Differential Geometry **Fall 2020**

Lecture 3

Lecturer:Dr. Muhammad Yaseen *Topic:Arc Length*

Recall that the arc length s of a curve γ starting at point $\gamma(t)$ is given by

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| du$$
$$\Rightarrow \frac{ds}{dt} = \|\dot{\gamma}(t)\| = \text{Speed}$$

This means that if we think of $\gamma(t)$ as position of a moving particle at time t, then $\frac{ds}{dt}$ is the speed of the particle. In view of this, we have the following definition.

Definition 3.1 *If $\gamma : I \to \mathbb{R}^3$ is a prametrized curve, then its speed at point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$ and γ is called a unit speed curve (or curve parametrized by arc length) if $\|\dot{\gamma}(t)\| = 1 \;\; \forall \;\; t \in I$.*

Result: Let $\vec{n}(t)$ be a unit vector that is a smooth function of the parameter t, then $\vec{n}(t).\dot{\vec{n}}(t) = 0 \;\forall t$. In particular if γ is a unit speed curve then $\dot{\gamma}.\ddot{\gamma} = 0$

Proof: Since $\vec{n}(t)$ is a unit vector, therefore,

$$\begin{aligned} \vec{n}(t).\vec{n}(t) &= 1 \\ \Rightarrow \vec{n}(t).\dot{\vec{n}}(t) + \dot{\vec{n}}(t).\vec{n}(t) &= 0 \\ \Rightarrow 2\vec{n}(t).\dot{\vec{n}}(t) &= 0 \\ \Rightarrow \vec{n}(t).\dot{\vec{n}}(t) &= 0 \end{aligned}$$

as required. If γ is a unit speed curve then $\dot{\gamma}$ is a unit vector. Taking $\vec{n}(t) = \dot{\gamma}$ in above result, we see that

$$\dot{\gamma}.\ddot{\gamma} = 0.$$

■

Definition 3.2 *A parametrized curve $\gamma : I \to \mathbb{R}^n$ is said to be regular if its velocity vector $\dot{\gamma}$ does not vanish i.e. $\dot{\gamma} \neq 0 \;\; \forall t$.*

Remark 3.3 *The condition $\dot{\gamma} \neq 0 \;\; \forall t$ ensures that the point $\gamma(t)$ moves at $t \in I$.*

Question: Calculate the arc length of Catenary $\gamma(t) = (t, \cosh t)$ starting at the point $(0, 1)$. ■

Answer: The point $(0, 1)$ corresponds to $t = 0$. We have

$$\begin{aligned}
\dot{\gamma}(t) &= (1, \sinh t) \\
\Rightarrow \|\dot{\gamma}(t)\| &= \sqrt{1 + \sinh^2 t} \\
&= \cosh t \\
\Rightarrow s(t) &= \int_0^t \|\dot{\gamma}(u)\| du \\
&= \int_0^t \cosh u du \\
&= \sinh |_0^t = \sinh t - \sinh 0 \\
&= \sinh t
\end{aligned}$$

■

Example 3.4 *Show that the curve* $\gamma(t) = (\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}})$ *is unit speed curve.*

Solution: We have

$$\begin{aligned}
\dot{\gamma}(t) &= (\frac{1}{2}(1+t)^{\frac{1}{2}}, \frac{-1}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}) \\
\Rightarrow \|\dot{\gamma}(t)\| &= \sqrt{\frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}} \\
&= 1.
\end{aligned}$$

So γ is a unit speed curve. ■

Example 3.5 *Show that the curve* $\gamma(t) = (\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t)$ *is unit speed curve.*

Solution: We have

$$\begin{aligned}
\dot{\gamma}(t) &= (\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t) \\
\Rightarrow \|\dot{\gamma}(t)\| &= \sqrt{\frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t} \\
&= \sqrt{\sin^2 t + \cos^2 t} \\
&= 1
\end{aligned}$$

which proves that γ is a unit speed curve. ■

Example 3.6 *Parametrize the curve* $\gamma(t) = (e^t \cos t, e^t \sin t)$ *by arc length and then show that the resulting curve is a unit speed curve.*

Solution: Recall that for the given curve we found the arc length to be $s = \sqrt{2}(e^t - 1)$. $\Rightarrow e^t = \frac{s}{\sqrt{2}} + 1$ $\Rightarrow t = \ln(\frac{s}{\sqrt{2}} + 1)$. Using this value of t in the given curve we obtain

$$\gamma(s) = ((\frac{s}{\sqrt{2}} + 1)\cos(\ln(\frac{s}{\sqrt{2}} + 1)), (\frac{s}{\sqrt{2}} + 1)\sin(\ln(\frac{s}{\sqrt{2}} + 1)))$$

which is the required arc length parametrization of γ. Now we show that γ is a unit speed curve by showing that $\|\dot{\gamma}(s)\| = 1$. We have

$\dot{\gamma}(s) = (-\frac{1}{\sqrt{2}}\sin(\ln(\frac{s}{\sqrt{2}} + 1)) + \frac{1}{\sqrt{2}}\cos(\ln(\frac{s}{\sqrt{2}} + 1)), \frac{1}{\sqrt{2}}\cos(\ln(\frac{s}{\sqrt{2}} + 1)) + \frac{1}{\sqrt{2}}\sin(\ln(\frac{s}{\sqrt{2}} + 1)))$ (Check it)

This implies that $\|\dot{\gamma}(s)\| = 1$ (Check it). ■

Definition 3.7 *A parametrized curve* $\gamma : I \to \mathbb{R}^n$ *is called smooth if* γ *is differentiable up to all orders.*

Definition 3.8 *Let* $\gamma : I \to \mathbb{R}^n$ *be a parametrized curve. Then* $\tilde{\gamma} : J \to \mathbb{R}^n$ *is called a reparametrization of* γ *if there exists a smooth bijection* $\phi : J \to I$ *(the reparametrization map) whose inverse* $\phi^{-1} : I \to J$ *is also smooth such that*

$$\tilde{\gamma} = \gamma \circ \phi \quad \text{i.e.} \quad \tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) \quad \text{for all} \quad \tilde{t} \in J.$$

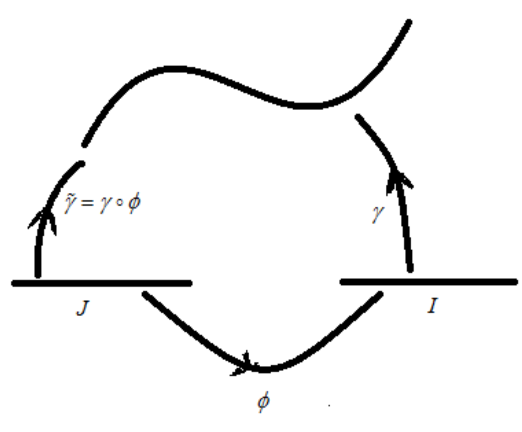

Figure 1:

Note 3.9 *If $\tilde{\gamma}$ is reparametrization of γ, then γ is reparametrization of $\tilde{\gamma}$ ($\because$ ϕ has a smooth inverse). To see this consider*

$$\gamma(t) = \gamma(\phi(\phi^{-1}(t))) = \tilde{\gamma}(\phi^{-1}(t)) \quad (\because \gamma \circ \phi = \tilde{\gamma}) \tag{1}$$

Definition 3.10 *A reparametrization map ϕ is orientation preserving if $\dot{\phi}(t) > 0 \; \forall \; t$ and is orientation reversing if $\dot{\phi}(t) < 0 \; \forall \; t$.*

Example 3.11 *Show that $\gamma_1 : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t, t)$ is reparametriztion of $\gamma_2 : (0, \infty) \to \mathbb{R}^2$ given by $\gamma(t) = (\ln t, \ln t)$.*

Solution: We have to find a mapping $\phi : (0, \infty) \to \mathbb{R}$ such that

$$\gamma_2(\phi(t)) = (\ln(\phi(t)), \ln(\phi(t))) = (t, t) = \gamma_1(t).$$

This suggests that $\phi(t) = e^t$. Then

$$\gamma_2(\phi(t)) = \gamma_2(e^t) = (\ln(e^t), \ln(e^t)) = (t, t) = \gamma_1(t)$$

which shows that γ_1 is reparametrization of γ_2. ■

Example 3.12 *Show that $\tilde{\gamma}(t) = (\sin t, \cos t)$ is a reparametriztion of $\gamma(t) = (\cos t, \sin t)$.*

Solution: We have to find a mapping $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma(\phi(t)) = (\cos(\phi(t)), \sin(\phi(t))) = (\sin t, \cos t) = \tilde{\gamma}(t).$$

This suggests that $\phi(t) = \frac{\pi}{2} - t$. Then

$$\gamma(\phi(t)) = \gamma(\frac{\pi}{2} - t) = (\cos(\frac{\pi}{2} - t), \sin(\frac{\pi}{2} - t)) = (\sin t, \cos t) = \tilde{\gamma}(t)$$

which shows that $\tilde{\gamma}$ is reparametrization of γ . ■

Curves parametrized by arc length are for many purposes convenient. But do they exist? The following proposition answers this question.

Proposition 3.13 *For every regular parametrized curve γ, there exists an orientation preserving reparametrization map ϕ such that the reparametrization $\gamma \circ \phi$ is parametrized by arc length.*

Proof: Let $\gamma : I \to \mathbb{R}^n$ be regular parametrized curve. Choose $s_0 \in I$ and define

$$\psi(s) = \int_{s_0}^{s} \|\dot{\gamma}(t)\| dt.$$

$\Rightarrow \quad \acute{\psi}(s) = \|\dot{\gamma}(s)\| > 0$ (by first fundamental theorem of calculus) so that ψ is increasing and hence injective. Thus $\psi : I \to J := \psi(I)$ is an orientation preserving reparamentrization map. Define the inverse map $\phi := \psi^{-1} : J \to I$. Then ϕ and ψ are smooth and we have

$$\dot{\phi}(t) = \dot{\psi^{-1}}(t) = \frac{1}{\acute{\psi}(\phi(t))} \quad (\because \acute{f^{-1}} = \frac{1}{\acute{f}(f^{-1}(t))})$$
$$= \frac{1}{\|\dot{\gamma}(\phi(t))\|} > 0$$

so that ϕ is orientation preserving. Now by chain rule, we have

$$\|\frac{d}{dt}(\gamma \circ \phi)(t)\| = \|\dot{\gamma}(\phi(t)) \cdot \dot{\phi}(t)\|$$
$$= \|\dot{\gamma}(\phi(t)) \cdot \frac{1}{\|\dot{\gamma}(\phi(t))\|}\| = \frac{\|\dot{\gamma}(\phi(t))\|}{\|\dot{\gamma}(\phi(t))\|} = 1$$

$\Rightarrow \quad \gamma \circ \phi$ is a unit speed curve and hence is parametrized by arc length. ■