Differential Geometry	**Fall 2020**

Lecture 4

Lecturer:Dr. Muhammad Yaseen	*Topic:Arc Length*

Lemma 4.1 *Let $\gamma_1 : I_1 \to \mathbb{R}^n$ and $\gamma_2 : I_2 \to \mathbb{R}^n$ be two different parametrizations by arc length of the same curve. Then the corresponding reparametrization map $\phi : I_1 \to I_2$ with $\gamma_1 = \gamma_2 \circ \phi$ is of the form $\phi(t) = t + t_0$ for $t_0 \in \mathbb{R}$ if γ_1 and γ_2 have same orientation and if γ_1 and γ_2 have opposite orientation, then it is of the form $\phi(t) = -t + t_0$.*

Proof: We have

$$\begin{aligned}
1 &= \|\dot{\gamma_1}(t)\| \quad \text{(because } \gamma_1 \text{ is a unit speed curve)} \\
&= \|\frac{d}{dt}(\gamma_2 \circ \phi)(t)\| \\
&= \|\dot{\gamma_2}(\phi(t)).\dot{\phi}(t)\| \\
&= \|\dot{\gamma_2}(\phi(t))\||\dot{\phi}(t)| \\
&= |\dot{\phi}(t)| \text{ (because } \gamma_2 \text{ is a unit speed curve, therefore } \|\dot{\gamma_2}(\phi(t))\| = 1)
\end{aligned}$$

This implies that

$$\begin{aligned}
&|\dot{\phi}(t)| = 1 \\
&\Rightarrow \dot{\phi}(t) = \pm 1 \\
&\Rightarrow \phi(t) = \pm t + t_0
\end{aligned}$$

If $\phi(t) = t + t_0$, then $\dot{\phi}(t) = 1 > 0$ so that ϕ is orientation preserving. If $\phi(t) = -t + t_0$, then $\dot{\phi}(t) = -1 < 0$ so that ϕ is orientation reversing. ■

Proposition 4.2 *Any reparmetrization of a regular curve is regular.*

Proof: Let $\tilde{\gamma} : J \to \mathbb{R}^n$ be a reparmetrization of a cuve $\gamma : I \to \mathbb{R}^n$. Then there exists a reparametrization map $\phi : J \to I$ whose inverse ϕ^{-1} is also smooth such that $\tilde{\gamma}(\tilde{t}) = \gamma \circ \phi(\tilde{t}) \;\; \forall \;\; \tilde{t} \in J$.
Let $\phi(\tilde{t}) = t$ and $\phi^{-1} = \psi$ so that $\tilde{t} = \phi^{-1}(t) = \psi(t)$. Now consider

$$\phi(\phi^{-1}(t)) = t$$

$$
\begin{aligned}
&\Rightarrow \phi(\psi(t)) = t \\
&\Rightarrow \frac{d}{dt}(\phi(\psi(t))) = 1 \\
&\Rightarrow \frac{d}{dt}(\phi(\tilde{t})) = 1 \\
&\Rightarrow \frac{d}{d\tilde{t}}(\phi(\tilde{t}))\frac{d\tilde{t}}{dt} = 1 \\
&\Rightarrow (\frac{d\phi}{d\tilde{t}})(\frac{d\psi}{dt}) = 1
\end{aligned}
$$

This implies that

$$
\frac{d\phi}{d\tilde{t}} \neq 0 \text{ and } \frac{d\psi}{dt} \neq 0 \tag{1}
$$

We have to show that $\frac{d\tilde{\gamma}}{dt} \neq 0 \ \forall \ \tilde{t} \ \in J$. For this consider

$$
\begin{aligned}
\tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\
\Rightarrow \frac{d\tilde{\gamma}}{dt} &= \frac{d}{d\tilde{t}}(\gamma(\phi(\tilde{t}))) \\
&= \frac{d}{d\tilde{t}}(\gamma(t)) \\
&= \frac{d}{dt}(\gamma(t))\frac{dt}{d\tilde{t}} \\
&= (\frac{d\gamma}{dt})(\frac{d\phi}{d\tilde{t}}) \\
&\neq 0 \text{ because } \gamma \text{ is regular and from (a), } \frac{d\phi}{d\tilde{t}} \neq 0
\end{aligned}
$$

This implies that $\tilde{\gamma}$ is regular. ■

Proposition 4.3 *If $\gamma(t)$ is a regular curve, then its arc length starting at any point of γ is a smooth function of t.*

Proof: We have

$$
\begin{aligned}
s(t) &= \int_{t_0}^{t} \|\dot{\gamma}(u) du \\
\Rightarrow \frac{ds}{dt} &= \|\dot{\gamma}(t)\|.
\end{aligned}
$$

To simplify notation, let us assume that γ is a plane curve, i.e. $\gamma(t) = (u(t), v(t))$, where u and v are smooth functions of t. Then $\frac{ds}{dt} = \sqrt{(\dot{u})^2 + (\dot{v})^2} > 0$. So

$$
\begin{aligned}
\frac{d^2 s}{dt^2} &= \frac{1}{2\sqrt{(\dot{u})^2 + (\dot{v})^2}}(2\dot{u}\ddot{u} + 2\dot{v}\ddot{v}) \\
&\frac{\dot{u}\ddot{u}}{\sqrt{(\dot{u})^2 + (\dot{v})^2}} + \frac{\dot{v}\ddot{v}}{\sqrt{(\dot{u})^2 + (\dot{v})^2}}
\end{aligned}
$$

which exists. Similarly for other derivatives. This proves that s is a smooth function of t. ∎

Theorem 4.4 *A parametrized curve has a unit speed reparametrization if and only if it is regular.*

Proof: Let $\tilde{\gamma} : J \to \mathbb{R}^n$ be a unit speed reparametrization of $\gamma : I \to \mathbb{R}^n$. Then there exists a smooth bijection $\phi : J \to I$ whose inverse $\phi^{-1} : I \to J$ is also smooth and

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})), \quad \forall \tilde{t} \in J. \tag{2}$$

As before, let $t = \phi(\tilde{t})$. Then differentiating both sides with respect to t, we obtain

$$\begin{aligned} 1 &= \frac{d}{dt}(\phi(\tilde{t})) \\ &= \frac{d}{d\tilde{t}}(\phi(\tilde{t})).\frac{d\tilde{t}}{dt} \\ &= \frac{d\phi}{d\tilde{t}}.\frac{d\tilde{t}}{dt} \end{aligned} \tag{3}$$

Now (2) implies

$$\begin{aligned} \frac{d\tilde{\gamma}}{d\tilde{t}} &= \frac{d}{d\tilde{t}}(\gamma(\phi(\tilde{t}))) \\ &= \frac{d}{d\tilde{t}}(\gamma(t)) \\ &= \frac{d\gamma}{dt}.\frac{dt}{d\tilde{t}} \\ \Rightarrow \|\frac{d\tilde{\gamma}}{dt}\| &= \|\frac{d\gamma}{dt}\|.|\frac{dt}{d\tilde{t}}| \\ \Rightarrow 1 &= \|\frac{d\gamma}{dt}\|.|\frac{dt}{d\tilde{t}}| \text{ (because } \tilde{\gamma} \text{ is a unit speed curve)} \\ \Rightarrow \|\frac{d\gamma}{dt}\| &\neq 0 \\ \Rightarrow \frac{d\gamma}{dt} &\neq 0. \end{aligned}$$

This proves that γ is regular.

Conversely suppose that γ is regular i.e. $\frac{d\gamma}{dt} \neq 0 \ \forall t$. Then $\frac{ds}{dt} = \|\dot{\gamma}(t)\| = \|\frac{d\gamma}{dt}\| > 0 \ \ \forall t$, where s is arc length of γ starting at any point. The inverse function theorem states that:

'If f is a smooth function and its first derivative is non zero at a point, then f is injective and has smooth inverse in the neighbourhood of that point.'

Since s is smooth and $\frac{ds}{dt} > 0$, therefore by inverse function theorem, $s : I \to J$ has a smooth inverse $s^{-1} : J \to I$. If $\tilde{\gamma}$ is the corresponding reparametrization map then

$$\tilde{\gamma}(s(t)) = \gamma(t)$$

$$\Rightarrow \frac{d}{dt}(\dot{\gamma}(s(t))) = \frac{d\gamma}{dt}$$
$$\Rightarrow \frac{d\dot{\gamma}}{ds}.\frac{ds}{dt} = \frac{d\gamma}{dt} \text{ (By chain rule)}$$
$$\Rightarrow \|\frac{d\tilde{\gamma}}{ds}\|.|\frac{ds}{dt}| = \|\frac{d\gamma}{dt}\|$$
$$\Rightarrow \|\frac{d\tilde{\gamma}}{ds}\| = 1 \text{ (because} \frac{ds}{dt} = \|\frac{d\gamma}{dt}\|.)$$

So $\tilde{\gamma}$ is a unit speed reparametrization of γ ■

The following example shows that a curve can have regular and non-regular parametrization.

Example 4.5 *For the parametrization,* $\gamma(t) = (t, t^2)$ *of the parabola* $y = x^2$*, we have* $\dot{\gamma}(t) = (1, 2t) \neq 0\ \forall t$*. But* $\tilde{\gamma}(t) = (t^3, t^6)$ *is a parametrization of the same parabola but at this time* $\dot{\tilde{\gamma}}(t) = (3t^2, 6t^5)$ *is zero at* $t = 0$*. So* $\tilde{\gamma}$ *is not regular.*