is a third-order partial differential equation.

LINEAR PDE

A PDE is said to be linear if it is linear in the unknown function and all its derivatives with coefficients depending only on the independent variables;

QUASILINEAR PDE

A PDE is said to be quasilinear if it is linear in the heighest-ordered derivative of the unknown function. e.g

$$y u_{xx} + 2xy u_{yy} + u = 1$$

is second order linear partial

differential equation, whereas

$$u_x u_{xx} + xu u_y = \sin y$$

is a second-order quasilinear partial differential equation.

NON LINEAR PDE

The equation which is not linear is called non-linear equation.

Most general second-order PDE (linear) in n-independent variables has the form

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Fu = G \quad \text{——} (1)$$

where we assume that $A_{ij} = A_{ji}$

Also B_i, F and G are functions of the n-independent variables x_i.

HOMOGENEOUS PDE

If $G=0$ in Eq. (1) the equation is said to be homogeneous; otherwise it is non-homogeneous.

SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION

The functions $u = u(x, y, ..)$ which satisfy Eq. (1), in a domain D are called solutions of Eq. (1).

For example, the functions

$$u(x,y) = (x+y)^3$$

and $u(x,y) = \sin(x-y)$

are solutions of the partial diff. equation

$$u_{xx} - u_{yy} = 0 \longrightarrow (2)$$

* The general solution of a linear ordinary differential equation (ODE) of nth order is a family of functions depending on n independent arbitrary constants. In the case of partial diff. eqs., the general solution depends on arbitrary functions rather than on arbitrary constants

To illustrates this, consider the eq.

$$u_{xy} = 0 \longrightarrow (3)$$

Integrate (3) w.r.t 'y', we get

$$u_x(x,y) = f(x) \longrightarrow (4)$$

Now integration of (4) w.r.t 'x' gives

$$u(x,y) = g(x) + h(y)$$

where $g(x)$ and $h(y)$ are arbitrary functions.

(2) $u_{yy} = 2$ (if $u = u(x, y, z)$)

General soln. of above equation is

$$u(x,y,z) = y^2 + y f(x,z) + g(x,z)$$

Recall that in case of ODE, the first tasks is to find the general solution, and then a particular solution is determined by finding the values of arbitrary constants from the prescribed conditions. But for PDEs, selecting a particular solution satisfying the supplementary conditions from the general soln. of a PDE may be as difficult as, or even more difficult than, the problem of finding the general soln. itself. This is so, because the general soln. of a PDE involves arbitrary fns.; the specialization of such a solution to the particular form which satisfies the supplementary conditions requires the determination of these arbitrary functions, rather than merely the determination of constants.

For linear homogeneous ordinary differential equations of order n, a linear combination of n linearly independent solutions is a soln. Unfortunately, this is not true, in ~~genral~~ general, in the case of PDEs. This is due to the fact that the soln. space of every homogeneous linear PDE is ~~ind~~ infinite dimensional.