Differential Geometry	**Fall 2020**
Lecture 6	
Lecturer:Dr. Muhammad Yaseen	*Topic:Curvature*

Exercise:

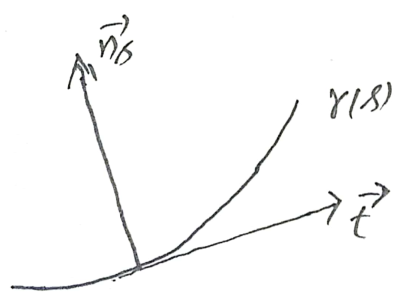
(i) Find unit speed reparametrization of $\gamma(t) = (2\cos t, 2\sin t)$, $0 < t < \frac{\pi}{2}$.

(ii) Compute the arc length of $\gamma(t) = (3\cosh(2t), 3\sinh(2t), 6t)$, $\quad 0 \leq t \leq \pi$.

(iii) Find arc length parametrization of of $\gamma(t) = (e^t \cos t, e^t \sin t, e^t)$.

Plane Curves

For a plane curve it is possible to refine the definition of curvature to give it an appealing geometric interpretation. Suppose that $\gamma(s)$ is a unit speed curve in $\mathbb{R}^3$. Denoting $\frac{d}{ds}$ by a dot, let $\vec{t} = \dot{\gamma}$ be a tangent vector of γ. Note that $\vec{t}$ is a unit vector (because γ is unit speed curve). There are two vector perpendicular



to $\vec{t}$. We make a choice by defining $\vec{n}_s$ (signed unit normal) to be a unit vector obtained by rotating $\vec{t}$ anticlockwise by $\frac{\pi}{2}$. Then $\dot{\vec{t}} \perp \vec{t}$ (because $\vec{t}$ is a unit vector) and hence is parallel to $\vec{n}_s$. Then there exists a scaler κ_s such that

$$\dot{\vec{t}} = \kappa_s \vec{n}_s \ i.e. \ \ddot{\vec{\gamma}} = \kappa_s \vec{n}_s. \tag{1}$$

Here κ_s is called the signed curvature of γ (it can be positive, negative or zero). Now

$$\begin{aligned}\kappa &= \|\ddot{\gamma}\| \\ &= \|\kappa_s \vec{n}_s\| \\ &= |\kappa_s|\|\vec{n}_s\| \\ &= |\kappa_s| \text{ (because } \|\vec{n}_s\| = 1).\end{aligned}$$

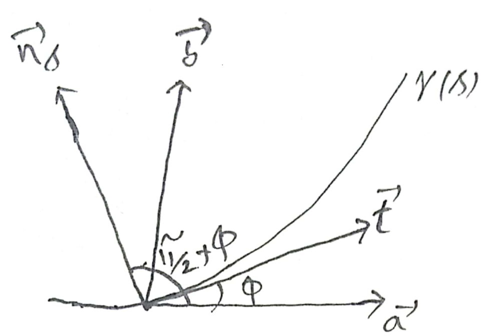
So the curvature of γ is the absolute value of its signed curvature.

Remark 6.1 *For clockwise rotating curves, the signed curvature is negative and for anticlockwise rotating curves the signed curvature is positive.*

Geometric Interpretation of Signed Curvature

Proposition 6.2 *Let $\gamma(s)$ be a unit speed plane curve and let $\phi(s)$ be the angle through which a fixed unit vector must be rotated anticlockwise to bring it into coincidence with unit vector $\vec{t}$ of γ, then $\kappa_s = \frac{d\phi}{ds}$.*

Proof: Let $\vec{a}$ be a fixed unit vector let $\vec{b}$ be the unit vector obtained by rotating $\vec{a}$ anticlockwise by $\frac{\pi}{2}$. Then



$$\begin{aligned}\vec{t} &= \vec{a}\cos\phi + \vec{b}\sin\phi \\ \Rightarrow \frac{d\vec{t}}{ds} &= -\vec{a}\sin\phi.\frac{d\phi}{ds} + \vec{b}\cos\phi.\frac{d\phi}{ds} \\ \Rightarrow \dot{\vec{t}} &= (-\vec{a}\sin\phi + \vec{b}\cos\phi).\frac{d\phi}{ds} \\ \Rightarrow \dot{\vec{t}}\cdot\vec{a} &= -\sin\phi.\frac{d\phi}{ds}\end{aligned}$$

$$\Rightarrow \kappa_s \overrightarrow{n}_s \cdot \overrightarrow{a} = -\sin\phi . \frac{d\phi}{ds}$$
$$\Rightarrow \kappa_s \|\overrightarrow{n}_s\| \cdot \|\overrightarrow{a}\| \cos(\frac{\pi}{2} + \phi) = -\sin\phi . \frac{d\phi}{ds}$$
$$\Rightarrow \kappa_s(-\sin\phi) = -\sin\phi . \frac{d\phi}{ds} \text{ (because } \|\overrightarrow{n}_s\| = 1 \& \|\overrightarrow{a}\| = 1)$$
$$\Rightarrow \kappa_s = \frac{d\phi}{ds}.$$

This completes the proof. ■

Definition 6.3 *A rigid motion in* $\mathbb{R}^2$ *is a map* $\mathbb{R}^2 \to \mathbb{R}^2$ *is of the form* $M = T_{\overrightarrow{a}} \circ R_\theta$*, where* $T_{\overrightarrow{a}}$ *is the translation by the vector* $\overrightarrow{a}$ *given by*

$$T_{\overrightarrow{a}}(\overrightarrow{v}) = \overrightarrow{v} + \overrightarrow{a}$$

for $\overrightarrow{v} \in \mathbb{R}^2$ *and* R_θ *is the anticlockwise rotation by an angle* θ *given by*

$$R_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$
$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

Theorem 6.4 *Let* $\kappa : (\alpha, \beta) \to \mathbb{R}^2$ *be any smooth function. Then there exists a unit speed curve* $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ *whose signed curvature is* κ*. Further if* $\tilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ *is any other unit speed curve whose signed curvature is* κ*, then there exists a rigid motion* M *such that* $\tilde{\gamma}(s) = M(\gamma(s)) \; \forall s \in (\alpha, \beta)$*.*

Proof: For first part, fix $s_0 \in (\alpha, \beta)$ and define

$$\phi(s) = \int_{s_0}^{s} \kappa(u) du \tag{2}$$

and

$$\gamma(s) = \left(\int_{s_0}^{s} \cos(\phi(t)) dt, \int_{s_0}^{s} \sin(\phi(t)) dt \right)$$

so that

$$\dot{\gamma}(s) = (\cos(\phi(s), \sin(\phi(s)))$$

which is a unit vector making angle $\phi(s)$ with positive $x-$axis. Thus γ is unit speed curve and by previous proposition, its signed curvature is

$$\kappa_s = \frac{d\phi}{ds} = \kappa \text{ by } (2).$$

For the second part, suppose that $\tilde{\phi}(s)$ is the angle between $x-$axis and the tangent vector $\dot{\tilde{\gamma}}$ of γ. Then

$$\dot{\tilde{\gamma}}(s) = \left(\cos(\tilde{\phi}(s)), \sin(\tilde{\phi}(s)) \right)$$

$$
\begin{aligned}
\Rightarrow \tilde{\gamma}(s)|_{s_0}^{s} &= \left(\int_{s_0}^{s} \cos(\tilde{\phi})(t)dt, \int_{s_0}^{s} \sin(\tilde{\phi}(t))dt\right) \\
\Rightarrow \tilde{\gamma}(s) &= \left(\int_{s_0}^{s} \cos(\tilde{\phi}(t))dt, \int_{s_0}^{s} \sin(\tilde{\phi}(t))dt\right) + \tilde{\gamma}(s_0). \qquad (3)
\end{aligned}
$$

We have

$$
\begin{aligned}
\kappa(s) &= \frac{d\tilde{\phi}(s)}{ds} \\
\Rightarrow d\tilde{\phi} &= \kappa(s)ds \\
\Rightarrow \tilde{\phi}(s) &= \int_{s_0}^{s} \kappa(u)du + \tilde{\phi}(s_0) \\
\Rightarrow \tilde{\phi}(s) &= \phi(s) + \tilde{\phi}(s_0). \qquad (4)
\end{aligned}
$$

Inserting (4) in (3) and writing $\overrightarrow{a}$ for $\tilde{\gamma}(s_0)$ and θ for $\tilde{\phi}(s_0)$, we obtain

$$
\begin{aligned}
\tilde{\gamma}(s) &= \left(\int_{s_0}^{s} \cos(\phi(t)) + \theta)dt, \int_{s_0}^{s} \sin(\phi(t)) + \theta)dt\right) + \overrightarrow{a} \\
&= T_{\overrightarrow{a}} \left(\int_{s_0}^{s} \left(\cos(\phi(t))\cos(\theta) - \sin(\phi(t))\sin(\theta)\right)dt, \int_{s_0}^{s} \left(\sin(\phi(t))\cos(\theta) + \cos(\phi(t))\sin(\theta)\right)dt\right) \\
&= T_{\overrightarrow{a}} \left(\cos(\theta)\int_{s_0}^{s} \cos(\phi(t))dt - \sin(\theta)\int_{s_0}^{s} \sin(\phi(t))dt, \cos(\theta)\int_{s_0}^{s} (\sin(\phi(t)dt + \sin(\theta)\int_{s_0}^{s} \cos(\phi(t))dt\right) \\
&= T_{\overrightarrow{a}} R_\theta \left(\int_{s_0}^{s} (\cos(\phi(t))dt, \int_{s_0}^{s} (\sin(\phi(t))dt\right) \\
&= T_{\overrightarrow{a}} R_\theta \\
&= M(\gamma(s))
\end{aligned}
$$

■