Finally, we make a comment about the lowest frequency we can fit to a set of data. If we had just six months of temperature readings from winter to summer it would not be clear if there was an upward trend in the observations or if winters are colder than summers. However, with one year's data it **would** become clear that winters are colder than summers. Thus if we are interested in variation at the low frequency of 1 cycle per year, then we must have at least one year's data. Thus the lower the frequency we are interested in, the longer the time period over which we need to take measurements, whereas the higher the frequency we are interested in, the more frequently must we take observations.

7.3 PERIODOGRAM ANALYSIS

Early attempts at discovering hidden periodicities in a given time series basically consisted of repeating the analysis of Section 7.2 at all the frequencies $2\pi/N, 4\pi/N, \ldots, \pi$. In view of (7.3)–(7.5) the different terms are orthogonal and we end up with the finite Fourier series representation of the $\{x_t\}$, namely

$$x_t = a_0 + \sum_{p=1}^{(N/2)-1} [a_p \cos(2\pi pt/N) + b_p \sin(2\pi pt/N)] + a_{N/2} \cos \pi t \qquad t = 1, 2, \ldots, N \quad (7.9)$$

where the coefficients $\{a_p, b_p\}$ are of the same form as equations (7.6) and (7.7), namely

$$\begin{aligned} a_0 &= \bar{x} \\ a_{N/2} &= \Sigma(-1)^t x_t/N \\ \left.\begin{aligned} a_p &= 2[\Sigma x_t \cos(2\pi pt/N)]/N \\ b_p &= 2[\Sigma x_t \sin(2\pi pt/N)]/N \end{aligned}\right\} & \quad p = 1, \ldots, (N/2)-1 \end{aligned} \qquad (7.10)$$

An analysis along these lines is sometimes called a Fourier analysis or a **harmonic** analysis. The Fourier series representation (7.9) has N parameters to describe N observations and so can be made to fit the data exactly (just as a polynomial of degree $N-1$ involving N parameters can be found which goes exactly through N observations in polynomial regression). This explains why there is no error term in (7.9) in contrast to (7.1). Also note that there is no term in $\sin \pi t$ in (7.9) as $\sin \pi t$ is zero for all integer t.

It is worth stressing that the Fourier series coefficients (7.10) at a given frequency ω are exactly the same as the least squares estimates for model (7.1).

The overall effect of the Fourier analysis of the data is to partition the variability of the series into components at frequencies $2\pi/N, 4\pi/N, \ldots, \pi$. The component at frequency $\omega_p = 2\pi p/N$ is often called the pth harmonic. For $p \neq N/2$, it is often useful to write the pth harmonic in the equivalent form

$$a_p \cos \omega_p t + b_p \sin \omega_p t = R_p \cos(\omega_p t + \phi_p) \tag{7.11}$$

where

$$R_p = \sqrt{(a_p^2 + b_p^2)} \tag{7.12}$$

is the **amplitude** of the pth harmonic, and

$$\phi_p = \tan^{-1}(-b_p/a_p) \tag{7.13}$$

is the **phase** of the pth harmonic.

We have already noted in Section 7.2 that, for $p \neq N/2$, the contribution of the pth harmonic to the total sum of squares is given by $N(a_p^2 + b_p^2)/2$. Using (7.12), this is equal to $NR_p^2/2$. Extending this result using (7.2)–(7.5) and (7.9), we have, after some algebra, that (Exercise 7.3)

$$\sum_{t=1}^{N} (x_t - \bar{x})^2 = N \sum_{p=1}^{(N/2)-1} R_p^2/2 + Na_{N/2}^2 \tag{7.14}$$

Dividing through by N we have

$$\Sigma(x_t - \bar{x})^2/N = \sum_{p=1}^{(N/2)-1} R_p^2/2 + a_{N/2}^2 \tag{7.15}$$

which is known as Parseval's theorem. The left-hand side of (7.15) is effectively the variance of the observations, although the divisor is N rather than the more usual $(N-1)$. Thus $R_p^2/2$ is the contribution of the pth harmonic to the variance, and (7.15) shows how the total variance is partitioned.

If we plot $R_p^2/2$ against $\omega_p = 2\pi p/N$ we obtain a line spectrum. A different type of line spectrum occurs in the physical sciences when light from molecules in a gas discharge tube is viewed through a spectroscope. The light has energy at discrete frequencies and this energy can be seen as bright lines. But most time series have continuous spectra, and then it is inappropriate to plot a line spectrum. If we regard $R_p^2/2$ as the contribution to variance in the range $\omega_p \pm \pi/N$, we can plot a histogram whose height in the range $\omega_p \pm \pi/N$ is such that

$$\begin{aligned} R_p^2/2 &= \text{area of histogram rectangle} \\ &= \text{height of histogram} \times 2\pi/N \end{aligned}$$

Thus the height of the histogram is given by

$$I(\omega_p) = NR_p^2/4\pi \tag{7.16}$$

As usual, (7.16) does not apply for $p = N/2$; we may regard $a_{N/2}^2$ as the contribution to variance in the range $[\pi(N-1)/N, \pi]$ so that

$$I(\pi) = Na_{N/2}^2/\pi$$

The plot of $I(\omega)$ against ω is usually called the **periodogram** even though $I(\omega)$ is a function of frequency rather than period. Other authors define the periodogram in a slightly different way, as some other multiple of R_p^2. Hannan (1970, equation (3.8)) defines the periodogram in terms of complex numbers as

$$\frac{1}{2\pi N}\left|\sum_{t=1}^{N} x_t e^{it\omega}\right|^2$$

which is $\frac{1}{2}\times$ expression (7.16). Anderson (1971, Section 4.3.2) describes the graph of R_p^2 against the **period** N/p as the periodogram, and suggests the term **spectrogram** to describe the graph of R_p^2 against frequency. An advantage of definition (7.16) is that the total area under the periodogram is equal to the variance of the time series. Expression (7.16) may readily be calculated directly from the data by

$$I(\omega_p)=[(\Sigma x_t \cos 2\pi pt/N)^2+(\Sigma x_t \sin 2\pi pt/N)^2]/N\pi \qquad (7.17)$$

Equation (7.17) also applies for $p=N/2$. Jenkins and Watts (1968) define a similar expression in terms of the variable $f=\omega/2\pi$, but call it the 'sample spectrum'.

The periodogram appears to be a natural way of estimating the power spectral density function, but we shall see that for a process with a **continuous** spectrum it provides a poor estimate and needs to be modified.

7.3.1 The relationship between the periodogram and the autocovariance function

The periodogram ordinate $I(\omega)$ and the autocovariance coefficient c_k are both quadratic forms of the data $\{x_t\}$. It is of interest to see how they are related. We will show that the periodogram is the finite Fourier transform of $\{c_k\}$.

Using (7.2) we may rewrite (7.17) for $p\neq N/2$ as

$$\begin{aligned} I(\omega_p)&=\{[\Sigma(x_t-\bar{x})\cos\omega_p t]^2+[\Sigma(x_t-\bar{x})\sin\omega_p t]^2\}/N\pi\\ &=\sum_{s,t=1}^{N}(x_t-\bar{x})(x_s-\bar{x})(\cos\omega_p t\cos\omega_p s+\sin\omega_p t\sin\omega_p s)/N\pi\end{aligned}$$

But (see (4.1))

$$\sum_{t=1}^{N-k}(x_t-\bar{x})(x_{t+k}-\bar{x})/N=c_k$$

and

$$\begin{aligned} &\cos\omega_p t\cos\omega_p(t+k)+\sin\omega_p t\sin\omega_p(t+k)\\ &=\cos\omega_p(t+k-t)\\ &=\cos\omega_p k\end{aligned}$$

so that

$$I(\omega_p) = (c_0 + 2 \sum_{k=1}^{N-1} c_k \cos \omega_p k)/\pi \tag{7.18}$$

$$= \sum_{k=-(N-1)}^{N-1} c_k \, e^{-i\omega_p k}/\pi \tag{7.19}$$

We recognize (7.19) as a finite Fourier transform (assuming that $c_k = 0$ for $|k| \geq N$).

7.3.2 Properties of the periodogram

When the periodogram is expresssed in the form (7.18), it appears to be the 'obvious' estimate of the power spectrum

$$f(\omega) = (\gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos \omega k)/\pi$$

simply replacing γ_k by its estimate c_k for values of k up to $(N-1)$, and putting subsequent estimates of γ_k equal to zero. But although we find

$$\underset{N\to\infty}{E} [I(\omega)] \to f(\omega) \tag{7.20}$$

so that the periodogram is asymptotically unbiased, we will see that the variance of $I(\omega)$ does not decrease as N increases. Thus $I(\omega)$ is **not a consistent** estimator for $f(\omega)$. An example of a periodogram is given in Figure 7.5(c), and it can be seen that the graph fluctuates wildly. The lack of consistency is perhaps not too surprising when one realizes that the Fourier series representation (7.9) requires one to evaluate N parameters from N observations however long the series. Thus in Section 7.4 we will consider alternative ways of estimating a power spectrum which are essentially ways of **smoothing** the periodogram.

We complete this section by proving that $I(\omega)$ is not a consistent estimator for $f(\omega)$ in the case where $(x_1, \ldots, x_N)$ are taken from a discrete purely random process, where the observations are independent $N(\mu, \sigma^2)$ variates. This result can be extended to other stationary processes with continuous spectra, but this will not be demonstrated here.

From (7.10) we see that a_p and b_p are linear combinations of normally distributed random variables and so will themselves be normally distributed. Using (7.2)–(7.4), it can be shown (Exercise 7.4) that a_p and b_p each have mean zero and variance $2\sigma^2/N$ for $p \neq N/2$. Furthermore we have

$$\text{Cov}(a_p, b_p) = 4 \, \text{Cov}[(\Sigma x_t \cos \omega_p t), (\Sigma x_t \sin \omega_p t)]/N^2$$

$$= 4\sigma^2(\Sigma \cos \omega_p t \sin \omega_p t)/N^2$$

since the $\{x_t\}$ are independent. Thus, using (7.5), we see that a_p and b_p are uncorrelated. Since (a_p, b_p) are bivariate normal, zero correlation implies that a_p and b_p are independent. Now a result from distribution theory says that if Y_1, Y_2 are independent N(0, 1) variables, then $(Y_1^2 + Y_2^2)$ has a χ^2 distribution with two degrees of freedom, which is written χ_2^2. Thus

$$\frac{N(a_p^2+b_p^2)}{2\sigma^2}=\frac{I(\omega_p)2\pi}{\sigma^2}$$

is χ_2^2. Now the variance of a χ^2 distribution with ν degrees of freedom is 2ν, so that

$$\text{Var}[I(\omega_p)2\pi/\sigma^2]=4$$

and

$$\text{Var}[I(\omega_p)]=\sigma^4/\pi^2$$

As this variance is a constant, it does **not** tend to zero as $N\to\infty$, and hence $I(\omega_p)$ is not a consistent estimator for $f(\omega_p)$. Furthermore it can be shown that neighbouring periodogram ordinates are asymptotically independent, which further explains the very irregular form of an observed periodogram. Thus the periodogram needs to be modified in order to obtain a good estimate of a continuous spectrum.

7.4 SPECTRAL ANALYSIS: SOME CONSISTENT ESTIMATION PROCEDURES

This section describes several alternative procedures for carrying out a spectral analysis. The different methods will be compared in Section 7.6. Each method provides a **consistent** estimate of the (power) spectral density function, in contrast to the periodogram. But although the periodogram is itself an inconsistent estimate, we shall see that the procedures described in this section are essentially based on the periodogram by using some sort of smoothing procedure.

Throughout the section we will assume that any obvious trend or seasonal variation has been removed from the data. If this is not done, the results of the spectral analysis are likely to be dominated by these effects, making any other effects difficult or impossible to see. Trend will produce a peak at zero frequency, while seasonal variation produces peaks at the seasonal frequency and at integer multiples of the seasonal frequency. These integer multiples of the fundamental frequency are called **harmonics** (see Section 7.8). For a non-stationary series, the estimated spectrum can depend rather crucially on the method chosen to remove trend and seasonality.

The methods described in this chapter are essentially non-parametric in that