| | |
|---|---|
| **Differential Geometry** | **Fall 2020** |

# Lecture 1

*Lecturer:Dr. Muhammad Yaseen* *Topic:Curves in Plane*

**Course Title: Differential Geometry**

**Course Code: MATH-303 (For BS) & MATH-601 (For MSc)**

**Question:** What is a curve? ■

**Answer:** One can draw a variety of curves.

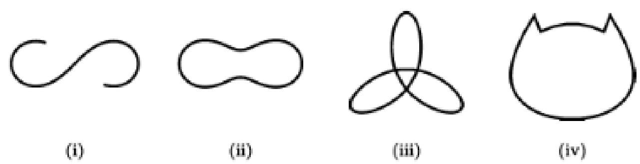


Figure 1: Various Curves

The curve in (i) is a smooth (sufficiently differentiable) curve. The curve in (ii) is also of a smooth curve, but now without endpoints, since traveling along the curve will bring you back to where you started. This type of curve is called a closed curve. In (iii), we have self intersections and we still regard it as a smooth closed curve. The curve in (iv) is a closed curve, but as it has sharp angles at particular points, it is not smooth at those points.

There are two ways to think of a curve:

**(a) The implicit function representation**

In general, if we take a function $\boldsymbol{F(x,y)}$ of two variables and collect together all points in the plane satisfying $\boldsymbol{F(x,y)=0}$ to produce a curve $\boldsymbol{C}$, we call this the implicit function representation for curve by $\boldsymbol{F(x,y)}$. So

$$\boldsymbol{C=\{(x,y)|F(x,y)=0\}.}$$

**Example 1.1** *The following are examples of curves that are represented implicitly by the solution sets of the following four equations.*

$$\textit{Ellipse:} \qquad \boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.}$$

$$
\begin{aligned}
&\textit{Hyperbola:} && \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \\
&\textit{Lemniscate:} && (x^2 + y^2)^2 - a^2(x^2 - y^2) = 0. \\
&\textit{Astroid:} && (a^2 - x^2 - y^2)^3 - 27a^2x^2y^2 = 0.
\end{aligned}
$$

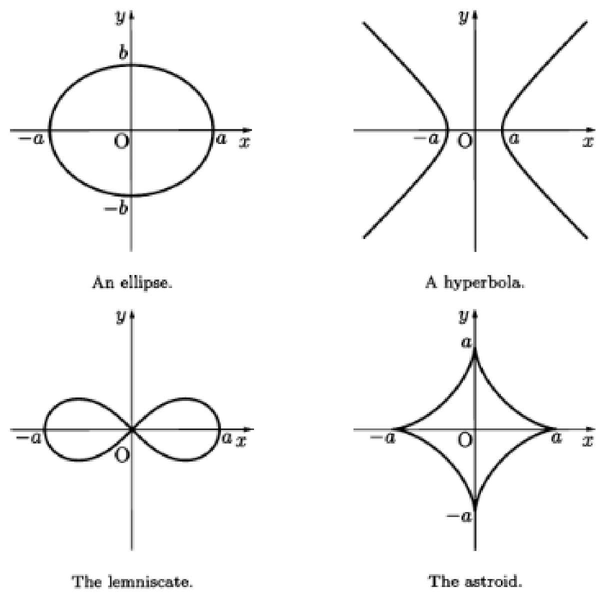


Figure 2:

**(b) As path of a moving particle**

If a particle is moving through the plane with respect to time variable $t$, then $(x(t), y(t))$ denotes the position of the particle at any time $t$, where $x(t),\ \ y(t)$ are smooth functions of $t$. Then the trajectory of these points can be imagined as a curve. Here, $t$ is called a parameter of the curve. This way of representing a curve is called a parametrization. Pairing up the two functions $x(t), y(t)$ and naming the pair $\gamma(t)$, we have the curve

$$\gamma(t) = (x(t), y(t)).$$

**Remark 1.2** *We are mostly interested in geometric shape of a curve as in **(a)** but **(b)** is more useful because we can bring in tools from calculus to describe the geometric behavior. It is like giving a coordinate system along a curve which allows calculations to be done.*

■

**Definition 1.3** *A **parametrized curve** in* $\mathbb{R}^3$ *is a map* $\gamma : \boldsymbol{I} \to \mathbb{R}^3$ *given by* $\gamma(\boldsymbol{t}) = (\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{y}(\boldsymbol{t}), \boldsymbol{z}(\boldsymbol{t}))$*, where* $\boldsymbol{I}$ *is usually taken to be an open interval which is either bounded or unbounded.*

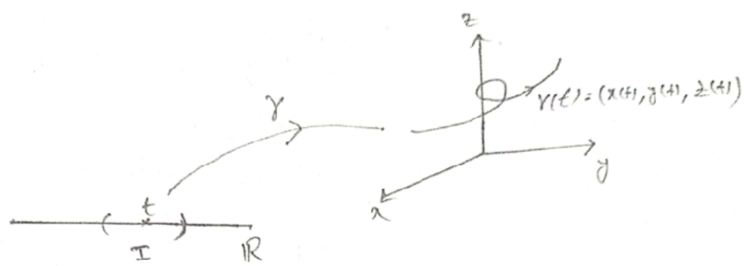


Figure 3:

**Definition 1.4** *If* $\gamma$ *is a parametrizeed curve in* $\mathbb{R}^3$ *given by* $\gamma(\boldsymbol{t}) = (\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{y}(\boldsymbol{t}), \boldsymbol{z}(\boldsymbol{t}))$*, then* $\dot{\gamma}(\boldsymbol{t}) = (\dot{\boldsymbol{x}}(\boldsymbol{t}), \dot{\boldsymbol{y}}(\boldsymbol{t}), \dot{\boldsymbol{z}}(\boldsymbol{t}))$ *is called the **tangent vector** of* $\gamma$ *at the point* $\gamma(\boldsymbol{t})$*.*

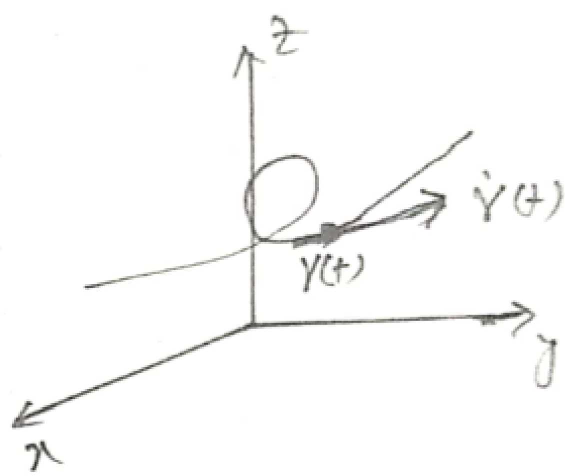


Figure 4:

**Definition 1.5** *The **trace of*** $\gamma$ *is defined as*

$$\mathit{Image}(\gamma) = \gamma(\boldsymbol{I}) \subset \mathbb{R}^3$$

**Definition 1.6** *We say that* $\gamma : \boldsymbol{I} \to \mathbb{R}^3$ *is a **plane curve** if there exists a plane* $\boldsymbol{P} \subset \mathbb{R}^3$ *such that* $\gamma(\boldsymbol{I}) \subset \boldsymbol{P}$*.*

**Remark 1.7** *The parametrization of a curve is not unique in general e.g. consider the parabola* $y = x^2$*. If we choose* $x = t$*, then* $y = t^2$ *so that* $\gamma(t) = (t, t^2)$ *is a parametrization of* $y = x^2$*. Another choice is* $\gamma(t) = (2t, 4t^2)$*. Yet another choice is* $\gamma(t) = (t^3, t^6)$ *and so on.*

**Question:** Find parametrization of the circle $x^2 + y^2 = 1$. ■

**Answer:** If we chose $x = t$, then $y = \pm\sqrt{1 - t^2}$. So we have the following parametrizations:

$$\gamma(t) = (t, \sqrt{1 - t^2}) \qquad \text{(Upper semi-circle)}$$
$$\gamma(t) = (t, -\sqrt{1 - t^2}) \qquad \text{(Lower semi-circle)}$$

To find parametriztions that cover the whole circle, we must find functions $x(t),\ y(t)$, such that $(x(t))^2 + (y(t))^2 = 1$. One can obviously choose $x(t) = \cos t, \quad y(t) = \sin t$ so that $\gamma(t) = (\cos t, \sin t)$ parmetrizes the whole circle if $t \in I$, where $I$ is an open interval whose length is greater than $2\pi$. ■