Differential Geometry	**Fall 2020**

Lecture 2

Lecturer:Dr. Muhammad Yaseen — *Topic:Curves in Plane*

There are three important curves that we will keep mentioning time to time:

(I) **Straight Line:** The curve $\gamma : I \to \mathbb{R}^2$ given by $\gamma(t) = \overrightarrow{a}t + \overrightarrow{b}$ is the straight line parallel to $\overrightarrow{a}$ and through $\overrightarrow{b}$.

(II) **Circle:** The curve $\gamma : I \to \mathbb{R}^2$ given by $\gamma(t) = \overrightarrow{C} + (r\cos t, r\sin t)$ is the circle of radius r with center $\overrightarrow{C}$.

(III) **Helix:** The curve $\gamma : I \to \mathbb{R}^2$ given by $\gamma(t) = (r\cos t, r\sin t, t)$ is the Helix.

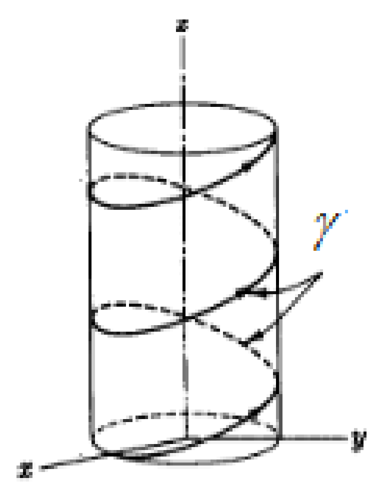

Figure 1: The Helix

Lemma 2.1 *If tangent vector of a parametrized curve γ is constant, then it is (part of) a straight line.*

Proof: Let $\dot{\gamma}(t) = \overrightarrow{a}$, where $\overrightarrow{a}$ is a constant vector. Then

$$\int \dot{\gamma}(t)dt = \overrightarrow{a}\int dt$$

$$\Rightarrow \gamma(t) = \overrightarrow{a}t + \overrightarrow{b}$$

which is equation of a straight line parallel to $\overrightarrow{a}$ and through $\overrightarrow{b}$. ■

Question: Find parametrization of the following curves

(i) $y^2 - x^2 = 1$

(ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1$

■

Answer:

(i) $\gamma(t) = (\tan t, \sec t)$ because $\sec^2 t - \tan^2 t = 1$.

(ii) $\gamma(t) = (2\cos t, 3\sin t)$ because for the ellipse, we have $x = a\cos t, \quad y = b\sin t$.

■

Question: Find Cartesian equation of the following parametrized curves.

(i) $\gamma(t) = (\cos^2 t, \sin^2 t)$

(ii) $\gamma(t) = (e^t, t^2)$

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Answer:

(i) We have $x = \cos^2 t,\ y = \sin^2 t$ so that $x + y = 1$, which is the required Cartesian form.

(ii) We have

$x = e^t \qquad (1)$

and

$y = t^2 \qquad (2)$.

(1) implies that $t = \ln x$. Using this value of t in (2), we get $y = (\ln x)^2$ which is the required Cartesian form.

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Arc Length (Length of Curves): Consider the parametrized curve $\gamma(t) = (x(t), y(t)), \ \ a \le t \le b$. Corresponding to two values t and $t + \Delta t$, with Δt close to zero, we get two points $\gamma(t)$ and $\gamma(t + \Delta t)$ on the curve.

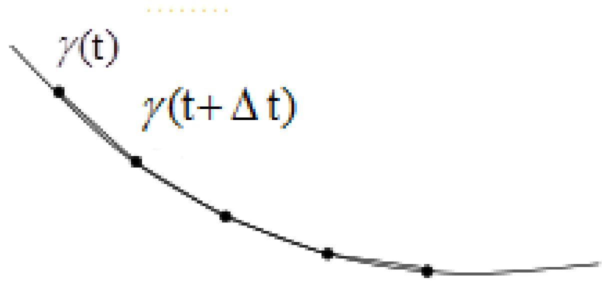

Figure 2:

Then the distance between $\gamma(t)$ and $\gamma(t+\Delta t)$ is given by

$$\begin{aligned}
|\gamma(t+\Delta t)-\gamma(t)| &= |(x(t+\Delta t), y(t+\Delta t)) - (x(t), y(t))| \\
&= |(x(t+\Delta t) - x(t), y(t+\Delta t) - y(t))| \\
&= |(\Delta x, \Delta y)|, \quad \text{where} \quad \Delta x = x(t+\Delta t) - x(t) \quad \text{and} \quad \Delta y = y(t+\Delta t) - y(t) \\
&= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\
&= \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2} \ \ \Delta t.
\end{aligned}$$

Adding up many such short distances and taking the limit as $\Delta t \to 0$, we arrive at the arc length

$$s(t) = \int_a^t \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \ \ dt = \int_a^t \sqrt{\dot{x}^2 + \dot{y}^2} \ \ dt. \tag{1}$$

of the curve γ starting at $t = a$. The integrand in (1) is the norm $\|\dot{\gamma}(t)\| := \sqrt{\dot{x}^2 + \dot{y}^2}$ of the velocity vector $\dot{\gamma}$, so we can write

$$s(t) = \int_a^t \|\dot{\gamma}(u)\| du.$$

Note 2.2 *$s(a) = 0$ and $s(t)$ is positive or negative depending upon whether t is larger or smaller than a.*

Example 2.3 *Consider the logarithmic spiral $\gamma(t) = (e^t \cos t, e^t \sin t)$. Find the arc length of γ starting at $\gamma(0)$.*

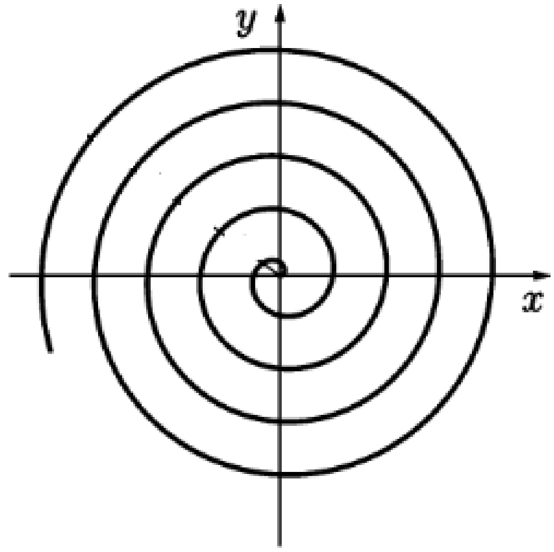

Figure 3: The logarithmic Spiral

Solution: We have $\gamma(t) = (e^t \cos t, e^t \sin t)$. Then

$$\dot{\gamma}(t) = (-e^t \sin t + e^t \cos t, e^t \sin t + e^t \cos t)$$

$$\begin{aligned} \dot{\gamma}(t) &= (-e^t \sin t + e^t \cos t, e^t \sin t + e^t \cos t) \\ \Rightarrow \|\dot{\gamma}(t)\| &= \sqrt{(e^{2t} \sin^2 t + e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t) + (e^{2t} \sin^2 t + e^{2t} \cos^2 t + 2e^{2t} \sin t \cos t)} \\ &= \sqrt{(2e^{2t} \sin^2 t + 2e^{2t} \cos^2 t} \\ &= \sqrt{2} e^t \end{aligned}$$

So,

$$\begin{aligned} s(t) &= \int_0^t \|\dot{\gamma}(u)\| du \\ &= \sqrt{2} \int_0^t e^u du = \sqrt{2}(e^t - 1) \end{aligned}$$

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