Differential Geometry	**Fall 2020**
Lecture 5	
Lecturer:Dr. Muhammad Yaseen	*Topic:Curvature*

Curvature

The curvature of a curve is the measure of how much a curve deviates from straight line. In other words, curvature measures the extent to which a curve is not contained in a straight line.

Note 5.1 *If γ is a curve with $\ddot{\gamma} = \mathbf{0}$, then γ is a straight line (because if γ is a straight line parallel to $\overrightarrow{a}$ and through $\overrightarrow{b}$, then $\gamma(t) = \overrightarrow{a}t + \overrightarrow{b}$ so that $\ddot{\gamma} = \mathbf{0}$). So we might define curvature of a curve γ to be $\|\ddot{\gamma}\|$ (We have taken norm because we want curvature to be a scaler not a vector).*

Definition 5.2 *If γ is a unit speed curve with parameter s, then its curvature $\kappa(s)$ at point $\gamma(s)$ is defined as $\|\ddot{\gamma}(s)\|$.*

Remark 5.3 *Since the curvature depends only on the shape of the curve, it should remain unchanged when curve is reparametrized.*

Proof: Recall that if γ is a unit speed curve, then the unit speed reparametrization of γ is of the form $\gamma(u)$, where $u = \pm s + c$, where c is a constant. Then

$$\begin{aligned}\frac{d\gamma}{ds} &= \frac{d\gamma}{du}.\frac{du}{ds} \\ &= \pm\frac{d\gamma}{du} \qquad (1)\end{aligned}$$

Now

$$\begin{aligned}\frac{d^2\gamma}{ds^2} &= \frac{d}{ds}(\frac{d\gamma}{ds}) \\ &= \frac{d}{ds}(\pm\frac{d\gamma}{du}) \quad \text{(from (1)} \\ &= \pm\frac{d}{du}(\pm\frac{d\gamma}{du}) \quad \text{(from (1))} \\ &= \pm\frac{d^2\gamma}{du^2}\end{aligned}$$

$$\Rightarrow \|\frac{d^2\gamma}{ds^2}\| = \|\frac{d^2\gamma}{du^2}\|.$$

This implies that the curvature of the curve computed using unit speed parameter s is same as that computed using the unit speed parameter u. ■

Remark 5.4 *Large circles should have smaller curvature than small circles.*

Proof: Consider a circle with center (x_0, y_0) and radius R given by

$$\begin{aligned} \gamma(t) &= (x_0, y_0) + (R\cos t, R\sin t) \\ &= (x_0 + R\cos t, y_0 + R\sin t). \end{aligned}$$

Then $\dot{\gamma}(t) = (-R\sin t, R\cos t)$ so that $\|\dot{\gamma}(t)\| = R$. Then $s = \int_0^t \|\dot{\gamma}(u)\|du = \int_0^t Rdu = Rt$ so that $t = \frac{s}{R}$. Now the unit speed reparametrization of γ is

$$\gamma(s) = (x_0 + R\cos(\frac{s}{R}), y_0 + R\sin(\frac{s}{R})).$$

Now

$$\begin{aligned} \gamma'(s) &= (-\sin(\frac{s}{R}), \cos(\frac{s}{R})) \\ \gamma''(s) &= (-\frac{1}{R}\cos(\frac{s}{R}), -\frac{1}{R}\sin(\frac{s}{R})) \\ \Rightarrow \|\gamma''(s)\| &= \sqrt{\frac{1}{R^2}\cos^2(\frac{s}{R}) + \frac{1}{R^2}\sin^2(\frac{s}{R})} \\ &= \sqrt{\frac{1}{R^2}} \\ \Rightarrow \kappa(s) = \frac{1}{R} \end{aligned}$$

which is the reciprocal of the radius of circle. This shows that small circles should have larger curvature and large circles should have smaller curvature. ■

Note 5.5 *It is not always possible to find unit speed reparametrization explicitly. So, we need a formula for curvature in terms of γ itself rather than in terms of its unit speed reparametrization.*

Proposition 5.6 *Let γ be a regular curve in $\mathbb{R}^3$, then its curvature is*

$$\kappa(s) = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}.$$

Proof: Since γ is regular, it has a unit speed reparametrization. If s is arc length parameter, then by chain rule, we have

$$\begin{aligned}\frac{d\gamma}{dt} &= \frac{d\gamma}{ds}.\frac{ds}{dt}\\ \Rightarrow \frac{d\gamma}{ds} &= \frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}.\end{aligned} \tag{2}$$

We have

$$\begin{aligned}\kappa &= \|\frac{d^2\gamma}{ds^2}\|\\ &= \|\frac{d}{ds}(\frac{d\gamma}{ds})\|\\ &= \|\frac{d}{ds}(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}})\| \quad (by(2))\\ &= \|\frac{\frac{d}{dt}(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}})}{\frac{ds}{dt}}\| \quad (by(2))\\ &= \|\frac{\frac{\frac{ds}{dt}.\frac{d^2\gamma}{dt^2}-\frac{d\gamma}{dt}.\frac{d^2s}{dt^2}}{(\frac{ds}{dt})^2}}{\frac{ds}{dt}}\|\\ &= \|\frac{(\frac{ds}{dt}.\frac{d^2\gamma}{dt^2}-\frac{d\gamma}{dt}.\frac{d^2s}{dt^2}}{(\frac{ds}{dt})^3}\|.\end{aligned} \tag{3}$$

We have $\frac{ds}{dt} = \|\dot{\gamma}\|$. So

$$\begin{aligned}(\frac{ds}{dt})^2 &= \|\dot{\gamma}\|^2 = \dot{\gamma}.\dot{\gamma}\\ \Rightarrow 2(\frac{ds}{dt})\frac{d^2s}{dt^2} &= \dot{\gamma}.\ddot{\gamma} + \ddot{\gamma}.\dot{\gamma} = 2\ddot{\gamma}.\dot{\gamma}\\ \Rightarrow \frac{ds}{dt}\frac{d^2s}{dt^2} &= \ddot{\gamma}.\dot{\gamma}.\end{aligned}$$

(3) implies

$$\begin{aligned}\kappa &= \|\frac{(\frac{ds}{dt})^2.\frac{d^2\gamma}{dt^2} - \frac{d\gamma}{dt}.(\frac{ds}{dt})\frac{d^2s}{dt^2}}{(\frac{ds}{dt})^4}.\|\\ &= \frac{(\dot{\gamma}.\dot{\gamma})\ddot{\gamma} - \dot{\gamma}((\ddot{\gamma}.\dot{\gamma}))}{(\frac{ds}{dt})^4}\\ &= \frac{\dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma})}{\|\dot{\gamma}\|^4} \ \text{(because } a\times(b\times c) = (a.c)b - (a.b)c)\\ &= \frac{\|\dot{\gamma}\|\|\ddot{\gamma}\times\dot{\gamma}\|\sin 90}{\|\dot{\gamma}\|^4}\\ &= \frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}.\end{aligned}$$

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Example 5.7 *Find the curvature of the circular helix* $\gamma(\theta) = (a\cos\theta, a\sin\theta), b\theta)$.

Solution: $\dot{\gamma}(\theta) = (-a\sin\theta, a\cos\theta), b)$. Then
$\|\dot{\gamma}(\theta)\| = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta + b^2} = \sqrt{a^2+b^2} \neq 1$ so that γ is not a unit speed curve. Now
$\ddot{\gamma}(\theta) = (-a\cos\theta, -a\sin\theta), 0)$. Then

$$\ddot{\gamma} \times \dot{\gamma} = \begin{vmatrix} i & j & k \\ -a\cos\theta & -a\sin\theta & 0 \\ -a\sin\theta & a\cos\theta & b \end{vmatrix}$$

$$(-ab\sin\theta, -ab\cos\theta, -a^2)$$

$$\begin{aligned} \Rightarrow \|\ddot{\gamma} \times \dot{\gamma}\| &= \sqrt{a^2b^2\sin^2\theta, a^2b^2\cos^2\theta + a^4} \\ &= \sqrt{a^2b^2 + a^4} = \sqrt{a^2}\sqrt{a^2+b^2} \\ &= |a|\sqrt{a^2+b^2} \end{aligned}$$

So

$$\begin{aligned} \kappa &= \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3} \\ &= \frac{|a|\sqrt{a^2+b^2}}{(a^2+b^2)^{\frac{3}{2}}} \\ &= \frac{|a|}{a^2+b^2} \end{aligned}$$

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Example 5.8 *Compute the curvature of the curve* $\gamma(t) = (\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}})$.

Solution: $\dot{\gamma}(t) = (\frac{1}{2}(1+t)^{\frac{1}{2}}, -\frac{1}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}})$. Then $\|\dot{\gamma}(t)\| = 1$ (already shown). Now

$$\begin{aligned} \ddot{\gamma}(t) &= (\frac{1}{4\sqrt{1+t}}, \frac{1}{4\sqrt{1-t}}, 0) \\ \Rightarrow \kappa = \|\ddot{\gamma}(t)\| &= \sqrt{\frac{1}{16(1+t)} + \frac{1}{16(1-t)}} \\ &= \sqrt{\frac{1-t+1-t}{16(1-t^2)}} = \frac{1}{\sqrt{8(1-t^2)}}. \end{aligned}$$

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Example 5.9 *Compute the curvature of the curve* $\gamma(t) = (\frac{4}{5}\cos t, 1-\sin t, \frac{-3}{5}\cos t)$.

Solution: $\dot{\gamma}(t) = (\frac{-4}{5}\sin t, -\cos t, \frac{3}{5}\sin t)$. Then $\|\dot{\gamma}(t)\| = 1$ (already shown). Now

$$\begin{aligned}
\ddot{\gamma}(t) &= (\frac{-4}{5}\cos t, \sin t, \frac{3}{5}\cos t) \\
\Rightarrow \kappa = \|\ddot{\gamma}(t)\| &= \sqrt{\frac{16}{25}\cos^2 t + \sin^2 t, \frac{9}{25}\cos^2 t} \\
&= 1
\end{aligned}$$

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Example 5.10 *Compute the curvature of the curve* $\gamma(t) = (t, \cosh t, 0)$.

Solution: $\dot{\gamma}(t) = (1, \sinh t, 0)$. Then $\|\dot{\gamma}(t)\| = \sqrt{1 + \sinh^2 t} = \cosh t \neq 0$. Now $\ddot{\gamma}(t) = (0, \cosh t, 0)$

$$\begin{aligned}
\ddot{\gamma}(t) \times \dot{\gamma} &= \begin{vmatrix} 1 & j & k \\ 0 & \cosh t & 0 \\ 1 & \cosh t & 0 \end{vmatrix} \\
&= (0, 0, \cosh t) \\
\Rightarrow \|\ddot{\gamma}(t) \times \dot{\gamma}\| &= \cosh t.
\end{aligned}$$

Thus

$$\kappa = \frac{\|\ddot{\gamma}(t) \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{\cosh t}{\cosh^2 t} = \frac{1}{\cosh t}.$$

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Example 5.11 *Compute the curvature of the curve* $\gamma(t) = (\cos^3 t, \sin^3 t, 0)$.

Solution: Do yourself (Answer: $\kappa = \frac{1}{3|\sin t \cos t|}$) ■