

Numerical Methods

Using MATLAB

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Interpolation and Polynomial Approximation

The computational procedures used in computer software for the evaluation of a library function, such as $\sin(x)$, $\cos(x)$, or e^x , involve polynomial approximation. The state-of-the-art methods use rational functions (which are the quotients of polynomials). However, the theory of polynomial approximation is suitable for a first course in numerical analysis, and we will mainly consider them in this chapter. Suppose that the function $f(x) = e^x$ is to be approximated by a polynomial of degree $n = 2$ over the interval $[-1, 1]$. The Taylor polynomial is shown in Figure 4.1(a) and can be con-

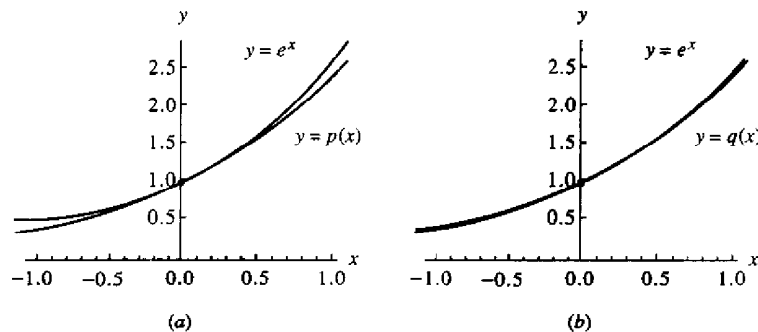


Figure 4.1 (a) The Taylor polynomial $p(x) = 1.000000 + 1.000000x + 0.500000x^2$ which approximates $f(x) = e^x$ over $[-1, 1]$. (b) The Chebyshev approximation $q(x) = 1.000000 + 1.129772x + 0.532042x^2$ for $f(x) = e^x$ over $[-1, 1]$.

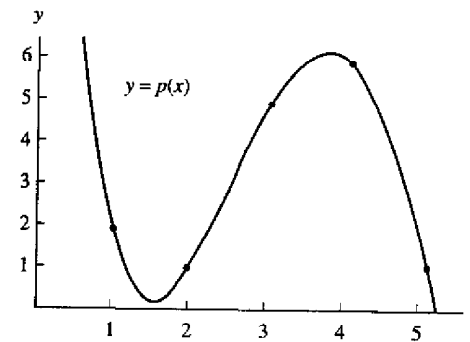


Figure 4.2 The graph of the collocation polynomial that passes through $(1, 2)$, $(2, 1)$, $(3, 5)$, $(4, 6)$, and $(5, 1)$.

trasted with the Chebyshev approximation in Figure 4.1(b). The maximum error for the Taylor approximation is 0.218282, whereas the maximum error for the Chebyshev polynomial is 0.056468. In this chapter we develop the basic theory needed to investigate these matters.

An associated problem involves the construction of the collocation polynomial. Given $n + 1$ points in the plane (no two of which are aligned vertically), the collocation polynomial is the unique polynomial of degree $\leq n$ that passes through the points. In cases where data are known to a high degree of precision, the collocation polynomial is sometimes used to find a polynomial that passes through the given data points. A variety of methods can be used to construct the collocation polynomial: solving a linear system for its coefficients, the use of Lagrange coefficient polynomials, and the construction of a divided differences table and the coefficients of the Newton polynomial. All three techniques are important for a practitioner of numerical analysis to know. For example, the collocation polynomial of degree $n = 4$ that passes through the five points $(1, 2)$, $(2, 1)$, $(3, 5)$, $(4, 6)$, and $(5, 1)$ is

$$P(x) = \frac{5x^4 - 82x^3 + 427x^2 - 806x + 504}{24},$$

and a graph showing both the points and the polynomial is given in Figure 4.2.

4.1 Taylor Series and Calculation of Functions

Limit processes are the basis of calculus. For example, the derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

is the limit of the difference quotient where both the numerator and the denominator go to zero. A Taylor series illustrates another type of limit process. In this case an

Table 4.1 Taylor Series Expansions for Some Common Functions

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	for all x
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	for all x
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	for all x
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 \leq x \leq 1$
$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$-1 \leq x \leq 1$
$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$	for $ x < 1$

infinite number of terms is added together by taking the limit of certain partial sums. An important application is their use to represent the elementary functions: $\sin(x)$, $\cos(x)$, e^x , $\ln(x)$, etc. Table 4.1 gives several of the common Taylor series expansions. The partial sums can be accumulated until an approximation to the function is obtained that has the accuracy specified. Series solutions are used in the areas of engineering and physics.

We want to learn how a finite sum can be used to obtain a good approximation to an infinite sum. For illustration we shall use the exponential series in Table 4.1 to compute the number $e = e^1$, which is the base of the natural logarithm and exponential functions. Here we choose $x = 1$ and use the series

$$e^1 = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \cdots + \frac{1^k}{k!} + \cdots$$

The definition for the sum of an infinite series in Section 1.1 requires that the partial sums S_N tend to a limit. The values of these sums are given in Table 4.2.

A natural way to think about the power series representation of a function is to view the expansion as the limiting case of polynomials of increasing degree. If enough terms are added, then an accurate approximation will be obtained. This needs to be made precise. What degree should be chosen for the polynomial, and how do we calculate the coefficients for the powers of x in the polynomial? Theorem 4.1 answers these questions.

Table 4.2 Partial Sums S_n Used to Determine e

n	$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$
0	1.0
1	2.0
2	2.5
3	2.666666666666...
4	2.7083333333333...
5	2.7166666666666...
6	2.7180555555555...
7	2.718253968254...
8	2.718278769841...
9	2.718281525573...
10	2.718281801146...
11	2.718281826199...
12	2.718281828286...
13	2.718281828447...
14	2.718281828458...
15	2.718281828459...

Theorem 4.1 (Taylor Polynomial Approximation). Assume that $f \in C^{N+1}[a, b]$ and $x_0 \in [a, b]$ is a fixed value. If $x \in [a, b]$, then

$$(1) \quad f(x) = P_N(x) + E_N(x),$$

where $P_N(x)$ is a polynomial that can be used to approximate $f(x)$:

$$(2) \quad f(x) \approx P_N(x) = \sum_{k=0}^N \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

The error term $E_N(x)$ has the form

$$(3) \quad E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} (x - x_0)^{N+1}$$

for some value $c = c(x)$ that lies between x and x_0 .

Proof. The proof is left as an exercise. •

Relation (2) indicates how the coefficients of the Taylor polynomial are calculated. Although the error term (3) involves a similar expression, notice that $f^{(N+1)}(c)$ is to be evaluated at an undetermined number c that depends on the value of x . For this reason we do not try to evaluate $E_N(x)$: it is used to determine a bound for the accuracy of the approximation.

Example 4.1. Show why 15 terms are all that are needed to obtain the 13-digit approximation $e = 2.718281828459$ in Table 4.2.

Expand $f(x) = e^x$ in a Taylor polynomial of degree 15 using the fixed value $x_0 = 0$ and involving the powers $(x - 0)^k = x^k$. The derivatives required are $f'(x) = f''(x) = \dots = f^{(16)}(x) = e^x$. The first 15 derivatives are used to calculate the coefficients $a_k = e^0/k!$ and are used to write

$$(4) \quad P_{15}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{15}}{15!}.$$

Setting $x = 1$ in (4) gives the partial sum $S_{15} = P_{15}(1)$. The remainder term is needed to show the accuracy of the approximation:

$$(5) \quad E_{15}(x) = \frac{f^{(16)}(c)x^{16}}{16!}.$$

Since we chose $x_0 = 0$ and $x = 1$, the value c lies between them (i.e., $0 < c < 1$), which implies that $e^c < e^1$. Notice that the partial sums in Table 4.2 are bounded above by 3. Combining these two inequalities yields $e^c < 3$, which is used in the following calculation

$$|E_{15}(1)| = \frac{|f^{(16)}(c)|}{16!} \leq \frac{e^c}{16!} < \frac{3}{16!} < 1.433844 \times 10^{-13}.$$

Therefore, all the digits in the approximation $e \approx 2.718281828459$ are correct, because the actual error (whatever it is) must be less than 2 in the thirteenth decimal place. ■

Instead of giving a rigorous proof of Theorem 4.1, we shall discuss some of the features of the approximation; the reader can look in any standard reference text on calculus for more details. For illustration, we again use the function $f(x) = e^x$ and the value $x_0 = 0$. From elementary calculus we know that the slope of the curve $y = e^x$ at the point (x, e^x) is $f'(x) = e^x$. Hence the slope at the point $(0, 1)$ is $f'(0) = 1$. Therefore, the tangent line to the curve at the point $(0, 1)$ is $y = 1 + x$. This is the same formula that would be obtained if we used $N = 1$ in Theorem 4.1, that is, $P_1(x) = f(0) + f'(0)x/1! = 1 + x$. Therefore, $P_1(x)$ is the equation of the tangent line to the curve. The graphs are shown in Figure 4.3.

Observe that the approximation $e^x \approx 1 + x$ is good near the center $x_0 = 0$ and that the distance between the curves grows as x moves away from 0. Notice that the slopes of the curves agree at $(0, 1)$. In calculus we learned that the second derivative indicates whether a curve is concave up or down. The study of curvature¹ shows that if two curves $y = f(x)$ and $y = g(x)$ have the property that $f(x_0) = g(x_0)$, $f'(x_0) = g'(x_0)$ and $f''(x_0) = g''(x_0)$ then they have the same curvature at x_0 . This property would be desirable for a polynomial function that approximates $f(x)$. Corollary 4.1 shows that the Taylor polynomial has this property for $N \geq 2$.

¹The curvature K of a graph $y = f(x)$ at (x_0, y_0) is defined by $K = |f''(x_0)|/(1 + [f'(x_0)]^2)^{3/2}$.

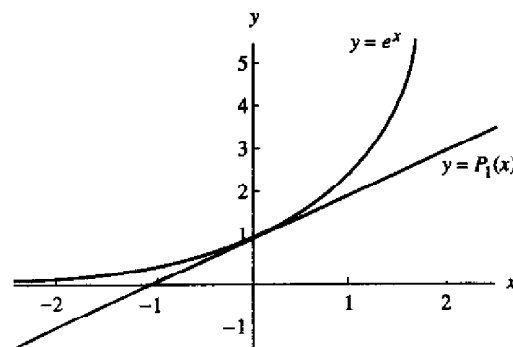


Figure 4.3 The graphs of $y = e^x$ and $y = P_1(x) = 1 + x$.

Corollary 4.1. If $P_N(x)$ is the Taylor polynomial of degree N given in Theorem 4.1, then

$$(6) \quad P_N^{(k)}(x_0) = f^{(k)}(x_0) \quad \text{for } k = 0, 1, \dots, N.$$

Proof. Set $x = x_0$ in equations (2) and (3), and the result is $P_N(x_0) = f(x_0)$. Thus statement (6) is true for $k = 0$. Now differentiate the right-hand side of (2) and get

$$(7) \quad P_N'(x) = \sum_{k=1}^N \frac{f^{(k)}(x_0)}{(k-1)!} (x - x_0)^{k-1} = \sum_{k=0}^{N-1} \frac{f^{(k+1)}(x_0)}{k!} (x - x_0)^k.$$

Set $x = x_0$ in (7) to obtain $P_N'(x_0) = f'(x_0)$. Thus statement (6) is true for $k = 1$. Successive differentiations of (7) will establish the other identities in (6). The details are left as an exercise. ■

Applying Corollary 4.1, we see that $y = P_2(x)$ has the properties $f(x_0) = P_2(x_0)$, $f'(x_0) = P_2'(x_0)$, and $f''(x_0) = P_2''(x_0)$; hence the graphs have the same curvature at x_0 . For example, consider $f(x) = e^x$ and $P_2(x) = 1 + x + x^2/2$. The graphs are shown in Figure 4.4 and it is seen that they curve up in the same fashion at $(0, 1)$.

In the theory of approximation, one seeks to find an accurate polynomial approximation to the analytic function² $f(x)$ over $[a, b]$. This is one technique used in developing computer software. The accuracy of a Taylor polynomial is increased when we choose N large. The accuracy of any given polynomial will generally decrease as the value of x moves away from the center x_0 . Hence we must choose N large enough and restrict the maximum value of $|x - x_0|$ so that the error does not exceed a specified bound. If we choose the interval width to be $2R$ and x_0 in the center (i.e., $|x - x_0| < R$),

²The function $f(x)$ is analytic at x_0 if it has continuous derivatives of all orders and can be represented as a Taylor series in an interval about x_0 .

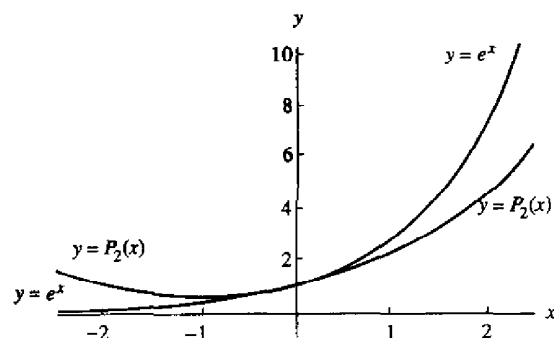


Figure 4.4 The graphs of $y = e^x$ and $y = P_2(x) \approx 1 + x + x^2/2$.

Table 4.3 Values for the Error Bound $|\text{error}| < e^R R^{N+1}/(N+1)!$ Using the Approximation $e^x \approx P_N(x)$ for $|x| \leq R$

	$R = 2.0,$ $ x \leq 2.0$	$R = 1.5,$ $ x \leq 1.5$	$R = 1.0,$ $ x \leq 1.0$	$R = 0.5,$ $ x \leq 0.5$
$e^x \approx P_5(x)$	0.65680499	0.07090172	0.00377539	0.00003578
$e^x \approx P_6(x)$	0.18765857	0.01519323	0.00053934	0.00000256
$e^x \approx P_7(x)$	0.04691464	0.00284873	0.00006742	0.00000016
$e^x \approx P_8(x)$	0.01042548	0.00047479	0.00000749	0.00000001

the absolute value of the error satisfies the relation

$$(8) \quad |\text{error}| = |E_N(x)| \leq \frac{MR^{N+1}}{(N+1)!},$$

where $M \leq \max\{|f^{(N+1)}(z)| : x_0 - R \leq z \leq x_0 + R\}$. If N is fixed and the derivatives are uniformly bounded, the error bound in (8) is proportional to $R^{N+1}/(N+1)!$ and decreases if R goes to zero as N gets large. Table 4.3 shows how the choices of these two parameters affect the accuracy of the approximation $e^x \approx P_N(x)$ over the interval $|x| \leq R$. The error is smallest when N is largest and R smallest. Graphs for P_2 , P_3 , and P_4 are given in Figure 4.5.

Example 4.2. Establish the error bounds for the approximation $e^x \approx P_8(x)$ on each of the intervals $|x| \leq 1.0$ and $|x| \leq 0.5$.

If $|x| \leq 1.0$, then letting $R = 1.0$ and $|f^{(9)}(c)| = |e^c| \leq e^{1.0} = M$ in (8) implies that

$$|\text{error}| = |E_8(x)| \leq \frac{e^{1.0}(1.0)^9}{9!} \approx 0.00000749.$$

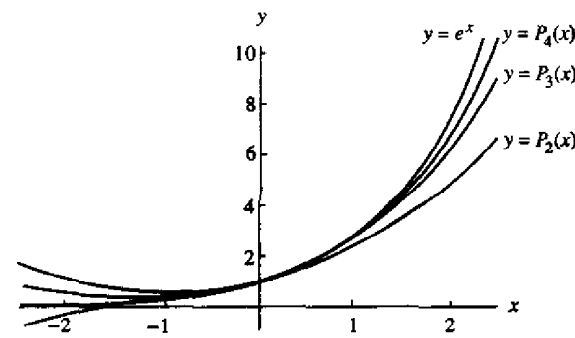


Figure 4.5 The graphs of $y = e^x$, $y = P_2(x)$, $y = P_3(x)$, and $y = P_4(x)$.

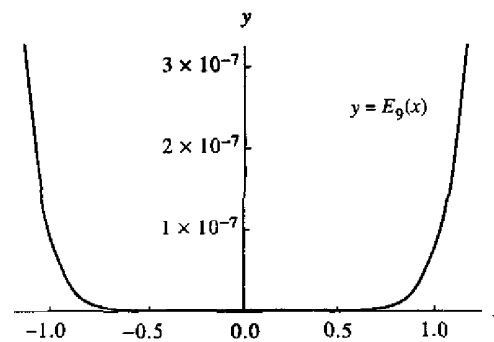


Figure 4.6 The graph of the error $y = E_9(x) = e^x - P_9(x)$.

If $|x| \leq 0.5$, then letting $R = 0.5$ and $|f^{(9)}(c)| = |e^c| \leq e^{0.5} = M$ in (8) implies that

$$|\text{error}| = |E_8(x)| \leq \frac{e^{0.5}(0.5)^9}{9!} \approx 0.00000001. \quad \blacksquare$$

Example 4.3. If $f(x) = e^x$, show that $N = 9$ is the smallest integer, so that the $|\text{error}| = |E_N(x)| \leq 0.0000005$ for x in $[-1, 1]$. Hence $P_9(x)$ can be used to compute approximate values of e^x that will be accurate in the sixth decimal place.

We need to find the smallest integer N so that

$$|\text{error}| = |E_N(x)| \leq \frac{e^1(1)^{N+1}}{(N+1)!} < 0.0000005.$$

In Example 4.2 we saw that $N = 8$ was too small, so we try $N = 9$ and discover that $|E_N(x)| \leq e^1(1)^{9+1}/(9+1)! \leq 0.000000749$. This value is slightly larger than

desired; hence we would be likely to choose $N = 10$. But we used $e^c \leq e^1$ as a crude estimate in finding the error bound. Hence 0.000000749 is a little larger than the actual error. Figure 4.6 shows a graph of $E_9(x) = e^x - P_9(x)$. Notice that the maximum vertical range is about 3×10^{-7} and occurs at the right end point $(1, E_9(1))$. Indeed, the maximum error on the interval is $E_9(1) = 2.718281828 - 2.718281526 \approx 3.024 \times 10^{-7}$. Therefore, $N = 9$ is justified. ■

Methods for Evaluating a Polynomial

There are several mathematically equivalent ways to evaluate a polynomial. Consider, for example the function

$$(9) \quad f(x) = (x - 1)^8.$$

The evaluation of f will require the use of an exponential function. Or the binomial formula can be used to expand $f(x)$ in powers of x :

$$(10) \quad f(x) = \sum_{k=0}^8 \binom{8}{k} x^{8-k} (-1)^k \\ = x^8 - 8x^7 + 28x^6 - 56x^5 + 70x^4 - 56x^3 + 28x^2 - 8x + 1.$$

Horner's method (see Section 1.1), which is also called **nested multiplication**, can now be used to evaluate the polynomial in (10). When applied to formula (10), nested multiplication permits us to write

$$(11) \quad f(x) = ((((((x - 8)x + 28)x - 56)x + 70)x - 56)x + 28)x - 8)x + 1.$$

To evaluate $f(x)$ now requires seven multiplications and eight additions or subtractions. The necessity of using an exponential function to evaluate the polynomial has now been eliminated.

We end this section with the theorem that relates the Taylor series in Table 4.1 and the Taylor polynomials of Theorem 4.1.

Theorem 4.2 (Taylor Series). Assume that $f(x)$ is analytic and has continuous derivatives of all order $N = 1, 2, \dots$, on an interval (a, b) containing x_0 . Suppose that the Taylor polynomials (2) tend to a limit

$$(12) \quad S(x) = \lim_{N \rightarrow \infty} P_N(x) = \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k,$$

then $f(x)$ has the Taylor series expansion

$$(13) \quad f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

Proof. This follows directly from the definition of convergence of series in Section 1.1. The limit condition is often stated by saying that the error term must go to zero as N goes to infinity. Therefore, a necessary and sufficient condition for (13) to hold is that

$$(14) \quad \lim_{N \rightarrow \infty} E_N(x) = \lim_{N \rightarrow \infty} \frac{f^{(N+1)}(c)(x - x_0)^{N+1}}{(N+1)!} = 0,$$

where c depends on N and x . ■

Exercises for Taylor Series and Calculation of Functions

1. Let $f(x) = \sin(x)$ and apply Theorem 4.1.

(a) Use $x_0 = 0$ and find $P_5(x)$, $P_7(x)$, and $P_9(x)$.

(b) Show that if $|x| \leq 1$ then the approximation

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

has the error bound $|E_9(x)| < 1/10! \leq 2.75574 \times 10^{-7}$.

(c) Use $x_0 = \pi/4$ and find $P_5(x)$, which involves powers of $(x - \pi/4)$.

2. Let $f(x) = \cos(x)$ and apply Theorem 4.1.

(a) Use $x_0 = 0$ and find $P_4(x)$, $P_6(x)$, and $P_8(x)$.

(b) Show that if $|x| \leq 1$ then the approximation

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

has the error bound $|E_8(x)| < 1/9! \leq 2.75574 \times 10^{-6}$.

(c) Use $x_0 = \pi/4$ and find $P_4(x)$, which involves powers of $(x - \pi/4)$.

3. Does $f(x) = x^{1/2}$ have a Taylor series expansion about $x_0 = 0$? Justify your answer. Does the function $f(x) = x^{1/2}$ have a Taylor series expansion about $x_0 = 1$? Justify your answer.

4. (a) Find a Taylor polynomial of degree $N = 5$ for $f(x) = 1/(1+x)$ expanded about $x_0 = 0$.

(b) Find the error term $E_5(x)$ for the polynomial in part (a).

5. Find the Taylor polynomial of degree $N = 3$ for $f(x) = e^{-x^2/2}$ expanded about $x_0 = 0$.

6. Find the Taylor polynomial of degree $N = 3$, $P_3(x)$, for $f(x) = x^3 - 2x^2 + 2x$ expanded about $x_0 = 1$. Show that $f(x) = P_3(x)$.

7. (a) Find the Taylor polynomial of degree $N = 5$ for $f(x) = x^{1/2}$ expanded about $x_0 = 4$.
 (b) Find the Taylor polynomial of degree $N = 5$ for $f(x) = x^{1/2}$ expanded about $x_0 = 9$.
 (c) Determine which of the polynomials in parts (a) and (b) best approximates $(6.5)^{1/2}$.
8. Use $f(x) = (2+x)^{1/2}$ and apply Theorem 4.1.
 (a) Find the Taylor polynomial $P_3(x)$ expanded about $x_0 = 2$.
 (b) Use $P_3(x)$ to find an approximation to $3^{1/2}$.
 (c) Find the maximum value of $|f^{(4)}(c)|$ on the interval $1 \leq c \leq 3$ and find a bound for $|E_3(x)|$.
9. Determine the degree of the Taylor polynomial $P_N(x)$ expanded about $x_0 = 0$ that should be used to approximate $e^{0.1}$ so that the error is less than 10^{-6} .
10. Determine the degree of the Taylor polynomial $P_N(x)$ expanded about $x_0 = \pi$ that should be used to approximate $\cos(33\pi/32)$ so that the error is less than 10^{-6} .
11. (a) Find the Taylor polynomial of degree $N = 4$ for $F(x) = \int_{-1}^x \cos(t^2) dt$ expanded about $x_0 = 0$.
 (b) Use the Taylor polynomial to approximate $F(0.1)$.
 (c) Find a bound on the error to the approximation in part (b).
12. (a) Use the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots \quad \text{for } |x| < 1,$$

and integrate both sides term by term to obtain

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad \text{for } |x| < 1.$$

- (b) Use $\pi/6 = \arctan(3^{-1/2})$ and the series in part (a) to show that

$$\pi = 3^{1/2} \times 2 \left(1 - \frac{3^{-1}}{3} + \frac{3^{-2}}{5} - \frac{3^{-3}}{7} + \frac{3^{-4}}{9} - \dots \right).$$

- (c) Use the series in part (b) to compute π accurate to eight digits.
 Fact. $\pi \approx 3.141592653589793284 \dots$
13. Use $f(x) = \ln(1+x)$ and $x_0 = 0$, and apply Theorem 4.1.
 (a) Show that $f^{(k)}(x) = (-1)^{k-1}((k-1)!)/(1+x)^k$.
 (b) Show that the Taylor polynomial of degree N is

$$P_N(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{N-1}x^N}{N}.$$

- (c) Show that the error term for $P_N(x)$ is

$$E_N(x) = \frac{(-1)^N x^{N+1}}{(N+1)(1+c)^{N+1}}.$$

- (d) Evaluate $P_3(0.5)$, $P_6(0.5)$, and $P_9(0.5)$. Compare with $\ln(1.5)$.
 (e) Show that if $0.0 \leq x \leq 0.5$ then the approximation

$$\ln(x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9}$$

has the error bound $|E_9| \leq 0.00009765 \dots$

14. **Binomial series.** Let $f(x) = (1+x)^p$ and $x_0 = 0$.

- (a) Show that $f^{(k)}(x) = p(p-1) \cdots (p-k+1)(1+x)^{p-k}$.
 (b) Show that the Taylor polynomial of degree N is

$$P_N(x) = 1 + px + \frac{p(p-1)x^2}{2!} + \dots + \frac{p(p-1) \cdots (p-N+1)x^N}{N!}.$$

- (c) Show that

$$E_N(x) = p(p-1) \cdots (p-N)x^{N+1}/((1+c)^{N+1-p}(N+1)!).$$

- (d) Set $p = 1/2$ and compute $P_2(0.5)$, $P_4(0.5)$, and $P_6(0.5)$. Compare with $(1.5)^{1/2}$.
 (e) Show that if $0.0 \leq x \leq 0.5$ then the approximation

$$(1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256}$$

has the error bound $|E_5| \leq (0.5)^6(21/1024) = 0.0003204 \dots$

- (f) Show that if $p = N$ is a positive integer, then

$$P_N(x) = 1 + Nx + \frac{N(N-1)x^2}{2!} + \dots + Nx^{N-1} + x^N.$$

Notice that this is the familiar binomial expansion.

15. Find c such that $|E_4| < 10^{-6}$ whenever $|x - x_0| < c$.
 (a) Let $f(x) = \cos(x)$ and $x_0 = 0$.
 (b) Let $f(x) = \sin(x)$ and $x_0 = \pi/2$.
 (c) Let $f(x) = e^x$ and $x_0 = 0$.
16. (a) Suppose that $y = f(x)$ is an even function (i.e., $f(-x) = f(x)$ for all x in the domain of f). What can be said about $P_N(x)$?
 (b) Suppose that $y = f(x)$ is an odd function (i.e., $f(-x) = -f(x)$ for all x in the domain of f). What can be said about $P_N(x)$?

17. Let $y = f(x)$ be a polynomial of degree N . If $f(x_0) > 0$ and $f'(x_0), \dots, f^{(N)}(x_0) \geq 0$, show that all the real roots of f are less than x_0 . *Hint.* Expand f in a Taylor polynomial of degree N about x_0 .
18. Let $f(x) = e^x$. Use Theorem 4.1 to find $P_N(x)$, for $N = 1, 2, 3, \dots$, expanded about $x_0 = 0$. Show that every real root of $P_N(x)$ has multiplicity less than or equal to one. *Note.* If p is a root of multiplicity M of the polynomial $P(x)$, then p is a root of multiplicity $M - 1$ of $P'(x)$.
19. Finish the proof of Corollary 4.1 by writing down the expression for $P_N^{(k)}(x)$ and showing that

$$P_N^{(k)}(x_0) = f^{(k)}(x_0) \quad \text{for } k = 2, 3, \dots, N.$$

Exercises 20 and 21 form a proof of Taylor's theorem.

20. Let $g(t)$ and its derivatives $g^{(k)}(t)$, for $k = 1, 2, \dots, N + 1$, be continuous on the interval (a, b) , which contains x_0 . Suppose that there exist two distinct points x and x_0 such that $g(x) = 0$, and $g(x_0) = g'(x_0) = \dots = g^{(N)}(x_0) = 0$. Prove that there exists a value c that lies between x_0 and x such that $g^{(N+1)}(c) = 0$.

Remark. Note that $g(t)$ is a function of t , and the values x and x_0 are to be treated as constants with respect to the variable t .

Hint. Use Rolle's theorem (Theorem 1.5, Section 1.1) on the interval with end points x_0 and x to find the number c_1 such that $g'(c_1) = 0$. Then use Rolle's theorem applied to the function $g'(t)$ on the interval with end points x_0 and c_1 to find the number c_2 such that $g''(c_2) = 0$. Inductively repeat the process until the number c_{N+1} is found such that $g^{(N+1)}(c_{N+1}) = 0$.

21. Use the result of Exercise 20 and the special function

$$g(t) = f(t) - P_N(t) - E_N(x) \frac{(t - x_0)^{N+1}}{(x - x_0)^{N+1}},$$

where $P_N(x)$ is the Taylor polynomial of degree N , to prove that the error term $E_N(x) = f(x) - P_N(x)$ has the form

$$E_N(x) = f^{(N+1)}(c) \frac{(x - x_0)^{N+1}}{(N + 1)!}.$$

Hint. Find $g^{(N+1)}(t)$ and evaluate it at $t = c$.

Algorithms and Programs

The matrix nature of MATLAB allows us to quickly evaluate functions at a large number of values. If $X = [-1 \ 0 \ 1]$, then $\sin(X)$ will produce $[\sin(-1) \ \sin(0) \ \sin(1)]$. Similarly, if $X = -1:0.1:1$, then $Y = \sin(X)$ will produce a matrix Y of the same dimension as X with the appropriate values of \sin . These two row matrices can be displayed in the

form of a table by defining the matrix $D = [X' \ Y']$ (*Note.* The matrices X and Y must be of the same length.)

1. (a) Use the plot command to plot $\sin(x)$, $P_5(x)$, $P_7(x)$, and $P_9(x)$ from Exercise 1 on the same graph using the interval $-1 \leq x \leq 1$.
(b) Create a table with columns that consist of $\sin(x)$, $P_5(x)$, $P_7(x)$, and $P_9(x)$ evaluated at 10 equally spaced values of x from the interval $[-1, 1]$.
2. (a) Use the plot command to plot $\cos(x)$, $P_4(x)$, $P_6(x)$, and $P_8(x)$ from Exercise 2 on the same graph using the interval $-1 \leq x \leq 1$.
(b) Create a table with columns that consist of $\cos(x)$, $P_4(x)$, $P_6(x)$, and $P_8(x)$ evaluated at 19 equally spaced values of x from the interval $[-1, 1]$.

4.2. Introduction to Interpolation

In Section 4.1 we saw how a Taylor polynomial can be used to approximate the function $f(x)$. The information needed to construct the Taylor polynomial is the value of f and its derivatives at x_0 . A shortcoming is that the higher-order derivatives must be known, and often they are either not available or they are hard to compute.

Suppose that the function $y = f(x)$ is known at the $N + 1$ points $(x_0, y_0), \dots, (x_N, y_N)$, where the values x_k are spread out over the interval $[a, b]$ and satisfy

$$a \leq x_0 < x_1 < \dots < x_N \leq b \quad \text{and} \quad y_k = f(x_k).$$

A polynomial $P(x)$ of degree N will be constructed that passes through these $N + 1$ points. In the construction, only the numerical values x_k and y_k are needed. Hence the higher-order derivatives are not necessary. The polynomial $P(x)$ can be used to approximate $f(x)$ over the entire interval $[a, b]$. However, if the error function $E(x) = f(x) - P(x)$ is required, then we will need to know $f^{(N+1)}(x)$ and a bound for its magnitude, that is

$$M = \max\{|f^{(N+1)}(x)| : a \leq x \leq b\}.$$

Situations in statistical and scientific analysis arise where the function $y = f(x)$ is available only at $N + 1$ tabulated points (x_k, y_k) , and a method is needed to approximate $f(x)$ at nontabulated abscissas. If there is a significant amount of error in the tabulated values, then the methods of curve fitting in Chapter 5 should be considered. On the other hand, if the points (x_k, y_k) are known to a high degree of accuracy, then the polynomial curve $y = P(x)$ that passes through them can be considered. When $x_0 < x < x_N$, the approximation $P(x)$ is called an *interpolated value*. If either $x < x_0$ or $x_N < x$, then $P(x)$ is called an *extrapolated value*. Polynomials are used to design software algorithms to approximate functions, for numerical differentiation, for numerical integration, and for making computer-drawn curves that must pass through specified points.

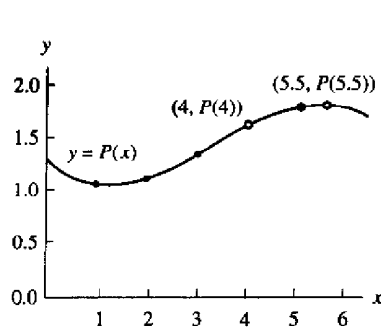


Figure 4.7 (a) The approximating polynomial $P(x)$ can be used for interpolation at the point $(4, P(4))$ and extrapolation at the point $(5.5, P(5.5))$.

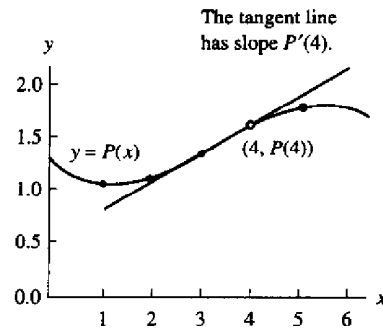


Figure 4.7 (b) The approximating polynomial $P(x)$ is differentiated and $P'(x)$ is used to find the slope at the interpolation point $(4, P(4))$.

Let us briefly mention how to evaluate the polynomial $P(x)$:

$$(1) \quad P(x) = a_N x^N + a_{N-1} x^{N-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Horner's method of synthetic division is an efficient way to evaluate $P(x)$. The derivative $P'(x)$ is

$$(2) \quad P'(x) = N a_N x^{N-1} + (N-1) a_{N-1} x^{N-2} + \cdots + 2 a_2 x + a_1$$

and the indefinite integral $I(x) = \int P(x) dx$, which satisfies $I'(x) = P(x)$, is

$$(3) \quad I(x) = \frac{a_N x^{N+1}}{N+1} + \frac{a_{N-1} x^N}{N} + \cdots + \frac{a_2 x^3}{3} + \frac{a_1 x^2}{2} + a_0 x + C,$$

where C is the constant of integration. **Algorithm 4.1** (end of Section 4.2) shows how to adapt Horner's method to $P'(x)$ and $I(x)$.

Example 4.4. The polynomial $P(x) = -0.02x^3 + 0.2x^2 - 0.4x + 1.28$ passes through the four points $(1, 1.06)$, $(2, 1.12)$, $(3, 1.34)$, and $(5, 1.78)$. Find (a) $P(4)$, (b) $P'(4)$.

(c) $\int_1^4 P(x) dx$, and (d) $P(5.5)$. Finally, (e) show how to find the coefficients of $P(x)$.

Use Algorithm 4.1(i)–(iii) (this is equivalent to the process in Table 1.2) with $x = 4$.

$$\begin{aligned} (a) \quad & b_3 = a_3 = -0.02 \\ & b_2 = a_2 + b_3 x = 0.2 + (-0.02)(4) = 0.12 \\ & b_1 = a_1 + b_2 x = -0.4 + (0.12)(4) = 0.08 \\ & b_0 = a_0 + b_1 x = 1.28 + (0.08)(4) = 1.60. \end{aligned}$$

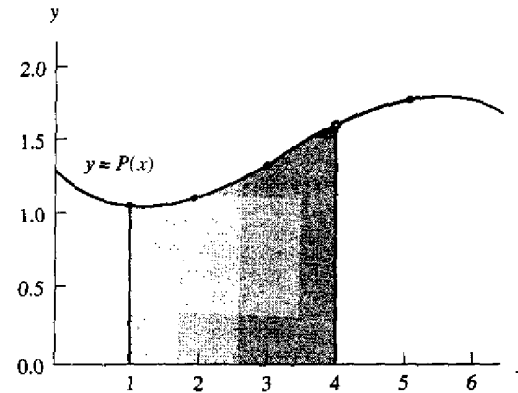


Figure 4.8 The approximating polynomial $P(x)$ is integrated and its antiderivative is used to find the area under the curve for $1 \leq x \leq 4$.

The interpolated value is $P(4) = 1.60$ (see Figure 4.7(a)).

$$\begin{aligned} (b) \quad & d_2 = 3a_3 = -0.06 \\ & d_1 = 2a_2 + d_2 x = 0.4 + (-0.06)(4) = 0.16 \\ & d_0 = a_1 + d_1 x = -0.4 + (0.16)(4) = 0.24. \end{aligned}$$

The numerical derivative is $P'(4) = 0.24$ (see Figure 4.7(b)).

$$\begin{aligned} (c) \quad & i_4 = \frac{a_3}{4} = -0.005 \\ & i_3 = \frac{a_2}{3} + i_4 x = 0.06666667 + (-0.005)(4) = 0.04666667 \\ & i_2 = \frac{a_1}{2} + i_3 x = -0.2 + (0.04666667)(4) = -0.01333333 \\ & i_1 = a_0 + i_2 x = 1.28 + (-0.01333333)(4) = 1.22666667 \\ & i_0 = 0 + i_1 x = 0 + (1.22666667)(4) = 4.90666667. \end{aligned}$$

Hence $I(4) = 4.90666667$. Similarly, $I(1) = 1.14166667$. Therefore, $\int_1^4 P(x) dx = I(4) - I(1) = 3.765$ (see Figure 4.8).

(d) Use Algorithm 4.1(i) with $x = 5.5$.

$$\begin{aligned} & b_3 = a_3 = -0.02 \\ & b_2 = a_2 + b_3 x = 0.2 + (-0.02)(5.5) = 0.09 \\ & b_1 = a_1 + b_2 x = -0.4 + (0.09)(5.5) = 0.095 \\ & b_0 = a_0 + b_1 x = 1.28 + (0.095)(5.5) = 1.8025. \end{aligned}$$

The extrapolated value is $P(5.5) = 1.8025$ (see Figure 4.7(a)).

Table 4.4 Values of the Taylor Polynomial $T(x)$ of Degree 5, and the Function $\ln(1+x)$ and the Error $\ln(1+x) - T(x)$ on $[0, 1]$

x	Taylor polynomial, $T(x)$	Function, $\ln(1+x)$	Error, $\ln(1+x) - T(x)$
0.0	0.00000000	0.00000000	0.00000000
0.2	0.18233067	0.18232156	-0.00000911
0.4	0.33698133	0.33647224	-0.00050909
0.6	0.47515200	0.47000363	-0.00514837
0.8	0.61380267	0.58778666	-0.02601601
1.0	0.78333333	0.69314718	-0.09018615

(e) The methods of Chapter 3 can be used to find the coefficients. Assume that $P(x) = A + Bx + Cx^2 + Dx^3$; then at each value $x = 1, 2, 3$, and 5 we get a linear equation involving A, B, C , and D .

$$\begin{aligned}
 \text{At } x = 1 : A + 1B + 1C + 1D &= 1.06 \\
 \text{At } x = 2 : A + 2B + 4C + 8D &= 1.12 \\
 \text{At } x = 3 : A + 3B + 9C + 27D &= 1.34 \\
 \text{At } x = 5 : A + 5B + 25C + 125D &= 1.78
 \end{aligned}
 \tag{4}$$

The solution to (4) is $A = 1.28$, $B = -0.4$, $C = 0.2$, and $D = -0.2$.

This method for finding the coefficients is mathematically sound, but sometimes the matrix is difficult to solve accurately. In this chapter we design algorithms specifically for polynomials.

Let us return to the topic of using a polynomial to calculate approximations to a known function. In Section 4.1 we saw that the fifth-degree Taylor polynomial for $f(x) = \ln(1+x)$ is

$$T(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.
 \tag{5}$$

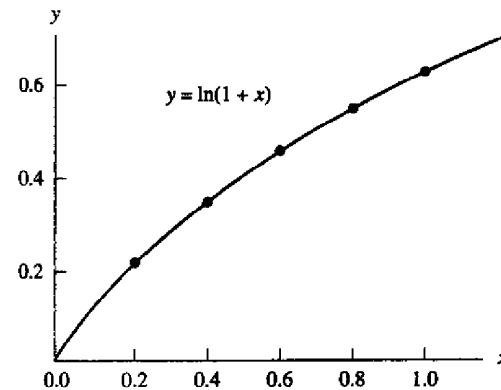
If $T(x)$ is used to approximate $\ln(1+x)$ on the interval $[0, 1]$, then the error is 0 at $x = 0$ and is largest when $x = 1$ (see Table 4.4). Indeed, the error between $T(1)$ and the correct value $\ln(1)$ is 13%. We seek a polynomial of degree 5 that will approximate $\ln(1+x)$ better over the interval $[0, 1]$. The polynomial $P(x)$ in Example 4.5 is an interpolating polynomial and will approximate $\ln(1+x)$ with an error no bigger than 0.00002385 over the interval $[0, 1]$.

Example 4.5. Consider the function $f(x) = \ln(1+x)$ and the polynomial

$$\begin{aligned}
 P(x) = & 0.02957206x^5 - 0.12895295x^4 + 0.28249626x^3 \\
 & - 0.48907554x^2 + 0.99910735x
 \end{aligned}$$

Table 4.5 Values of the Approximating Polynomial $P(x)$ of Example 4.5 and the Function $f(x) = \ln(1+x)$ and the Error $E(x) = f(x) - P(x)$ on $[-0.1, 1.1]$

x	Approximating polynomial, $P(x)$	Function, $f(x) = \ln(1+x)$	Error, $E(x) = f(x) - P(x)$
-0.1	-0.10509718	-0.10536052	-0.00026334
0.0	0.00000000	0.00000000	0.00000000
0.1	0.09528988	0.09531018	0.00002030
0.2	0.18232156	0.18232156	0.00000000
0.3	0.26237015	0.26236426	-0.00000589
0.4	0.33647224	0.33647224	0.00000000
0.5	0.40546139	0.40546511	0.00000372
0.6	0.47000363	0.47000363	0.00000000
0.7	0.53063292	0.53062825	-0.00000467
0.8	0.58778666	0.58778666	0.00000000
0.9	0.64184118	0.64185389	0.00001271
1.0	0.69314718	0.69314718	0.00000000
1.1	0.74206529	0.74193734	-0.00012795

**Figure 4.9** The graph of $y = P(x)$, which “lies on top” of the graph $y = \ln(1+x)$.

based on the six nodes $x_k = k/5$ for $k = 0, 1, 2, 3, 4$, and 5 . The following are empirical descriptions of the approximation $P(x) \approx \ln(1+x)$.

1. $P(x_k) = f(x_k)$ at each node (see Table 4.5).
2. The maximum error on the interval $[-0.1, 1.1]$ occurs at $x = -0.1$ and $|\text{error}| \leq 0.00026334$ for $-0.1 \leq x \leq 1.1$ (see Figure 4.10). Hence the graph of $y = P(x)$ would appear identical to that of $y = \ln(1+x)$ (see Figure 4.9).
3. The maximum error on the interval $[0, 1]$ occurs at $x = 0.06472456$ and $|\text{error}| \leq 0.00002385$ for $0 \leq x \leq 1$ (see Figure 4.10).

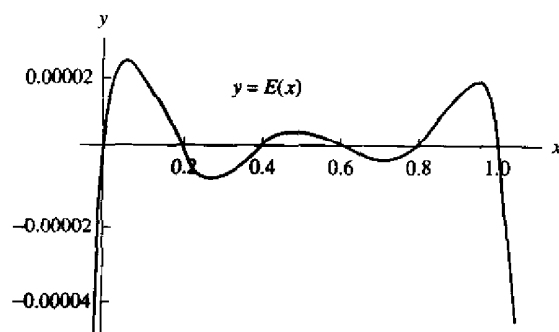


Figure 4.10 The graph of the error $y = E(x) = \ln(1+x) - P(x)$.

Remark. At a node x_k we have $f(x_k) = P(x_k)$. Hence $E(x_k) = 0$ at a node. The graph of $E(x) = f(x) - P(x)$ looks like a vibrating string, with the nodes being the abscissa where there is no displacement.

Algorithm 4.1 (Polynomial Calculus). To evaluate the polynomial $P(x)$, its derivative $P'(x)$, and its integral $\int P(x) dx$ by performing synthetic division.

INPUT N {Degree of $P(x)$ }
 INPUT $A(0), A(1), \dots, A(N)$ {Coefficients of $P(x)$ }
 INPUT C {Constant of integration}
 INPUT X {Independent variable}

(i) Algorithm to Evaluate $P(x)$ $B(N) := A(N)$ FOR $K = N - 1$ DOWNTO 0 DO $B(K) := A(K) + B(K+1) * X$ PRINT "The value $P(x)$ is", $B(0)$	Space-saving version: $Poly := A(N)$ FOR $K = N - 1$ DOWNTO 0 DO $Poly := A(K) + Poly * X$ PRINT "The value $P(x)$ is", $Poly$
(ii) Algorithm to Evaluate $P'(x)$ $D(N-1) := N * A(N)$ FOR $K = N - 1$ DOWNTO 1 DO $D(K-1) := K * A(K) + D(K) * X$ PRINT "The value $P'(x)$ is", $D(0)$	Space-saving version: $Deriv := N * A(N)$ FOR $K = N - 1$ DOWNTO 1 DO $Deriv := K * A(K) + Deriv * X$ PRINT "The value $P'(x)$ is", $Deriv$
(iii) Algorithm to Evaluate $I(x)$ $I(N+1) := A(N)/(N+1)$ FOR $K = N$ DOWNTO 1 DO $I(K) := A(K-1)/K + I(K+1) * X$ $I(0) := C + I(1) * X$ PRINT "The value $I(x)$ is", $I(0)$	Space-saving version: $Integ := A(N)/(N+1)$ FOR $K = N$ DOWNTO 1 DO $Integ := A(K-1)/K + Integ * X$ $Integ := C + Integ * X$ PRINT "The value $I(x)$ is", $Integ$

Exercises for Introduction to Interpolation

- Consider $P(x) = -0.02x^3 + 0.1x^2 - 0.2x + 1.66$, which passes through the four points $(1, 1.54)$, $(2, 1.5)$, $(3, 1.42)$, and $(5, 0.66)$.
 - Find $P(4)$.
 - Find $P'(4)$.
 - Find the definite integral of $P(x)$ taken over $[1, 4]$.
 - Find the extrapolated value $P(5.5)$.
 - Show how to find the coefficients of $P(x)$.
- Consider $P(x) = -0.04x^3 + 0.14x^2 - 0.16x + 2.08$, which passes through the four points $(0, 2.08)$, $(1, 2.02)$, $(2, 2.00)$, and $(4, 1.12)$.
 - Find $P(3)$.
 - Find $P'(3)$.
 - Find the definite integral of $P(x)$ taken over $[0, 3]$.
 - Find the extrapolated value $P(4.5)$.
 - Show how to find the coefficients of $P(x)$.
- Consider $P(x) = -0.0292166667x^3 + 0.275x^2 - 0.570833333x - 1.375$, which passes through the four points $(1, 1.05)$, $(2, 1.10)$, $(3, 1.35)$, and $(5, 1.75)$.
 - Show that the ordinates 1.05, 1.10, 1.35, and 1.75 differ from those of Example 4.4 by less than 1.8%, yet the coefficients of x^3 and x differ by more than 42%.
 - Find $P(4)$ and compare with Example 4.4.
 - Find $P'(4)$ and compare with Example 4.4.
 - Find the definite integral of $P(x)$ taken over $[1, 4]$ and compare with Example 4.4.
 - Find the extrapolated value $P(5.5)$ and compare with Example 4.4.

Remark. Part (a) shows that the computation of the coefficients of an interpolating polynomial is an ill-conditioned problem.

Algorithms and Programs

- Write a program in MATLAB that will implement Algorithm 4.1. The program should accept the coefficients of the polynomial $P(x) = a_N x^N + a_{N-1} x^{N-1} + \dots + a_2 x^2 + a_1 x + a_0$ as an $1 \times N$ matrix: $P = [a_N \ a_{N-1} \ \dots \ a_2 \ a_1 \ a_0]$.
- For each of the given functions, the fifth-degree polynomial $P(x)$ passes through the six points $(0, f(0))$, $(0.2, f(0.2))$, $(0.4, f(0.4))$, $(0.6, f(0.6))$, $(0.8, f(0.8))$, $(1, f(1))$. The six coefficients of $P(x)$ are a_0, a_1, \dots, a_5 , where

$$P(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0.$$

- (i) Find the coefficients of $P(x)$ by solving the 6×6 system of linear equations

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 = f(x_j)$$

using $x_j = (j - 1)/5$ and $j = 1, 2, 3, 4, 5, 6$ for the six unknowns $\{a_k\}_{k=0}^5$.

- (ii) Use your MATLAB program from Problem 1 to compute the interpolated values $P(0.3)$, $P(0.4)$, and $P(0.5)$ and compare with $f(0.3)$, $f(0.4)$, and $f(0.5)$ respectively.
- (iii) Use your MATLAB program to compute the extrapolated values $P(-0.1)$ and $P(1.1)$ and compare with $f(-0.1)$ and $f(1.1)$, respectively.
- (iv) Use your MATLAB program to find the integral of $P(x)$ taken over $[0, 1]$ and compare with the integral of $f(x)$ taken over $[0, 1]$. Plot $f(x)$ and $P(x)$ over $[0, 1]$ on the same graph.
- (v) Make a table of values for $P(x_k)$, $f(x_k)$, and $E(x_k) = f(x_k) - P(x_k)$, where $x_k = k/100$ for $k = 0, 1, \dots, 100$.
- (a) $f(x) = e^x$
 (b) $f(x) = \sin(x)$
 (c) $f(x) = (x + 1)^{(x+1)}$
3. A portion of an amusement park ride is to be modeled using three polynomials. The first section is to be a first-degree polynomial, $P_1(x)$, that covers a horizontal distance of 100 feet, starts at a height of 110 feet, and ends at a height of 60 feet. The third section is to also be a first-degree polynomial, $Q_1(x)$, that covers a horizontal distance of 50 feet, starts at a height of 65 feet, and ends at a height of 70 feet. The middle section is to be a polynomial, $P(x)$ (of smallest possible degree), that covers a horizontal distance of 150 feet.
- (a) Find expressions for $P(x)$, $P_1(x)$, and $Q_1(x)$ such that $P(100) = P_1(100)$, $P'(100) = P'_1(100)$, $P(250) = Q_1(250)$, and $P'(250) = Q'_1(250)$ and the curvature of $P(x)$ equals the curvature of $P_1(x)$ at $x = 100$ and equals the curvature of $Q_1(x)$ at $x = 250$.
- (b) Plot the graphs of $P_1(x)$, $P(x)$, and $Q_1(x)$ on the same coordinate system.
- (c) Use Algorithm 4.1(iii) to find the average height of the ride over the given horizontal distance.

3 Lagrange Approximation

Interpolation means to estimate a missing function value by taking a weighted average of known function values at neighboring points. Linear interpolation uses a line segment that passes through two points. The slope between (x_0, y_0) and (x_1, y_1) is $m = (y_1 - y_0)/(x_1 - x_0)$, and the point-slope formula for the line $y = m(x - x_0) + y_0$ can be rearranged as

$$(1) \quad y = P(x) = y_0 + (y_1 - y_0) \frac{x - x_0}{x_1 - x_0}.$$

When formula (1) is expanded, the result is a polynomial of degree ≤ 1 . Evaluation of $P(x)$ at x_0 and x_1 produces y_0 and y_1 , respectively:

$$(2) \quad \begin{aligned} P(x_0) &= y_0 + (y_1 - y_0)(0) = y_0, \\ P(x_1) &= y_0 + (y_1 - y_0)(1) = y_1. \end{aligned}$$

The French mathematician Joseph Louis Lagrange used a slightly different method to find this polynomial. He noticed that it could be written as

$$(3) \quad y = P_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}.$$

Each term on the right side of (3) involves a linear factor; hence the sum is a polynomial of degree ≤ 1 . The quotients in (3) are denoted by

$$(4) \quad L_{1,0}(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_{1,1}(x) = \frac{x - x_0}{x_1 - x_0}.$$

Computation reveals that $L_{1,0}(x_0) = 1$, $L_{1,0}(x_1) = 0$, $L_{1,1}(x_0) = 0$, and $L_{1,1}(x_1) = 1$ so that the polynomial $P_1(x)$ in (3) also passes through the two given points:

$$(5) \quad P_1(x_0) = y_0 + y_1(0) = y_0 \quad \text{and} \quad P_1(x_1) = y_0(0) + y_1 = y_1.$$

The terms $L_{1,0}(x)$ and $L_{1,1}(x)$ in (4) are called **Lagrange coefficient polynomials** based on the nodes x_0 and x_1 . Using this notation, (3) can be written in summation form

$$(6) \quad P_1(x) = \sum_{k=0}^1 y_k L_{1,k}(x).$$

Suppose that the ordinates y_k are computed with the formula $y_k = f(x_k)$. If $P_1(x)$ is used to approximate $f(x)$ over the interval $[x_0, x_1]$, we call the process **interpolation**. If $x < x_0$ (or $x_1 < x$), then using $P_1(x)$ is called **extrapolation**. The next example illustrates these concepts.

Example 4.6. Consider the graph $y = f(x) = \cos(x)$ over $[0.0, 1.2]$.

- (a) Use the nodes $x_0 = 0.0$ and $x_1 = 1.2$ to construct a linear interpolation polynomial $P_1(x)$.
- (b) Use the nodes $x_0 = 0.2$ and $x_1 = 1.0$ to construct a linear approximating polynomial $Q_1(x)$.

Using (3) with the abscissas $x_0 = 0.0$ and $x_1 = 1.2$ and the ordinates $y_0 = \cos(0.0) = 1.000000$ and $y_1 = \cos(1.2) = 0.362358$ produces

$$\begin{aligned} P_1(x) &= 1.000000 \frac{x - 1.2}{0.0 - 1.2} + 0.362358 \frac{x - 0.0}{1.2 - 0.0} \\ &= -0.833333(x - 1.2) + 0.301965(x - 0.0). \end{aligned}$$

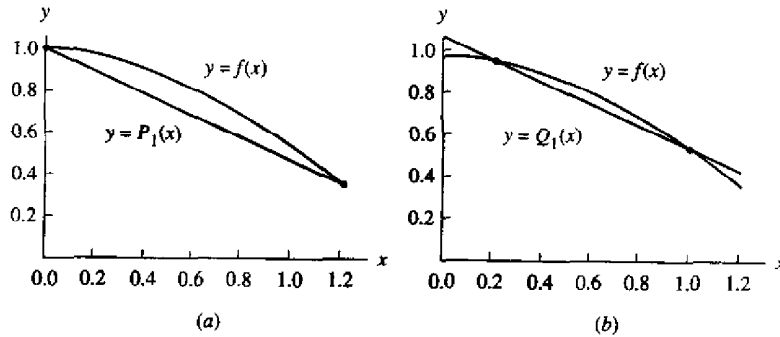


Figure 4.11 (a) The linear approximation of $y = P_1(x)$ where the nodes $x_0 = 0.0$ and $x_1 = 1.2$ are the end points of the interval $[a, b]$. (b) The linear approximation of $y = Q_1(x)$ where the nodes $x_0 = 0.2$ and $x_1 = 1.0$ lie inside the interval $[a, b]$.

When the nodes $x_0 = 0.2$ and $x_1 = 1.0$ with $y_0 = \cos(0.2) = 0.980067$ and $y_1 = \cos(1.0) = 0.540302$ are used, the result is

$$\begin{aligned} Q_1(x) &= 0.980067 \frac{x - 1.0}{0.2 - 1.0} + 0.540302 \frac{x - 0.2}{1.0 - 0.2} \\ &= -1.225083(x - 1.0) + 0.675378(x - 0.2). \end{aligned}$$

Figure 4.11(a) and (b) show the graph of $y = \cos(x)$ and compares it with $y = P_1(x)$ and $y = Q_1(x)$, respectively. Numerical computations are given in Table 4.6 and reveal that $Q_1(x)$ has less error at the points x_k that satisfy $0.1 \leq x_k \leq 1.1$. The largest tabulated error, $f(0.6) - P_1(0.6) = 0.144157$, is reduced to $f(0.6) - Q_1(0.6) = 0.065151$ by using $Q_1(x)$. ■

The generalization of (6) is the construction of a polynomial $P_N(x)$ of degree at most N that passes through the $N + 1$ points $(x_0, y_0), (x_1, y_1), \dots, (x_N, y_N)$ and has the form

$$(7) \quad P_N(x) = \sum_{k=0}^N y_k L_{N,k}(x),$$

where $L_{N,k}$ is the Lagrange coefficient polynomial based on these nodes:

$$(8) \quad L_{N,k}(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_N)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_N)}.$$

It is understood that the terms $(x - x_k)$ and $(x_k - x_k)$ do not appear on the right side of

Table 4.6 Comparison of $f(x) = \cos(x)$ and the Linear Approximations $P_1(x)$ and $Q_1(x)$

x_k	$f(x_k) = \cos(x_k)$	$P_1(x_k)$	$f(x_k) - P_1(x_k)$	$Q_1(x_k)$	$f(x_k) - Q_1(x_k)$
0.0	1.000000	1.000000	0.000000	1.090008	-0.090008
0.1	0.995004	0.946863	0.048141	1.035037	-0.040033
0.2	0.980067	0.893726	0.086340	0.980067	0.000000
0.3	0.955336	0.840589	0.114747	0.925096	0.030240
0.4	0.921061	0.787453	0.133608	0.870126	0.050935
0.5	0.877583	0.734316	0.143267	0.815155	0.062428
0.6	0.825336	0.681179	0.144157	0.760184	0.065151
0.7	0.764842	0.628042	0.136800	0.705214	0.059628
0.8	0.696707	0.574905	0.121802	0.650243	0.046463
0.9	0.621610	0.521768	0.099842	0.595273	0.026337
1.0	0.540302	0.468631	0.071671	0.540302	0.000000
1.1	0.453596	0.415495	0.038102	0.485332	-0.031736
1.2	0.362358	0.362358	0.000000	0.430361	-0.068003

equation (8). It is appropriate to introduce the product notation for (8), and we write

$$(9) \quad L_{N,k}(x) = \frac{\prod_{\substack{j=0 \\ j \neq k}}^N (x - x_j)}{\prod_{\substack{j=0 \\ j \neq k}}^N (x_k - x_j)}.$$

Here the notation in (9) indicates that in the numerator the product of the linear factors $(x - x_j)$ is to be formed, but the factor $(x - x_k)$ is to be left out (or skipped). A similar construction occurs in the denominator.

A straightforward calculation shows that, for each fixed k , the Lagrange coefficient polynomial $L_{N,k}(x)$ has the property

$$(10) \quad L_{N,k}(x_j) = 1 \text{ when } j = k \quad \text{and} \quad L_{N,k}(x_j) = 0 \text{ when } j \neq k.$$

Then direct substitution of these values into (7) is used to show that the polynomial curve $y = P_N(x)$ goes through (x_j, y_j) :

$$(11) \quad \begin{aligned} P_N(x_j) &= y_0 L_{N,0}(x_j) + \cdots + y_j L_{N,j}(x_j) + \cdots + y_N L_{N,N}(x_j) \\ &= y_0(0) + \cdots + y_j(1) + \cdots + y_N(0) = y_j. \end{aligned}$$

To show that $P_N(x)$ is unique, we invoke the fundamental theorem of algebra, which states that a polynomial $T(x)$ of degree $\leq N$ has at most N roots. In other words, if $T(x)$ is zero at $N + 1$ distinct abscissas, it is identically zero. Suppose that $P_N(x)$ is not unique and that there exists another polynomial $Q_N(x)$ of degree $\leq N$ that also passes through the $N + 1$ points. Form the difference polynomial $T(x) =$

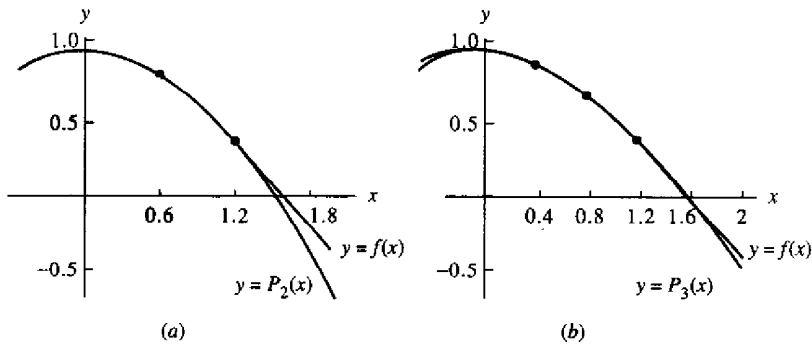


Figure 4.12 (a) The quadratic approximation polynomial $y = P_2(x)$ based on the nodes $x_0 = 0.0$, $x_1 = 0.6$, and $x_2 = 1.2$. (b) The cubic approximation polynomial $y = P_3(x)$ based on the nodes $x_0 = 0.0$, $x_1 = 0.4$, $x_2 = 0.8$, and $x_3 = 1.2$.

$P_N(x) - Q_N(x)$. Observe that the polynomial $T(x)$ has degree $\leq N$ and that $T(x_j) = P_N(x_j) - Q_N(x_j) = y_j - y_j = 0$, for $j = 0, 1, \dots, N$. Therefore, $T(x) \equiv 0$ and it follows that $Q_N(x) = P_N(x)$.

When (7) is expanded, the result is similar to (3). The Lagrange quadratic interpolating polynomial through the three points (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) is

$$(12) \quad P_2(x) = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

The Lagrange cubic interpolating polynomial through the four points (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) is

$$(13) \quad P_3(x) = y_0 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + y_1 \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\ + y_2 \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}.$$

Example 4.7. Consider $y = f(x) = \cos(x)$ over $[0.0, 1.2]$.

(a) Use the three nodes $x_0 = 0.0$, $x_1 = 0.6$, and $x_2 = 1.2$ to construct a quadratic interpolation polynomial $P_2(x)$.

(b) Use the four nodes $x_0 = 0.0$, $x_1 = 0.4$, $x_2 = 0.8$, and $x_3 = 1.2$ to construct a cubic interpolation polynomial $P_3(x)$.

Using $x_0 = 0.0$, $x_1 = 0.6$, $x_2 = 1.2$ and $y_0 = \cos(0.0) = 1$, $y_1 = \cos(0.6) = 0.825336$

and $y_2 = \cos(1.2) = 0.362358$ in equation (12) produces

$$P_2(x) = 1.0 \frac{(x - 0.6)(x - 1.2)}{(0.0 - 0.6)(0.0 - 1.2)} + 0.825336 \frac{(x - 0.0)(x - 1.2)}{(0.6 - 0.0)(0.6 - 1.2)} \\ + 0.362358 \frac{(x - 0.0)(x - 0.6)}{(1.2 - 0.0)(1.2 - 0.6)} \\ = 1.388889(x - 0.6)(x - 1.2) - 2.292599(x - 0.0)(x - 1.2) \\ + 0.503275(x - 0.0)(x - 0.6).$$

Using $x_0 = 0.0$, $x_1 = 0.4$, $x_2 = 0.8$, $x_3 = 1.2$ and $y_0 = \cos(0.0) = 1.0$, $y_1 = \cos(0.4) = 0.921061$, $y_2 = \cos(0.8) = 0.696707$, and $y_3 = \cos(1.2) = 0.362358$ in equation (13) produces

$$P_3(x) = 1.000000 \frac{(x - 0.4)(x - 0.8)(x - 1.2)}{(0.0 - 0.4)(0.0 - 0.8)(0.0 - 1.2)} \\ + 0.921061 \frac{(x - 0.0)(x - 0.8)(x - 1.2)}{(0.4 - 0.0)(0.4 - 0.8)(0.4 - 1.2)} \\ + 0.696707 \frac{(x - 0.0)(x - 0.4)(x - 1.2)}{(0.8 - 0.0)(0.8 - 0.4)(0.8 - 1.2)} \\ + 0.362358 \frac{(x - 0.0)(x - 0.4)(x - 0.8)}{(1.2 - 0.0)(1.2 - 0.4)(1.2 - 0.8)} \\ = -2.604167(x - 0.4)(x - 0.8)(x - 1.2) \\ + 7.195789(x - 0.0)(x - 0.8)(x - 1.2) \\ - 5.443021(x - 0.0)(x - 0.4)(x - 1.2) \\ + 0.943641(x - 0.0)(x - 0.4)(x - 0.8).$$

The graphs of $y = \cos(x)$ and the polynomials $y = P_2(x)$ and $y = P_3(x)$ are shown in Figure 4.12(a) and (b), respectively. ■

Error Terms and Error Bounds

It is important to understand the nature of the error term when the Lagrange polynomial is used to approximate a continuous function $f(x)$. It is similar to the error term for the Taylor polynomial, except that the factor $(x - x_0)^{N+1}$ is replaced with the product $(x - x_0)(x - x_1) \cdots (x - x_N)$. This is expected because interpolation is exact at each of the $N + 1$ nodes x_k , where we have $E_N(x_k) = f(x_k) - P_N(x_k) = y_k - y_k = 0$ for $k = 0, 1, 2, \dots, N$.

Theorem 4.3 (Lagrange Polynomial Approximation). Assume that $f \in C^{N+1}[a, b]$ and that $x_0, x_1, \dots, x_N \in [a, b]$ are $N + 1$ nodes. If $x \in [a, b]$, then

$$(14) \quad f(x) = P_N(x) + E_N(x),$$

where $P_N(x)$ is a polynomial that can be used to approximate $f(x)$:

$$(15) \quad f(x) \approx P_N(x) = \sum_{k=0}^N f(x_k) L_{N,k}(x).$$

The error term $E_N(x)$ has the form

$$(16) \quad E_N(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_N)f^{(N+1)}(c)}{(N+1)!},$$

for some value $c = c(x)$ that lies in the interval $[a, b]$.

Proof. As an example of the general method, we establish (16) when $N = 1$. The general case is discussed in the exercises. Start by defining the special function $g(t)$ as follows

$$(17) \quad g(t) = f(t) - P_1(t) - E_1(x) \frac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)}.$$

Notice that x , x_0 and x_1 are constants with respect to the variable t and that $g(t)$ evaluates to be zero at these three values; that is,

$$\begin{aligned} g(x) &= f(x) - P_1(x) - E_1(x) \frac{(x-x_0)(x-x_1)}{(x-x_0)(x-x_1)} = f(x) - P_1(x) - E_1(x) = 0, \\ g(x_0) &= f(x_0) - P_1(x_0) - E_1(x) \frac{(x_0-x_0)(x_0-x_1)}{(x-x_0)(x-x_1)} = f(x_0) - P_1(x_0) = 0, \\ g(x_1) &= f(x_1) - P_1(x_1) - E_1(x) \frac{(x_1-x_0)(x_1-x_1)}{(x-x_0)(x-x_1)} = f(x_1) - P_1(x_1) = 0. \end{aligned}$$

Suppose that x lies in the open interval (x_0, x_1) . Applying Rolle's theorem to $g(t)$ on the interval $[x_0, x]$ produces a value d_0 , with $x_0 < d_0 < x$, such that

$$(18) \quad g'(d_0) = 0.$$

A second application of Rolle's theorem to $g(t)$ on $[x, x_1]$ will produce a value d_1 , with $x < d_1 < x_1$, such that

$$(19) \quad g'(d_1) = 0.$$

Equations (18) and (19) show that the function $g'(t)$ is zero at $t = d_0$ and $t = d_1$. A third use of Rolle's theorem, but this time applied to $g'(t)$ over $[d_0, d_1]$, produces a value c for which

$$(20) \quad g^{(2)}(c) = 0.$$

Now go back to (17) and compute the derivatives $g'(t)$ and $g''(t)$:

$$(21) \quad g'(t) = f'(t) - P_1'(t) - E_1(x) \frac{(t-x_0) + (t-x_1)}{(x-x_0)(x-x_1)},$$

$$(22) \quad g''(t) = f''(t) - 0 - E_1(x) \frac{2}{(x-x_0)(x-x_1)}.$$

In (22) we have used the fact the $P_1(t)$ is a polynomial of degree $N = 1$; hence its second derivative is $P_1''(t) \equiv 0$. Evaluation of (22) at the point $t = c$ and using (20) yields

$$(23) \quad 0 = f''(c) - E_1(x) \frac{2}{(x-x_0)(x-x_1)}.$$

Solving (23) for $E_1(x)$ results in the desired form (16) for the remainder:

$$(24) \quad E_1(x) = \frac{(x-x_0)(x-x_1)f^{(2)}(c)}{2!},$$

and the proof is complete. •

The next result addresses the special case when the nodes for the Lagrange polynomial are equally spaced $x_k = x_0 + hk$, for $k = 0, 1, \dots, N$, and the polynomial $P_N(x)$ is used only for interpolation inside the interval $[x_0, x_N]$.

Theorem 4.4 (Error Bounds for Lagrange Interpolation, Equally Spaced Nodes).

Assume that $f(x)$ is defined on $[a, b]$, which contains equally spaced nodes $x_k = x_0 + hk$. Additionally, assume that $f(x)$ and the derivatives of $f(x)$, up to the order $N + 1$, are continuous and bounded on the special subintervals $[x_0, x_1]$, $[x_0, x_2]$, and $[x_0, x_3]$, respectively; that is,

$$(25) \quad |f^{(N+1)}(x)| \leq M_{N+1} \quad \text{for } x_0 \leq x \leq x_N,$$

for $N = 1, 2, 3$. The error terms (16) corresponding to the cases $N = 1, 2$, and 3 have the following useful bounds on their magnitude:

$$(26) \quad |E_1(x)| \leq \frac{h^2 M_2}{8} \quad \text{valid for } x \in [x_0, x_1],$$

$$(27) \quad |E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} \quad \text{valid for } x \in [x_0, x_2],$$

$$(28) \quad |E_3(x)| \leq \frac{h^4 M_4}{24} \quad \text{valid for } x \in [x_0, x_3].$$

Proof. We establish (26) and leave the others for the reader. Using the change of variables $x - x_0 = t$ and $x - x_1 = t - h$, the error term $E_1(x)$ can be written as

$$(29) \quad E_1(x) = E_1(x_0 + t) = \frac{(t^2 - ht)f^{(2)}(c)}{2!} \quad \text{for } 0 \leq t \leq h.$$

The bound for the derivative for this case is

$$(30) \quad |f^{(2)}(c)| \leq M_2 \quad \text{for } x_0 \leq c \leq x_1.$$

Now determine a bound for the expression $(t^2 - ht)$ in the numerator of (29); call this term $\Phi(t) = t^2 - ht$. Since $\Phi'(t) = 2t - h$, there is one critical point $t = h/2$ that is the solution to $\Phi'(t) = 0$. The extreme values of $\Phi(t)$ over $[0, h]$ occur either at an end point $\Phi(0) = 0$, $\Phi(h) = 0$ or at the critical point $\Phi(h/2) = -h^2/4$. Since the latter value is the largest, we have established the bound

$$(31) \quad |\Phi(t)| = |t^2 - ht| \leq \frac{|-h^2|}{4} = \frac{h^2}{4} \quad \text{for } 0 \leq t \leq h.$$

Using (30) and (31) to estimate the magnitude of the product in the numerator in (29) results in

$$(32) \quad |E_1(x)| = \frac{|\Phi(t)||f^{(2)}(c)|}{2!} \leq \frac{h^2 M_2}{8},$$

and formula (26) is established.

Comparison of Accuracy and $O(h^{N+1})$

The significance of Theorem 4.4 is to understand a simple relationship between the size of the error terms for linear, quadratic, and cubic interpolation. In each case the error bound $|E_N(x)|$ depends on h in two ways. First, h^{N+1} is explicitly present so that $|E_N(x)|$ is proportional to h^{N+1} . Second, the values M_{N+1} generally depend on h and tend to $|f^{(N+1)}(x_0)|$ as h goes to zero. Therefore, as h goes to zero, $|E_N(x)|$ converges to zero with the same rapidity that h^{N+1} converges to zero. The notation $O(h^{N+1})$ is used when discussing this behavior. For example, the error bound (26) can be expressed as

$$|E_1(x)| = O(h^2) \quad \text{valid for } x \in [x_0, x_1].$$

The notation $O(h^2)$ stands in place of $h^2 M_2/8$ in relation (26) and is meant to convey the idea that the bound for the error term is approximately a multiple of h^2 ; that is,

$$|E_1(x)| \leq Ch^2 \approx O(h^2).$$

As a consequence, if the derivatives of $f(x)$ are uniformly bounded on the interval $|h| < 1$, then choosing N large will make h^{N+1} small, and the higher-degree approximating polynomial will have less error.

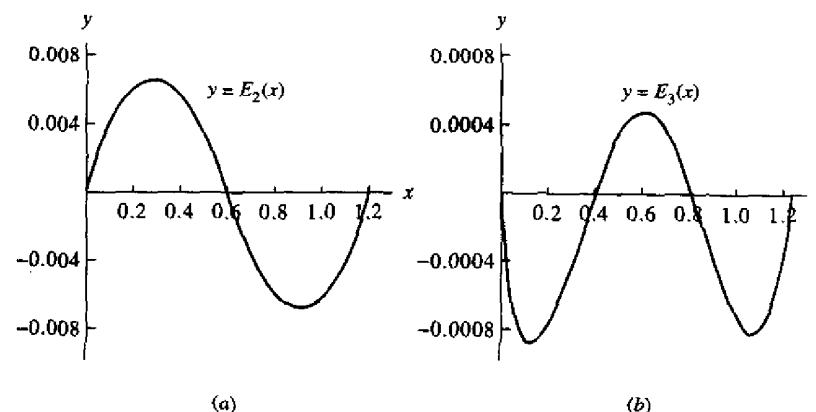


Figure 4.13 (a) The error function $E_2(x) = \cos(x) - P_2(x)$. (b) The error function $E_3(x) = \cos(x) - P_3(x)$.

Example 4.8. Consider $y = f(x) = \cos(x)$ over $[0.0, 1.2]$. Use formulas (26) through (28) and determine the error bounds for the Lagrange polynomials $P_1(x)$, $P_2(x)$, and $P_3(x)$ that were constructed in Examples 4.6 and 4.7.

First, determine the bounds M_2 , M_3 , and M_4 for the derivatives $|f^{(2)}(x)|$, $|f^{(3)}(x)|$, and $|f^{(4)}(x)|$, respectively, taken over the interval $[0.0, 1.2]$:

$$|f^{(2)}(x)| = |-\cos(x)| \leq |-\cos(0.0)| = 1.000000 = M_2,$$

$$|f^{(3)}(x)| = |\sin(x)| \leq |\sin(1.2)| = 0.932039 = M_3,$$

$$|f^{(4)}(x)| = |\cos(x)| \leq |\cos(0.0)| = 1.000000 = M_4.$$

For $P_1(x)$ the spacing of the nodes is $h = 1.2$, and its error bound is

$$(33) \quad |E_1(x)| \leq \frac{h^2 M_2}{8} \leq \frac{(1.2)^2 (1.000000)}{8} = 0.180000.$$

For $P_2(x)$ the spacing of the nodes is $h = 0.6$, and its error bound is

$$(34) \quad |E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} \leq \frac{(0.6)^3 (0.932039)}{9\sqrt{3}} = 0.012915.$$

For $P_3(x)$ the spacing of the nodes is $h = 0.4$, and its error bound is

$$(35) \quad |E_3(x)| \leq \frac{h^4 M_4}{24} \leq \frac{(0.4)^4 (1.000000)}{24} = 0.001067. \quad \blacksquare$$

From Example 4.6 we saw that $|E_1(0.6)| = |\cos(0.6) - P_1(0.6)| = 0.144157$, so the bound 0.180000 in (33) is reasonable. The graphs of the error functions $E_2(x) = \cos(x) - P_2(x)$ and $E_3(x) = \cos(x) - P_3(x)$ are shown in Figure 4.13(a) and (b),

Table 4.7 Comparison of $f(x) = \cos(x)$ and the Quadratic and Cubic Polynomial Approximations $P_2(x)$ and $P_3(x)$

x_k	$f(x_k) = \cos(x_k)$	$P_2(x_k)$	$E_2(x_k)$	$P_3(x_k)$	$E_3(x_k)$
0.0	1.000000	1.000000	0.0	1.000000	0.0
0.1	0.995004	0.990911	0.004093	0.995835	-0.000831
0.2	0.980067	0.973813	0.006253	0.980921	-0.000855
0.3	0.955336	0.948707	0.006629	0.955812	-0.000476
0.4	0.921061	0.915592	0.005469	0.921061	0.0
0.5	0.877583	0.874468	0.003114	0.877221	0.000361
0.6	0.825336	0.825336	0.0	0.824847	0.00089
0.7	0.764842	0.768194	-0.003352	0.764491	0.00035
0.8	0.696707	0.703044	-0.006338	0.696707	0.0
0.9	0.621610	0.629886	-0.008276	0.622048	-0.000438
1.0	0.540302	0.548719	-0.008416	0.541068	-0.00076
1.1	0.453596	0.459542	-0.005946	0.454320	-0.00072
1.2	0.362358	0.362358	0.0	0.362358	0.0

respectively, and numerical computations are given in Table 4.7. Using values in the table, we find that $|E_2(1.0)| = |\cos(1.0) - P_2(1.0)| = 0.008416$ and $|E_3(0.2)| = |\cos(0.2) - P_3(0.2)| = 0.000855$, which is in reasonable agreement with the bounds 0.012915 and 0.001607 given in (34) and (35), respectively.

MATLAB

The following program finds the collocation polynomial through a given set of points by constructing a vector whose entries are the coefficients of the Lagrange interpolatory polynomial. The program uses the commands `poly` and `conv`. The `poly` command creates a vector whose entries are the coefficients of a polynomial with specified roots. The `conv` command produces a vector whose entries are the coefficients of a polynomial that is the product of two other polynomials.

Example 4.9. Find the product of two first-degree polynomials, $P(x)$ and $Q(x)$, with roots 2 and 3, respectively.

```
>>P=poly(2)
P=
    1    -2
>>Q=poly(3)
Q=
    1    -3
>>conv(P,Q)
ans=
    1    -5    6
```

Thus the product of $P(x)$ and $Q(x)$ is $x^2 - 5x + 6$

Program 4.1 (Lagrange Approximation). To evaluate the Lagrange polynomial $P(x) = \sum_{k=0}^N y_k L_{N,k}(x)$ based on $N+1$ points (x_k, y_k) for $k = 0, 1, \dots, N$.

```
function [C,L]=lagran(X,Y)
%Input - X is a vector that contains a list of abscissas
%       - Y is a vector that contains a list of ordinates
%Output - C is a matrix that contains the coefficients of
%         the Lagrange interpolatory polynomial
%       - L is a matrix that contains the Lagrange
%         coefficient polynomials
w=length(X);
n=w-1;
L=zeros(w,w);
%Form the Lagrange coefficient polynomials
for k=1:n+1
    V=1;
    for j=1:n+1
        if k~=j
            V=conv(V,poly(X(j)))/(X(k)-X(j));
        end
    end
    L(k,:)=V;
end
%Determine the coefficients of the Lagrange interpolating
%polynomial
C=Y*L;
```

Exercises for Lagrange Approximation

1. Find Lagrange polynomials that approximate $f(x) = x^3$.
 - (a) Find the linear interpolation polynomial $P_1(x)$ using the nodes $x_0 = -1$ and $x_1 = 0$.
 - (b) Find the quadratic interpolation polynomial $P_2(x)$ using the nodes $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.
 - (c) Find the cubic interpolation polynomial $P_3(x)$ using the nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, and $x_3 = 2$.
 - (d) Find the linear interpolation polynomial $P_1(x)$ using the nodes $x_0 = 1$ and $x_1 = 2$.
 - (e) Find the quadratic interpolation polynomial $P_2(x)$ using the nodes $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.

2. Let $f(x) = x + 2/x$.
 - (a) Use quadratic Lagrange interpolation based on the nodes $x_0 = 1$, $x_1 = 2$, and $x_2 = 2.5$ to approximate $f(1.5)$ and $f(1.2)$.
 - (b) Use cubic Lagrange interpolation based on the nodes $x_0 = 0.5$, $x_1 = 1$, $x_2 = 2$, and $x_3 = 2.5$ to approximate $f(1.5)$ and $f(1.2)$.
3. Let $f(x) = 2 \sin(\pi x/6)$, where x is in radians.
 - (a) Use quadratic Lagrange interpolation based on the nodes $x_0 = 0$, $x_1 = 1$, and $x_2 = 3$ to approximate $f(2)$ and $f(2.4)$.
 - (b) Use cubic Lagrange interpolation based on the nodes $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, and $x_3 = 5$ to approximate $f(2)$ and $f(2.4)$.
4. Let $f(x) = 2 \sin(\pi x/6)$, where x is in radians.
 - (a) Use quadratic Lagrange interpolation based on the nodes $x_0 = 0$, $x_1 = 1$, and $x_2 = 3$ to approximate $f(4)$ and $f(3.5)$.
 - (b) Use cubic Lagrange interpolation based on the nodes $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, and $x_3 = 5$ to approximate $f(4)$ and $f(3.5)$.
5. Write down the error term $E_3(x)$ for cubic Lagrange interpolation to $f(x)$, where interpolation is to be exact at the four nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 3$, and $x_3 = 4$ and $f(x)$ is given by
 - (a) $f(x) = 4x^3 - 3x + 2$
 - (b) $f(x) = x^4 - 2x^3$
 - (c) $f(x) = x^5 - 5x^4$
6. Let $f(x) = x^x$.
 - (a) Find the quadratic Lagrange polynomial $P_2(x)$ using the nodes $x_0 = 1$, $x_1 = 1.25$, and $x_2 = 1.5$.
 - (b) Use the polynomial from part (a) to estimate the average value of $f(x)$ over the interval $[1, 1.5]$.
 - (c) Use expression (27) of Theorem 4.4 to obtain a bound on the error in approximating $f(x)$ with $P_2(x)$.
7. Consider the Lagrange coefficient polynomials $L_{2,k}(x)$ that are used for quadratic interpolation at the nodes x_0 , x_1 , and x_2 . Define $g(x) = L_{2,0}(x) + L_{2,1}(x) + L_{2,2}(x) - 1$.
 - (a) Show that g is a polynomial of degree ≤ 2 .
 - (b) Show that $g(x_k) = 0$ for $k = 0, 1, 2$.
 - (c) Show that $g(x) = 0$ for all x . *Hint.* Use the fundamental theorem of algebra.
8. Let $L_{N,0}(x)$, $L_{N,1}(x)$, \dots , and $L_{N,N}(x)$ be the Lagrange coefficient polynomials based on the $N + 1$ nodes x_0, x_1, \dots, x_N . Show that $\sum_{k=0}^N L_{N,k}(x) = 1$ for any real number x .
9. Let $f(x)$ be a polynomial of degree $\leq N$. Let $P_N(x)$ be the Lagrange polynomial of degree $\leq N$ based on the $N + 1$ nodes x_0, x_1, \dots, x_N . Show that $f(x) = P_N(x)$ for all x . *Hint.* Show that the error term $E_N(x)$ is identically zero.

10. Consider the function $f(x) = \sin(x)$ on the interval $[0, 1]$. Use Theorem 4.4 to determine the step size h so that
 - (a) linear Lagrange interpolation has an accuracy of 10^{-6} (i.e., find h such that $|E_1(x)| < 5 \times 10^{-7}$).
 - (b) quadratic Lagrange interpolation has an accuracy of 10^{-6} (i.e., find h such that $|E_2(x)| < 5 \times 10^{-7}$).
 - (c) cubic Lagrange interpolation has an accuracy of 10^{-6} (i.e., find h such that $|E_3(x)| < 5 \times 10^{-7}$).
11. Start with equation (16) and $N = 2$, and prove inequality (27). Let $x_1 = x_0 + h$, $x_2 = x_0 + 2h$. Prove that if $x_0 \leq x \leq x_2$ then

$$|x - x_0||x - x_1||x - x_2| \leq \frac{2h^3}{3 \times 3^{1/2}}.$$

Hint. Use the substitutions $t = x - x_1$, $t + h = x - x_0$, and $t - h = x - x_2$ and the function $v(t) = t^3 - th^2$ on the interval $-h \leq t \leq h$. Set $v'(t) = 0$ and solve for t in terms of h .

12. *Linear interpolation in two dimensions.* Consider the polynomial $z = P(x, y) = A + Bx + Cy$ that passes through the three points (x_0, y_0, z_0) , (x_1, y_1, z_1) , and (x_2, y_2, z_2) . Then A , B , and C are the solution values for the linear system of equations

$$A + Bx_0 + Cy_0 = z_0$$

$$A + Bx_1 + Cy_1 = z_1$$

$$A + Bx_2 + Cy_2 = z_2.$$

- (a) Find A , B , and C so that $z = P(x, y)$ passes through the points $(1, 1, 5)$, $(2, 1, 3)$, and $(1, 2, 9)$.
 - (b) Find A , B , and C so that $z = P(x, y)$ passes through the points $(1, 1, 2.5)$, $(2, 1, 0)$, and $(1, 2, 4)$.
 - (c) Find A , B , and C so that $z = P(x, y)$ passes through the points $(2, 1, 5)$, $(1, 3, 7)$, and $(3, 2, 4)$.
 - (d) Can values A , B , and C be found so that $z = P(x, y)$ passes through the points $(1, 2, 5)$, $(3, 2, 7)$, and $(1, 2, 0)$? Why?
13. Use Theorem 1.7, the Generalized Rolle's Theorem, and the special function

$$g(t) = f(t) - P_N(t) - E_N(x) \frac{(t - x_0)(t - x_1) \cdots (t - x_N)}{(x - x_0)(x - x_1) \cdots (x - x_N)},$$

where $P_N(x)$ is the Lagrange polynomial of degree N , to prove that the error term $E_N(x) = f(x) - P_N(x)$ has the form

$$E_N(x) = (x - x_0)(x - x_1) \cdots (x - x_N) \frac{f^{(N+1)}(c)}{(N+1)!}.$$

Hint. Find $g^{(N+1)}(t)$ and then evaluate it at $t = c$.

Algorithms and Programs

1. Use Program 4.1 to find the coefficients of the interpolatory polynomials in Problem 2(i) a, b, and c in the Algorithms and Programs in Section 4.2. Plot the graphs of each function and the associated interpolatory polynomial on the same coordinate system.
2. The measured temperatures during a 5-hour period in a suburb of Los Angeles on November 8 are given in the following table.
 - (a) Use Program 4.1 to construct a Lagrange interpolatory polynomial for the data in the table.
 - (b) Use Algorithm 4.1(iii) to estimate the average temperature during the given 5-hour period.
 - (c) Graph the data in the table and the polynomial from part (a) on the same coordinate system. Discuss the possible error that can result from using the polynomial in part (a) to estimate the average temperature.

Time, P.M.	Degrees Fahrenheit
1	66
2	66
3	65
4	64
5	63
6	63

4.4 Newton Polynomials

It is sometimes useful to find several approximating polynomials $P_1(x)$, $P_2(x)$, ..., $P_N(x)$ and then choose the one that suits our needs. If the Lagrange polynomials are used, there is no constructive relationship between $P_{N-1}(x)$ and $P_N(x)$. Each polynomial has to be constructed individually, and the work required to compute the higher-degree polynomials involves many computations. We take a new approach and construct Newton polynomials that have the recursive pattern

$$\begin{aligned} (1) \quad P_1(x) &= a_0 + a_1(x - x_0), \\ (2) \quad P_2(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1), \end{aligned}$$

$$\begin{aligned} (3) \quad P_3(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &\quad + a_3(x - x_0)(x - x_1)(x - x_2), \\ &\quad \vdots \\ (4) \quad P_N(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &\quad + a_3(x - x_0)(x - x_1)(x - x_2) \\ &\quad + a_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + \cdots \\ &\quad + a_N(x - x_0) \cdots (x - x_{N-1}). \end{aligned}$$

Here the polynomial $P_N(x)$ is obtained from $P_{N-1}(x)$ using the recursive relationship

$$(5) \quad P_N(x) = P_{N-1}(x) + a_N(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{N-1}).$$

The polynomial (4) is said to be a Newton polynomial with N centers x_0, x_1, \dots, x_{N-1} . It involves sums of products of linear factors up to

$$a_N(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{N-1}),$$

so $P_N(x)$ will simply to be an ordinary polynomial of degree $\leq N$.

Example 4.10. Given the centers $x_0 = 1$, $x_1 = 3$, $x_2 = 4$, and $x_3 = 4.5$ and the coefficients $a_0 = 5$, $a_1 = -2$, $a_2 = 0.5$, $a_3 = -0.1$, and $a_4 = 0.003$, find $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$ and evaluate $P_k(2.5)$ for $k = 1, 2, 3, 4$.

Using formulas (1) through (4), we have

$$\begin{aligned} P_1(x) &= 5 - 2(x - 1), \\ P_2(x) &= 5 - 2(x - 1) + 0.5(x - 1)(x - 3), \\ P_3(x) &= P_2(x) - 0.1(x - 1)(x - 3)(x - 4), \\ P_4(x) &= P_3(x) + 0.003(x - 1)(x - 3)(x - 4)(x - 4.5). \end{aligned}$$

Evaluating the polynomials at $x = 2.5$ results in

$$\begin{aligned} P_1(2.5) &= 5 - 2(1.5) = 2, \\ P_2(2.5) &= P_1(2.5) + 0.5(1.5)(-0.5) = 1.625, \\ P_3(2.5) &= P_2(2.5) - 0.1(1.5)(-0.5)(-1.5) = 1.5125, \\ P_4(2.5) &= P_3(2.5) + 0.003(1.5)(-0.5)(-1.5)(-2.0) = 1.50575. \end{aligned}$$

Nested Multiplication

If N is fixed and the polynomial $P_N(x)$ is evaluated many times, then nested multiplication should be used. The process is similar to nested multiplication for ordinary polynomials, except that the centers x_k must be subtracted from the independent variable x . The nested multiplication form for $P_3(x)$ is

$$(6) \quad P_3(x) = ((a_3(x - x_2) + a_2)(x - x_1) + a_1)(x - x_0) + a_0.$$

To evaluate $P_3(x)$ for a given value of x , start with the innermost grouping and form successively the quantities

$$(7) \quad \begin{aligned} S_3 &= a_3, \\ S_2 &= S_3(x - x_2) + a_2, \\ S_1 &= S_2(x - x_1) + a_1, \\ S_0 &= S_1(x - x_0) + a_0. \end{aligned}$$

The quantity S_0 is now $P_3(x)$.

Example 4.11. Compute $P_3(2.5)$ in Example 4.10 using nested multiplication. Using (6), we write

$$P_3(x) = ((-0.1(x - 4) + 0.5)(x - 3) - 2)(x - 1) + 5.$$

The values in (7) are

$$\begin{aligned} S_3 &= -0.1, \\ S_2 &= -0.1(2.5 - 4) + 0.5 = 0.65, \\ S_1 &= 0.65(2.5 - 3) - 2 = -2.325, \\ S_0 &= -2.325(2.5 - 1) + 5 = 1.5125. \end{aligned}$$

Therefore, $P_3(2.5) = 1.5125$. ■

Polynomial Approximation, Nodes, and Centers

Suppose that we want to find the coefficients a_k for all the polynomials $P_1(x), \dots, P_N(x)$ that approximate a given function $f(x)$. Then $P_k(x)$ will be based on the centers x_0, x_1, \dots, x_k and have the nodes x_0, x_1, \dots, x_{k+1} . For the polynomial $P_1(x)$ the coefficients a_0 and a_1 have a familiar meaning. In this case

$$(8) \quad P_1(x_0) = f(x_0) \quad \text{and} \quad P_1(x_1) = f(x_1).$$

Using (1) and (8) to solve for a_0 , we find that

$$(9) \quad f(x_0) = P_1(x_0) = a_0 + a_1(x_0 - x_0) = a_0.$$

Hence $a_0 = f(x_0)$. Next, using (1), (8), and (9), we have

$$f(x_1) = P_1(x_1) = a_0 + a_1(x_1 - x_0) = f(x_0) + a_1(x_1 - x_0),$$

which can be solved for a_1 , and we get

$$(10) \quad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Hence a_1 is the slope of the secant line passing through the two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

The coefficients a_0 and a_1 are the same for both $P_1(x)$ and $P_2(x)$. Evaluating (2) at the node x_2 , we find that

$$(11) \quad f(x_2) = P_2(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1).$$

The values for a_0 and a_1 in (9) and (10) can be used in (11) to obtain

$$\begin{aligned} a_2 &= \frac{f(x_2) - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \\ &= \left(\frac{f(x_2) - f(x_0)}{x_2 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) / (x_2 - x_1). \end{aligned}$$

For computational purposes we prefer to write this last quantity as

$$(12) \quad a_2 = \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) / (x_2 - x_0).$$

The two formulas for a_2 can be shown to be equivalent by writing the quotients over the common denominator $(x_2 - x_1)(x_2 - x_0)(x_1 - x_0)$. The details are left for the reader. The numerator in (12) is the difference between the first-order divided differences. In order to proceed, we need to introduce the idea of divided differences.

Definition 4.1 (Divided Differences). The divided differences for a function $f(x)$ are defined as follows:

$$(13) \quad \begin{aligned} f[x_k] &= f(x_k), \\ f[x_{k-1}, x_k] &= \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}}, \\ f[x_{k-2}, x_{k-1}, x_k] &= \frac{f[x_{k-1}, x_k] - f[x_{k-2}, x_{k-1}]}{x_k - x_{k-2}}, \\ f[x_{k-3}, x_{k-2}, x_{k-1}, x_k] &= \frac{f[x_{k-2}, x_{k-1}, x_k] - f[x_{k-3}, x_{k-2}, x_{k-1}]}{x_k - x_{k-3}}. \end{aligned}$$

The recursive rule for constructing higher-order divided differences is

$$(14) \quad f[x_{k-j}, x_{k-j+1}, \dots, x_k] = \frac{f[x_{k-j+1}, \dots, x_k] - f[x_{k-j}, \dots, x_{k-1}]}{x_k - x_{k-j}}$$

and is used to construct the divided differences in Table 4.8. ▲

The coefficients a_k of $P_N(x)$ depend on the values $f(x_j)$, for $j = 0, 1, \dots, k$. The next theorem shows that a_k can be computed using divided differences:

$$(15) \quad a_k = f[x_0, x_1, \dots, x_k].$$

Table 4.8 Divided-difference Table for $y = f(x)$

x_k	$f[x_k]$	$f[\ , \]$	$f[\ , \ , \]$	$f[\ , \ , \ , \]$	$f[\ , \ , \ , \ , \]$
x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0, x_1]$			
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$		
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$	
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$f[x_0, x_1, x_2, x_3, x_4]$

Theorem 4.5 (Newton Polynomial). Suppose that x_0, x_1, \dots, x_N are $N + 1$ distinct numbers in $[a, b]$. There exists a unique polynomial $P_N(x)$ of degree at most N with the property that

$$f(x_j) = P_N(x_j) \quad \text{for } j = 0, 1, \dots, N.$$

The Newton form of this polynomial is

$$(16) \quad P_N(x) = a_0 + a_1(x - x_0) + \dots + a_N(x - x_0)(x - x_1) \cdots (x - x_{N-1}),$$

where $a_k = f[x_0, x_1, \dots, x_k]$, for $k = 0, 1, \dots, N$.

Remark. If $\{(x_j, y_j)\}_{j=0}^N$ is a set of points whose abscissas are distinct, the values $f(x_j) = y_j$ can be used to construct the unique polynomial of degree $\leq N$ that passes through the $N + 1$ points.

Corollary 4.2 (Newton Approximation). Assume that $P_N(x)$ is the Newton polynomial given in Theorem 4.5 and is used to approximate the function $f(x)$, that is,

$$(17) \quad f(x) = P_N(x) + E_N(x).$$

If $f \in C^{N+1}[a, b]$, then for each $x \in [a, b]$ there corresponds a number $c = c(x)$ in (a, b) , so that the error term has the form

$$(18) \quad E_N(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_N)f^{(N+1)}(c)}{(N + 1)!}.$$

Remark. The error term $E_N(x)$ is the same as the one for Lagrange interpolation, which was introduced in equation (16) of Section 4.3.

It is of interest to start with a known function $f(x)$ that is a polynomial of degree N and compute its divided-difference table. In this case we know that $f^{(N+1)}(x) = 0$ for all x , and calculation will reveal that the $(N + 1)$ st divided difference is zero. This will happen because the divided difference (14) is proportional to a numerical approximation for the j th derivative.

Table 4.9

x_k	$f[x_k]$	First divided difference	Second divided difference	Third divided difference	Fourth divided difference	Fifth divided difference
$x_0 = 1$	-3					
$x_1 = 2$	0	3				
$x_2 = 3$	15	15	6			
$x_3 = 4$	48	33	9	1		
$x_4 = 5$	105	57	12	1	0	
$x_5 = 6$	192	87	15	1	0	0

Table 4.10 Divided-Difference Table Used for Constructing the Newton Polynomials $P_k(x)$ in Example 4.13

x_k	$f[x_k]$	$f[\ , \]$	$f[\ , \ , \]$	$f[\ , \ , \ , \]$	$f[\ , \ , \ , \ , \]$
$x_0 = 0.0$	1.0000000				
$x_1 = 1.0$	0.5403023	-0.4596977			
$x_2 = 2.0$	-0.4161468	-0.9564491	-0.2483757		
$x_3 = 3.0$	-0.9899925	-0.5738457	0.1913017	0.1465592	
$x_4 = 4.0$	-0.6536436	0.3363499	0.4550973	0.0879318	-0.0146568

Example 4.12. Let $f(x) = x^3 - 4x$. Construct the divided-difference table based on the nodes $x_0 = 1, x_1 = 2, \dots, x_5 = 6$, and find the Newton polynomial $P_3(x)$ based on x_0, x_1, x_2 , and x_3 .

See Table 4.9.

The coefficients $a_0 = -3, a_1 = 3, a_2 = 6$, and $a_3 = 1$ of $P_3(x)$ appear on the diagonal of the divided-difference table. The centers $x_0 = 1, x_1 = 2$, and $x_2 = 3$ are the values in the first column. Using formula (3), we write

$$P_3(x) = -3 + 3(x - 1) + 6(x - 1)(x - 2) + (x - 1)(x - 2)(x - 3).$$

Example 4.13. Construct a divided-difference table for $f(x) = \cos(x)$ based on the five points $(k, \cos(k))$, for $k = 0, 1, 2, 3, 4$. Use it to find the coefficients a_k and the four Newton interpolating polynomials $P_k(x)$, for $k = 1, 2, 3, 4$.

For simplicity we round off the values to seven decimal places, which are displayed in Table 4.10. The nodes x_0, x_1, x_2, x_3 and the diagonal elements a_0, a_1, a_2, a_3, a_4 in

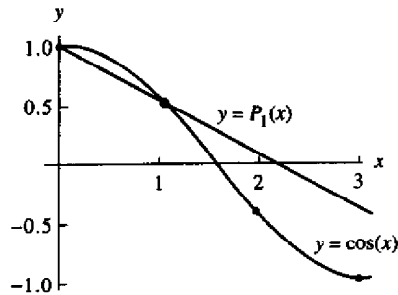


Figure 4.14 (a) Graphs of $y = \cos(x)$ and the linear Newton polynomial $y = P_1(x)$ based on the nodes $x_0 = 0.0$ and $x_1 = 1.0$.

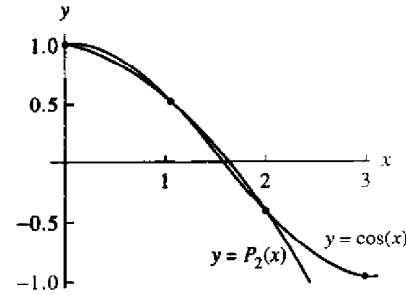


Figure 4.14 (b) Graphs of $y = \cos(x)$ and the quadratic Newton polynomial $y = P_2(x)$ based on the nodes $x_0 = 0.0$, $x_1 = 1.0$, and $x_2 = 2.0$.

Table 4.10 are used in formula (16), and we write down the first four Newton polynomials:

$$\begin{aligned} P_1(x) &= 1.0000000 - 0.4596977(x - 0.0), \\ P_2(x) &= 1.0000000 - 0.4596977(x - 0.0) - 0.2483757(x - 0.0)(x - 1.0), \\ P_3(x) &= 1.0000000 - 0.4596977(x - 0.0) - 0.2483757(x - 0.0)(x - 1.0) \\ &\quad + 0.1465592(x - 0.0)(x - 1.0)(x - 2.0), \\ P_4(x) &= 1.0000000 - 0.4596977(x - 0.0) - 0.2483757(x - 0.0)(x - 1.0) \\ &\quad + 0.1465592(x - 0.0)(x - 1.0)(x - 2.0) \\ &\quad - 0.0146568(x - 0.0)(x - 1.0)(x - 2.0)(x - 3.0). \end{aligned}$$

The following sample calculation shows how to find the coefficient a_2 .

$$\begin{aligned} f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{0.5403023 - 1.0000000}{1.0 - 0.0} = -0.4596977, \\ f[x_1, x_2] &= \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{-0.4161468 - 0.5403023}{2.0 - 1.0} = -0.9564491, \\ a_2 = f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{-0.9564491 + 0.4596977}{2.0 - 0.0} = -0.2483757. \end{aligned}$$

The graphs of $y = \cos(x)$ and $y = P_1(x)$, $y = P_2(x)$, and $y = P_3(x)$ are shown in Figure 4.14(a), (b), and (c), respectively.

For computational purposes the divided differences in Table 4.8 need to be stored in an array which is chosen to be $D(k, j)$. Thus (15) becomes

$$(19) \quad D(k, j) = f[x_{k-j}, x_{k-j+1}, \dots, x_k] \quad \text{for } j \leq k.$$

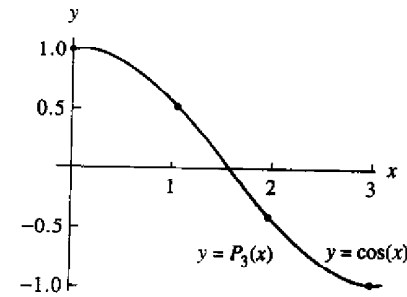


Figure 4.14 (c) Graphs of $y = \cos(x)$ and the cubic Newton polynomial $y = P_3(x)$ based on the nodes $x_0 = 0.0$, $x_1 = 1.0$, $x_2 = 2.0$, and $x_3 = 3.0$.

Relation (14) is used to obtain the formula to recursively compute the entries in the array:

$$(20) \quad D(k, j) = \frac{D(k, j-1) - D(k-1, j-1)}{x_k - x_{k-j}}.$$

Notice that the value a_k in (15) is the diagonal element $a_k = D(k, k)$. The algorithm for computing the divided differences and evaluating $P_N(x)$ is now given. We remark that Problem 2 in Algorithms and Programs investigates how to modify the algorithm so that the values $\{a_k\}$ are computed using a one-dimensional array. ■

Program 4.2 (Newton Interpolation Polynomial). To construct and evaluate the Newton polynomial of degree $\leq N$ that passes through $(x_k, y_k) = (x_k, f(x_k))$ for $k = 0, 1, \dots, N$:

$$(21) \quad P(x) = d_{0,0} + d_{1,1}(x - x_0) + d_{2,2}(x - x_0)(x - x_1) + \dots + d_{N,N}(x - x_0)(x - x_1) \cdots (x - x_{N-1}),$$

where

$$d_{k,0} = y_k \quad \text{and} \quad d_{k,j} = \frac{d_{k,j-1} - d_{k-1,j-1}}{x_k - x_{k-j}}.$$

```
function [C,D]=newpoly(X,Y)
%Input - X is a vector that contains a list of abscissas
%       - Y is a vector that contains a list of ordinates
%Output - C is a vector that contains the coefficients
%         of the Newton interpolatory polynomial
%       - D is the divided-difference table

n=length(X);
D=zeros(n,n);
D(:,1)=Y';
```

```

% Use formula (20) to form the divided-difference table
for j=2:n
    for k=j:n
        D(k,j)=(D(k,j-1)-D(k-1,j-1))/(X(k)-X(k-j+1));
    end
end
%Determine the coefficients of the Newton interpolating
%polynomial
C=D(n,n);
for k=(n-1):-1:1
    C=conv(C,poly(X(k)));
    m=length(C);
    C(m)=C(m)+D(k,k);
end

```

Exercises for Newton Polynomials

In Exercises 1 through 4, use the centers x_0, x_1, x_2 , and x_3 and the coefficients a_0, a_1, a_2, a_3 , and a_4 to find the Newton polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$, and evaluate them at the value $x = c$. *Hint:* Use equations (1) through (4) and the techniques of Example 4.9.

1. $a_0 = 4$ $a_1 = -1$ $a_2 = 0.4$ $a_3 = 0.01$ $a_4 = -0.002$
 $x_0 = 1$ $x_1 = 3$ $x_2 = 4$ $x_3 = 4.5$ $c = 2.5$
2. $a_0 = 5$ $a_1 = -2$ $a_2 = 0.5$ $a_3 = -0.1$ $a_4 = 0.003$
 $x_0 = 0$ $x_1 = 1$ $x_2 = 2$ $x_3 = 3$ $c = 2.5$
3. $a_0 = 7$ $a_1 = 3$ $a_2 = 0.1$ $a_3 = 0.05$ $a_4 = -0.04$
 $x_0 = -1$ $x_1 = 0$ $x_2 = 1$ $x_3 = 4$ $c = 3$
4. $a_0 = -2$ $a_1 = 4$ $a_2 = -0.04$ $a_3 = 0.06$ $a_4 = 0.005$
 $x_0 = -3$ $x_1 = -1$ $x_2 = 1$ $x_3 = 4$ $c = 2$

In Exercises 5 through 8:

- (a) Compute the divided-difference table for the tabulated function.
- (b) Write down the Newton polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$.
- (c) Evaluate the Newton polynomials in part (b) at the given values of x .
- (d) Compare the values in part (c) with the actual function value $f(x)$.

5. $f(x) = x^{1/2}$
 $x = 4.5, 7.5$

k	x_k	$f(x_k)$
0	4.0	2.00000
1	5.0	2.23607
2	6.0	2.44949
3	7.0	2.64575
4	8.0	2.82843

6. $f(x) = 3.6/x$
 $x = 2.5, 3.5$

k	x_k	$f(x_k)$
0	1.0	3.60
1	2.0	1.80
2	3.0	1.20
3	4.0	0.90
4	5.0	0.72

7. $f(x) = 3 \sin^2(\pi x/6)$
 $x = 1.5, 3.5$

k	x_k	$f(x_k)$
0	0.0	0.00
1	1.0	0.75
2	2.0	2.25
3	3.0	3.00
4	4.0	2.25

8. $f(x) = e^{-x}$
 $x = 0.5, 1.5$

k	x_k	$f(x_k)$
0	0.0	1.00000
1	1.0	0.36788
2	2.0	0.13534
3	3.0	0.04979
4	4.0	0.01832

9. Consider the $M + 1$ points $(x_0, y_0), \dots, (x_M, y_M)$.

- (a) If the $(N + 1)$ st divided differences are zero, then show that the $(N + 2)$ nd up to the M th divided differences are zero.
- (b) If the $(N + 1)$ st divided differences are zero, then show that there exists a polynomial $P_N(x)$ of degree N such that

$$P_N(x_k) = y_k \quad \text{for } k = 0, 1, \dots, M.$$

In Exercises 10 through 12, use the result of Exercise 9 to find the polynomial $P_N(x)$ that goes through the $M + 1$ points ($N < M$).

10.

x_k	y_k
0	-2
1	2
2	4
3	4
4	2
5	-2

11.

x_k	y_k
1	8
2	17
3	24
4	29
5	32
6	33

12.

x_k	y_k
0	5
1	5
2	3
3	5
4	17
5	45
6	95

13. Use Corollary 4.2 to find a bound on the maximum error ($|E_2(x)|$) on the interval $[0, \pi]$, when the Newton interpolatory polynomial $P_2(x)$ is used to approximate $f(x) = \cos(\pi x)$ at the centers $x_0 = 0$, $x_1 = \pi/2$, and $x_2 = \pi$.

Algorithms and Programs

1. Use Program 4.2 and repeat Problem 2 in Programs and Algorithms from Section 4.3.
2. In Program 4.2 the matrix D is used to store the divided-difference table.
 - (a) Verify that the following modification of Program 4.2 is an equivalent way to compute the Newton interpolatory polynomial.

```

for k=0:N
    A(k)=Y(k);
end
for j=1:N
    for k=N:-1:j
        A(k)=(A(k)-A(k-1))/(X(k)-X(k-j));
    end
end
      
```

 - (b) Repeat Problem 1 using this modification of Program 4.2

4.5 Chebyshev Polynomials (Optional)

We now turn our attention to polynomial interpolation for $f(x)$ over $[-1, 1]$ based on the nodes $-1 \leq x_0 < x_1 < \cdots < x_N \leq 1$. Both the Lagrange and Newton polynomials satisfy

$$f(x) = P_N(x) + E_N(x),$$

where

$$(1) \quad E_N(x) = Q(x) \frac{f^{(N+1)}(\xi)}{(N+1)!}$$

and $Q(x)$ is the polynomial of degree $N+1$:

$$(2) \quad Q(x) = (x - x_0)(x - x_1) \cdots (x - x_N).$$

Using the relationship

$$|E_N(x)| \leq |Q(x)| \frac{\max_{-1 \leq x \leq 1} \{|f^{(N+1)}(x)|\}}{(N+1)!}.$$

Table 4.11 Chebyshev Polynomials $T_0(x)$ through $T_7(x)$

$T_0(x) = 1$
$T_1(x) = x$
$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$
$T_4(x) = 8x^4 - 8x^2 + 1$
$T_5(x) = 16x^5 - 20x^3 + 5x$
$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

our task is to follow Chebyshev's derivation on how to select the set of nodes $\{x_k\}_{k=0}^N$ that minimizes $\max_{-1 \leq x \leq 1} \{|Q(x)|\}$. This leads us to a discussion of Chebyshev polynomials and some of their properties. To begin, the first eight Chebyshev polynomials are listed in Table 4.11.

Properties of Chebyshev Polynomials

Property 1. Recurrence relation

Chebyshev polynomials can be generated in the following way. Set $T_0(x) = 1$ and $T_1(x) = x$ and use the recurrence relation

$$(3) \quad T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x) \quad \text{for } k = 2, 3, \dots$$

Property 2. Leading Coefficient

The coefficient of x^N in $T_N(x)$ is 2^{N-1} when $N \geq 1$.

Property 3. Symmetry

When $N = 2M$, $T_{2M}(x)$ is an even function, that is,

$$(4) \quad T_{2M}(-x) = T_{2M}(x).$$

When $N = 2M + 1$, $T_{2M+1}(x)$ is an odd function, that is,

$$(5) \quad T_{2M+1}(-x) = -T_{2M+1}(x).$$

Property 4. Trigonometric Representation on $[-1, 1]$

$$(6) \quad T_N(x) = \cos(N \arccos(x)) \quad \text{for } -1 \leq x \leq 1.$$

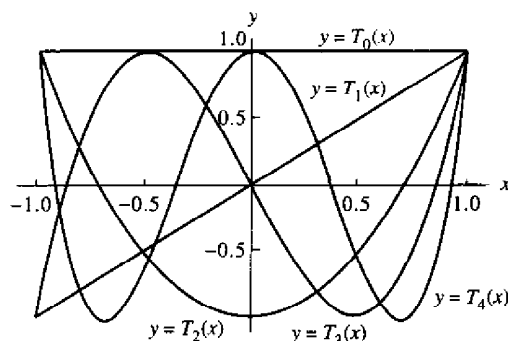


Figure 4.15 Graphs of the Chebyshev polynomials $T_0(x)$, $T_1(x)$, \dots , $T_4(x)$ over $[-1, 1]$.

Property 5. Distinct Zeros in $[-1, 1]$

$T_N(x)$ has N distinct zeros x_k that lie in the interval $[-1, 1]$ (see Figure 4.15):

$$(7) \quad x_k = \cos\left(\frac{(2k+1)\pi}{2N}\right) \quad \text{for } k = 0, 1, \dots, N-1.$$

These values are called the **Chebyshev abscissas (nodes)**.

Property 6. Extreme Values

$$(8) \quad |T_N(x)| \leq 1 \quad \text{for } -1 \leq x \leq 1.$$

Property 1 is often used as the definition for higher-order Chebyshev polynomials. Let us show that $T_3(x) = 2xT_2(x) - T_1(x)$. Using the expressions for $T_1(x)$ and $T_2(x)$ in Table 4.11, we obtain

$$2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x = T_3(x).$$

Property 2 is proved by observing that the recurrence relation doubles the leading coefficient of $T_{N-1}(x)$ to get the leading coefficient of $T_N(x)$.

Property 3 is established by showing that $T_{2M}(x)$ involves only even powers of x and $T_{2M+1}(x)$ involves only odd powers of x . The details are left for the reader.

The proof of property 4 uses the trigonometric identity

$$\cos(k\theta) = \cos(2\theta)\cos((k-2)\theta) - \sin(2\theta)\sin((k-2)\theta)$$

Substitute $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and get

$$\cos(k\theta) = 2\cos(\theta)(\cos(\theta)\cos((k-2)\theta) - \sin(\theta)\sin((k-2)\theta)) - \cos((k-2)\theta)$$

§4.5 CHEBYSHEV POLYNOMIALS

which is simplified as

$$\cos(k\theta) = 2\cos(\theta)\cos((k-1)\theta) - \cos((k-2)\theta).$$

Finally, substitute $\theta = \arccos(x)$ and obtain

$$(9) \quad 2x\cos((k-1)\arccos(x)) - \cos((k-2)\arccos(x)) \\ = \cos(k\arccos(x)) \quad \text{for } -1 \leq x \leq 1.$$

The first two Chebyshev polynomials are $T_0(x) = \cos(0\arccos(x)) = 1$ and $T_1(x) = \cos(1\arccos(x)) = x$. Now assume that $T_k(x) = \cos(k\arccos(x))$ for $k = 2, 3, \dots, N-1$. Formula (3) is used with (9) to establish the general case:

$$T_N(x) = 2xT_{N-1}(x) - T_{N-2}(x) \\ = 2x\cos((N-1)\arccos(x)) - \cos((N-2)\arccos(x)) \\ = \cos(N\arccos(x)) \quad \text{for } -1 \leq x \leq 1.$$

Properties 5 and 6 are consequences of Property 4.

Minimax

The Russian mathematician Chebyshev studied how to minimize the upper bound for $|E_N(x)|$. One upper bound can be formed by taking the product of the maximum value of $|Q(x)|$ over all x in $[-1, 1]$ and the maximum value $|f^{(N+1)}(x)/(N+1)!|$ over all x in $[-1, 1]$. To minimize the factor $\max\{|Q(x)|\}$, Chebyshev discovered that x_0, x_1, \dots, x_N should be chosen so that $Q(x) = (1/2^N)T_{N+1}(x)$.

Theorem 4.6. Assume that N is fixed. Among all possible choices for $Q(x)$ in equation (2), and thus among all possible choices for the distinct nodes $\{x_k\}_{k=0}^N$ in $[-1, 1]$, the polynomial $T(x) = T_{N+1}(x)/2^N$ is the unique choice that has the property

$$\max_{-1 \leq x \leq 1} \{|T(x)|\} \leq \max_{-1 \leq x \leq 1} \{|Q(x)|\}.$$

Moreover,

$$(10) \quad \max_{-1 \leq x \leq 1} \{|T(x)|\} = \frac{1}{2^N}.$$

Proof. The proof can be found in Reference [29].

Table 4.12 Lagrange Coefficient Polynomials Used to Form $P_3(x)$ Based on Equally Spaced Nodes $x_k = -1 + 2k/3$

$L_{3,0}(x) = -0.06250000 + 0.06250000x + 0.56250000x^2 - 0.56250000x^3$
$L_{3,1}(x) = 0.56250000 - 1.68750000x - 0.56250000x^2 + 1.68750000x^3$
$L_{3,2}(x) = 0.56250000 + 1.68750000x - 0.56250000x^2 - 1.68750000x^3$
$L_{3,3}(x) = -0.06250000 - 0.06250000x + 0.56250000x^2 + 0.56250000x^3$

The consequence of this result can be stated by saying that, for Lagrange interpolation $f(x) = P_N(x) + E_N(x)$ on $[-1, 1]$, the minimum value of the error bound

$$(\max\{|Q(x)|\})(\max\{|f^{(N+1)}(x)/(N+1)!\|})$$

is achieved when the nodes $\{x_k\}$ are the Chebyshev abscissas of $T_{N+1}(x)$. As an illustration, we look at the Lagrange coefficient polynomials that are used in forming $P_3(x)$. First we use equally spaced nodes and then the Chebyshev nodes. Recall that the Lagrange polynomial of degree $N = 3$ has the form

$$(11) \quad P_3(x) = f(x_0)L_{3,0}(x) + f(x_1)L_{3,1}(x) + f(x_2)L_{3,2}(x) + f(x_3)L_{3,3}(x).$$

Equally Spaced Nodes

If $f(x)$ is approximated by a polynomial of degree at most $N = 3$ on $[-1, 1]$, the equally spaced nodes $x_0 = -1$, $x_1 = -1/3$, $x_2 = 1/3$, and $x_3 = 1$ are easy to use for calculations. Substitution of these values into formula (8) of Section 4.3 and simplifying will produce the coefficient polynomials $L_{3,k}(x)$ in Table 4.12.

Chebyshev Nodes

When $f(x)$ is to be approximated by a polynomial of degree at most $N = 3$, using the Chebyshev nodes $x_0 = \cos(7\pi/8)$, $x_1 = \cos(5\pi/8)$, $x_2 = \cos(3\pi/8)$, and $x_3 = \cos(\pi/8)$, the coefficient polynomials are tedious to find (but this can be done by a computer). The results after simplification are shown in Table 4.13.

Example 4.14. Compare the Lagrange polynomials of degree $N = 3$ for $f(x) = e^x$ that are obtained by using the coefficient polynomials in Tables 4.12 and 4.13, respectively.

Using equally spaced nodes, we get the polynomial

$$P(x) = 0.99519577 + 0.99904923x + 0.54788486x^2 + 0.17615196x^3.$$

This is obtained by finding the function values

$$\begin{aligned} f(x_0) &= e^{(-1)} = 0.36787944, & f(x_1) &= e^{(-1/3)} = 0.71653131, \\ f(x_2) &= e^{(1/3)} = 1.39561243, & f(x_3) &= e^{(1)} = 2.71828183, \end{aligned}$$

Table 4.13 Coefficient Polynomials Used to Form $P_3(x)$ Based on the Chebyshev Nodes $x_k = \cos((7 - 2k)\pi/8)$

$C_0(x) = -0.10355339 + 0.11208538x + 0.70710678x^2 - 0.76536686x^3$
$C_1(x) = 0.60355339 - 1.57716102x - 0.70710678x^2 + 1.84775906x^3$
$C_2(x) = 0.60355339 + 1.57716102x - 0.70710678x^2 - 1.84775906x^3$
$C_3(x) = -0.10355339 - 0.11208538x + 0.70710678x^2 + 0.76536686x^3$

and using the coefficient polynomials $L_{3,k}(x)$ in Table 4.12, and forming the linear combination

$$P(x) = 0.36787944L_{3,0}(x) + 0.71653131L_{3,1}(x) + 1.39561243L_{3,2}(x) + 2.71828183L_{3,3}(x).$$

Similarly, when the Chebyshev nodes are used, we obtain

$$V(x) = 0.99461532 + 0.99893323x + 0.54290072x^2 + 0.17517569x^3.$$

Notice that the coefficients are different from those of $P(x)$. This is a consequence of using different nodes and function values:

$$\begin{aligned} f(x_0) &= e^{-0.92387953} = 0.39697597, \\ f(x_1) &= e^{-0.38268343} = 0.68202877, \\ f(x_2) &= e^{0.38268343} = 1.46621380, \\ f(x_3) &= e^{0.92387953} = 2.51904417. \end{aligned}$$

Then the alternative set of coefficient polynomials $C_k(x)$ in Table 4.13 is used to form the linear combination

$$V(x) = 0.39697597C_0(x) + 0.68202877C_1(x) + 1.46621380C_2(x) + 2.51904417C_3(x).$$

For a comparison of the accuracy of $P(x)$ and $V(x)$, the error functions are graphed in Figure 4.16(a) and (b), respectively. The maximum error $|e^x - P(x)|$ occurs at $x = 0.75490129$, and

$$|e^x - P(x)| \leq 0.00998481 \quad \text{for } -1 \leq x \leq 1.$$

The maximum error $|e^x - V(x)|$ occurs at $x = 1$, and we get

$$|e^x - V(x)| \leq 0.00665687 \quad \text{for } -1 \leq x \leq 1.$$

Notice that the maximum error in $V(x)$ is about two-thirds the maximum error in $P(x)$. Also, the error is spread out more evenly over the interval. ■

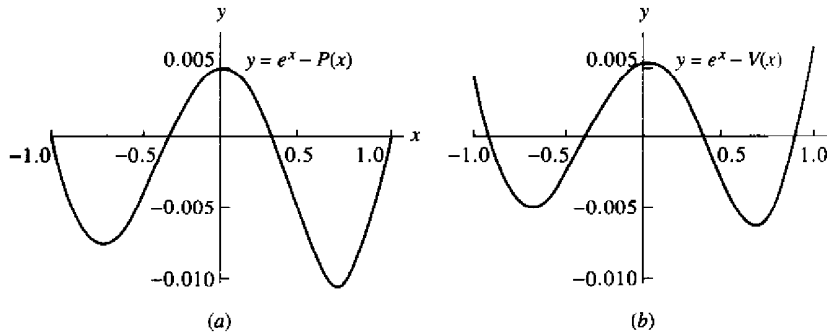


Figure 4.16 (a) The error function $y = e^x - P(x)$ for Lagrange approximation over $[-1, 1]$
 (b) The error function $y = e^x - V(x)$ for Lagrange approximation over $[-1, 1]$.

Runge Phenomenon

We now look deeper to see the advantage of using the Chebyshev interpolation nodes. Consider Lagrange interpolating to $f(x)$ over the interval $[-1, 1]$ based on equally spaced nodes. Does the error $E_N(x) = f(x) - P_N(x)$ tend to zero as N increases? For functions like $\sin(x)$ or e^x , where all the derivatives are bounded by the same constant M , the answer is yes. In general, the answer to this question is no, and it is easy to find functions for which the sequence $\{P_N(x)\}$ does not converge. If $f(x) = 1/(1 + 12x^2)$, the maximum of the error term $E_N(x)$ grows when $N \rightarrow \infty$. This nonconvergence is called the **Runge phenomenon** (see Reference [90], pp. 275–278). The Lagrange polynomial of degree 10 based on 11 equally spaced nodes for this function is shown in Figure 4.17(a). Wild oscillations occur near the end of the interval. If the number of nodes is increased, then the oscillations become larger. This problem occurs because the nodes are equally spaced!

If the Chebyshev nodes are used to construct an interpolating polynomial of degree 10 to $f(x) = 1/(1 + 12x^2)$, the error is much smaller, as seen in Figure 4.17(b). Under the condition that Chebyshev nodes be used, the error $E_N(x)$ will go to zero as $N \rightarrow \infty$. In general, if $f(x)$ and $f'(x)$ are continuous on $[-1, 1]$, then it can be proved that Chebyshev interpolation will produce a sequence of polynomials $\{P_N(x)\}$ that converges uniformly to $f(x)$ over $[-1, 1]$.

Transforming the Interval

Sometimes it is necessary to take a problem stated on an interval $[a, b]$ and reformulate the problem on the interval $[c, d]$ where the solution is known. If the approximation $P_N(x)$ to $f(x)$ is to be obtained on the interval $[a, b]$, then we change the variable

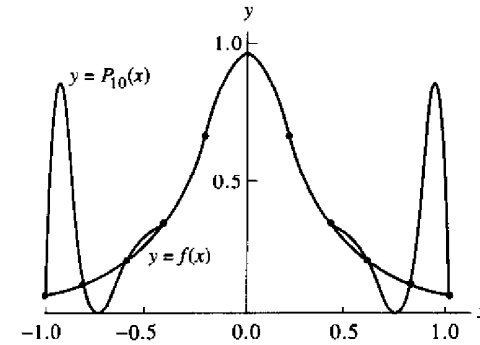


Figure 4.17 (a) The polynomial approximation to $y = 1/(1 + 12x^2)$ based on 11 equally spaced nodes over $[-1, 1]$.

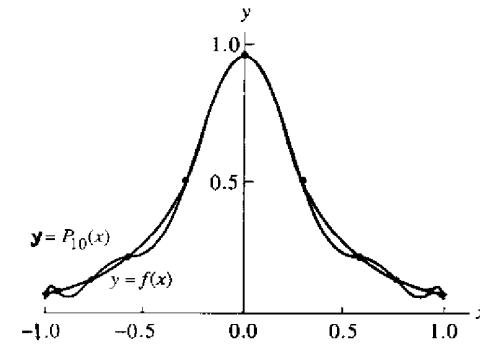


Figure 4.17 (b) The polynomial approximation to $y = 1/(1 + 12x^2)$ based on 11 Chebyshev nodes over $[-1, 1]$.

so that the problem is reformulated on $[-1, 1]$:

$$(12) \quad x = \left(\frac{b-a}{2}\right)t + \frac{a+b}{2} \quad \text{or} \quad t = 2\frac{x-a}{b-a} - 1.$$

where $a \leq x \leq b$ and $-1 \leq t \leq 1$.

The required Chebyshev nodes of $T_{N+1}(t)$ on $[-1, 1]$ are

$$(13) \quad t_k = \cos\left((2N+1-2k)\frac{\pi}{2N+2}\right) \quad \text{for } k = 0, 1, \dots, N$$

and the interpolating nodes on $[a, b]$ are obtained by using (12):

$$x_k = t_k \frac{b-a}{2} + \frac{a+b}{2} \quad \text{for } k = 0, 1, \dots, N.$$

Theorem 4.7 (Lagrange-Chebyshev Approximation Polynomial). Assume that $P_N(x)$ is the Lagrange polynomial that is based on the Chebyshev nodes given in (14). If $f \in C^{N+1}[a, b]$, then

$$(15) \quad |f(x) - P_N(x)| \leq \frac{2(b-a)^{N+1}}{4^{N+1}(N+1)!} \max_{a \leq x \leq b} \{|f^{(N+1)}(x)|\}.$$

Example 4.15. For $f(x) = \sin(x)$ on $[0, \pi/4]$, find the Chebyshev nodes and the error bound (15) for the Lagrange polynomial $P_5(x)$.

Formulas (12) and (13) are used to find the nodes;

$$x_k = \cos\left(\frac{(11-2k)\pi}{12}\right) \frac{\pi}{8} + \frac{\pi}{8} \quad \text{for } k = 0, 1, \dots, 5.$$

Using the bound $|f^{(6)}(x)| \leq |\sin(\pi/4)| = 2^{-1/2} = M$ in (15), we get

$$|f(x) - P_N(x)| \leq \left(\frac{\pi}{8}\right)^6 \left(\frac{2}{6!}\right) 2^{-1/2} \leq 0.00000720. \quad \blacksquare$$

Orthogonal Property

In Example 4.14, the Chebyshev nodes were used to find the Lagrange interpolating polynomial. In general, this implies that the Chebyshev polynomial of degree N can be obtained by Lagrange interpolation based on the $N+1$ nodes that are the $N+1$ zeros of $T_{N+1}(x)$. However, a direct approach to finding the approximation polynomial is to express $P_N(x)$ as a linear combination of the polynomials $T_k(x)$, which were given in Table 4.11. Therefore, the Chebyshev interpolating polynomial can be written in the form

$$(16) \quad P_N(x) = \sum_{k=0}^N c_k T_k(x) = c_0 T_0(x) + c_1 T_1(x) + \dots + c_N T_N(x).$$

The coefficients $\{c_k\}$ in (16) are easy to find. The technical proof requires the use of the following orthogonality properties. Let

$$(17) \quad x_k = \cos\left(\pi \frac{2k+1}{2N+2}\right) \quad \text{for } k = 0, 1, \dots, N;$$

$$(18) \quad \sum_{k=0}^N T_i(x_k) T_j(x_k) = 0 \quad \text{when } i \neq j,$$

$$(19) \quad \sum_{k=0}^N T_i(x_k) T_j(x_k) = \frac{N+1}{2} \quad \text{when } i = j \neq 0,$$

$$(20) \quad \sum_{k=0}^N T_0(x_k) T_0(x_k) = N+1.$$

Property 4 and the identities (18) and (20) can be used to prove the following theorem.

Theorem 4.8 (Chebyshev Approximation). The Chebyshev approximation polynomial $P_N(x)$ of degree $\leq N$ for $f(x)$ over $[-1, 1]$ can be written as a sum of $\{T_j(x)\}$:

$$(21) \quad f(x) \approx P_N(x) = \sum_{j=1}^N c_j T_j(x).$$

The coefficients $\{c_j\}$ are computed with the formulas

$$(22) \quad c_0 = \frac{1}{N+1} \sum_{k=0}^N f(x_k) T_0(x_k) = \frac{1}{N+1} \sum_{k=0}^N f(x_k)$$

and

$$(23) \quad c_j = \frac{2}{N+1} \sum_{k=0}^N f(x_k) T_j(x_k) \\ = \frac{2}{N+1} \sum_{k=0}^N f(x_k) \cos\left(\frac{j\pi(2k+1)}{2N+2}\right) \quad \text{for } j = 1, 2, \dots, N.$$

Example 4.16. Find the Chebyshev polynomial $P_3(x)$ that approximates the function $f(x) = e^x$ over $[-1, 1]$.

The coefficients are calculated using formulas (22) and (23), and the nodes $x_k = \cos(\pi(2k+1)/8)$ for $k = 0, 1, 2, 3$.

$$c_0 = \frac{1}{4} \sum_{k=0}^3 e^{x_k} T_0(x_k) = \frac{1}{4} \sum_{k=0}^3 e^{x_k} = 1.26606568,$$

$$c_1 = \frac{1}{2} \sum_{k=0}^3 e^{x_k} T_1(x_k) = \frac{1}{2} \sum_{k=0}^3 e^{x_k} x_k = 1.13031500,$$

$$c_2 = \frac{1}{2} \sum_{k=0}^3 e^{x_k} T_2(x_k) = \frac{1}{2} \sum_{k=0}^3 e^{x_k} \cos\left(2\pi \frac{2k+1}{8}\right) = 0.27145036,$$

$$c_3 = \frac{1}{2} \sum_{k=0}^3 e^{x_k} T_3(x_k) = \frac{1}{2} \sum_{k=0}^3 e^{x_k} \cos\left(3\pi \frac{2k+1}{8}\right) = 0.04379392.$$

Therefore, the Chebyshev polynomial $P_3(x)$ for e^x is

$$(24) \quad P_3(x) = 1.26606568 T_0(x) + 1.13031500 T_1(x) \\ + 0.27145036 T_2(x) + 0.04379392 T_3(x).$$

If the Chebyshev polynomial (24) is expanded in powers of x , the result is

$$P_3(x) = 0.99461532 + 0.99893324x + 0.54290072x^2 + 0.17517568x^3,$$

which is the same as the polynomial $V(x)$ in Example 4.14. If the goal is to find the Chebyshev polynomial, formulas (22) and (23) are preferred. \blacksquare

MATLAB

The following program uses the `eval` command instead of the `feval` command used in earlier programs. The `eval` command interprets a MATLAB text string as an expression or statement. For example, the following commands will quickly evaluate cosine at the values $x = k/10$ for $k = 0, 1, \dots, 5$:

```
>> x=0:.1:.5;
>> eval('cos(x)')
ans =
    1.0000    0.9950    0.9801    0.9553    0.9211    0.8776
```

Program 4.3 (Chebyshev Approximation). To construct and evaluate the Chebyshev interpolating polynomial of degree N over the interval $[-1, 1]$, where

$$P(x) = \sum_{j=0}^N c_j T_j(x)$$

is based on the nodes

$$x_k = \cos\left(\frac{(2k+1)\pi}{2N+2}\right).$$

```
function [C,X,Y]=cheby(fun,n,a,b)
%Input - fun is the string function to be approximated
%       - N is the degree of the Chebyshev interpolating
%       polynomial
%       - a is the left end point
%       - b is the right end point
%Output - C is the coefficient list for the polynomial
%       - X contains the abscissas
%       - Y contains the ordinates
if nargin==2, a=-1;b=1;end
d=pi/(2*n+2);
C=zeros(1,n+1);
for k=1:n+1
    X(k)=cos((2*k-1)*d);
end
X=(b-a)*X/2+(a+b)/2;
x=X;
Y=eval(fun);
for k =1:n+1
    z=(2*k-1)*d;
    for j=1:n+1
```

```
        C(j)=C(j)+Y(k)*cos((j-1)*z);
    end
end
C=2*C/(n+1);
C(1)=C(1)/2;
```

Exercises for Chebyshev Polynomials (Optional)

- Use property 1 and
 - construct $T_4(x)$ from $T_3(x)$ and $T_2(x)$.
 - construct $T_5(x)$ from $T_4(x)$ and $T_3(x)$.
- Use property 1 and
 - construct $T_6(x)$ from $T_5(x)$ and $T_4(x)$.
 - construct $T_7(x)$ from $T_6(x)$ and $T_5(x)$.
- Use mathematical induction to prove property 2.
- Use mathematical induction to prove property 3.
- Find the maximum and minimum values of $T_2(x)$.
- Find the maximum and minimum values of $T_3(x)$.
Hint: $T_3'(1/2) = 0$ and $T_3'(-1/2) = 0$.
- Find the maximum and minimum values of $T_4(x)$.
Hint: $T_4'(0) = 0$, $T_4'(2^{-1/2}) = 0$, and $T_4'(-2^{-1/2}) = 0$.
- Let $f(x) = \sin(x)$ on $[-1, 1]$.
 - Use the coefficient polynomials in Table 4.13 to obtain the Lagrange-Chebyshev polynomial approximation $P_3(x)$.
 - Find the error bound for $|\sin(x) - P_3(x)|$.
- Let $f(x) = \ln(x+2)$ on $[-1, 1]$.
 - Use the coefficient polynomials in Table 4.13 to obtain the Lagrange-Chebyshev polynomial approximation $P_3(x)$.
 - Find the error bound for $|\ln(x+2) - P_3(x)|$.
- The Lagrange polynomial of degree $N = 2$ has the form

$$f(x) = f(x_0)L_{2,0}(x) + f(x_1)L_{2,1}(x) + f(x_2)L_{2,2}(x).$$

If the Chebyshev nodes $x_0 = \cos(5\pi/6)$, $x_1 = 0$, and $x_2 = \cos(\pi/6)$ are used, show

that the coefficient polynomials are

$$L_{2,0}(x) = -\frac{x}{\sqrt{3}} + \frac{2x^2}{3},$$

$$L_{2,1}(x) = 1 - \frac{4x^2}{3},$$

$$L_{2,2}(x) = \frac{x}{\sqrt{3}} + \frac{2x^2}{3}.$$

11. Let $f(x) = \cos(x)$ on $[-1, 1]$.
 - (a) Use the coefficient polynomials in Exercise 10 to get the Lagrange-Chebyshev polynomial approximation $P_2(x)$.
 - (b) Find the error bound for $|\cos(x) - P_2(x)|$.
12. Let $f(x) = e^x$ on $[-1, 1]$.
 - (a) Use the coefficient polynomials in Exercise 10 to get the Lagrange-Chebyshev polynomial approximation $P_2(x)$.
 - (b) Find the error bound for $|e^x - P_2(x)|$.

In Exercises 13 through 15, compare the Taylor polynomial and the Lagrange-Chebyshev approximates to $f(x)$ on $[-1, 1]$. Find their error bounds.

13. $f(x) = \sin(x)$ and $N = 7$; the Lagrange-Chebyshev polynomial is

$$\sin(x) \approx 0.99999998x - 0.16666599x^2 + 0.00832995x^5 - 0.00019297x^7.$$
14. $f(x) = \cos(x)$ and $N = 6$; the Lagrange-Chebyshev polynomial is

$$\cos(x) \approx 1 - 0.49999734x^2 + 0.04164535x^4 - 0.00134608x^6.$$
15. $f(x) = e^x$ and $N = 7$; the Lagrange-Chebyshev polynomial is

$$\begin{aligned} e^x \approx & 0.99999980 + 0.99999998x + 0.50000634x^2 \\ & + 0.16666737x^3 + 0.04163504x^4 + 0.00832984x^5 \\ & + 0.00143925x^6 + 0.00020399x^7. \end{aligned}$$

16. Prove equation (18).
17. Prove equation (19).

Algorithms and Programs

In Problems 1 through 6, use Program 4.3 to compute the coefficients $\{c_k\}$ for the Chebyshev polynomial approximation $P_N(x)$ to $f(x)$ over $[-1, 1]$, when (a) $N = 4$, (b) $N = 5$, (c) $N = 6$, and (d) $N = 7$. In each case, plot $f(x)$ and $P_N(x)$ on the same coordinate system.

1. $f(x) = e^x$
2. $f(x) = \sin(x)$

3. $f(x) = \cos(x)$
4. $f(x) = \ln(x+2)$
5. $f(x) = (x+2)^{1/2}$
6. $f(x) = (x+2)^{(x+2)}$
7. Use Program 4.3 ($N = 5$) to obtain an approximation for $\int_0^1 \cos(x^2) dx$.

4.6 Padé Approximations

In this section we introduce the notion of rational approximations for functions. The function $f(x)$ will be approximated over a small portion of its domain. For example, if $f(x) = \cos(x)$, it is sufficient to have a formula to generate approximations on the interval $[0, \pi/2]$. Then trigonometric identities can be used to compute $\cos(x)$ for any value x that lies outside $[0, \pi/2]$.

A rational approximation to $f(x)$ on $[a, b]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M , respectively. We use the notation $R_{N,M}(x)$ to denote this quotient:

$$(1) \quad R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \leq x \leq b.$$

Our goal is to make the maximum error as small as possible. For a given amount of computational effort, one can usually construct a rational approximation that has a smaller overall error on $[a, b]$ than a polynomial approximation. Our development is an introduction and will be limited to Padé approximations.

The *method of Padé* requires that $f(x)$ and its derivative be continuous at $x = 0$. There are two reasons for the arbitrary choice of $x = 0$. First, it makes the manipulations simpler. Second, a change of variable can be used to shift the calculations over to an interval that contains zero. The polynomials used in (1) are

$$(2) \quad P_N(x) = p_0 + p_1x + p_2x^2 + \cdots + p_Nx^N$$

and

$$(3) \quad Q_M(x) = 1 + q_1x + q_2x^2 + \cdots + q_Mx^M.$$

The polynomials in (2) and (3) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x = 0$ and their derivatives up to $N + M$ agree at $x = 0$. In the case $Q_0(x) = 1$, the approximation is just the Maclaurin expansion for $f(x)$. For a fixed value of $N + M$ the error is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher than $Q_M(x)$.

Notice that the constant coefficient of Q_M is $q_0 = 1$. This is permissible, because it cannot be 0 and $R_{N,M}(x)$ is not changed when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence the rational function $R_{N,M}(x)$ has $N + M + 1$ unknown coefficients. Assume that $f(x)$ is analytic and has the Maclaurin expansion

$$(4) \quad f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots,$$

and form the difference $f(x)Q_M(x) - P_N(x) = Z(x)$:

$$(5) \quad \left(\sum_{j=0}^{\infty} a_j x^j \right) \left(\sum_{j=0}^M q_j x^j \right) - \sum_{j=0}^N p_j x^j = \sum_{j=N+M+1}^{\infty} c_j x^j.$$

The lower index $j = M + N + 1$ in the summation on the right side of (5) is chosen because the first $N + M$ derivatives of $f(x)$ and $R_{N,M}(x)$ are to agree at $x = 0$.

When the left side of (5) is multiplied out and the coefficients of the powers of x are set equal to zero for $k = 0, 1, \dots, N + M$, the result is a system of $N + M + 1$ linear equations:

$$(6) \quad \begin{aligned} a_0 - p_0 &= 0 \\ q_1 a_0 + a_1 - p_1 &= 0 \\ q_2 a_0 + q_1 a_1 + a_2 - p_2 &= 0 \\ q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - p_3 &= 0 \\ &\vdots \\ q_M a_{N-M} + q_{M-1} a_{N-M+1} + \cdots + a_N - p_N &= 0 \end{aligned}$$

and

$$(7) \quad \begin{aligned} q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + \cdots + q_1 a_N + a_{N+1} &= 0 \\ q_M a_{N-M+2} + q_{M-1} a_{N-M+3} + \cdots + q_1 a_{N+1} + a_{N+2} &= 0 \\ &\vdots \\ q_M a_N + q_{M-1} a_{N+1} + \cdots + q_1 a_{N+M-1} + a_{N+M} &= 0. \end{aligned}$$

Notice that in each equation the sum of the subscripts on the factors of each product is the same, and this sum increases consecutively from 0 to $N + M$. The M equations in (7) involve only the unknowns q_1, q_2, \dots, q_M and must be solved first. Then the equations in (6) are used successively to find p_0, p_1, \dots, p_N .

Example 4.17. Establish the Padé approximation

$$(8) \quad \cos(x) \approx R_{4,4}(x) = \frac{15,120 - 6900x^2 + 313x^4}{15,120 + 660x^2 + 13x^4}.$$

See Figure 4.18 for the graphs of $\cos(x)$ and $R_{4,4}(x)$ over $[-5, 5]$.

If the Maclaurin expansion for $\cos(x)$ is used, we will obtain nine equations in nine unknowns. Instead, notice that both $\cos(x)$ and $R_{4,4}(x)$ are even functions and involve powers of x^2 . We can simplify the computations if we start with $f(x) = \cos(x^{1/2})$:

$$(9) \quad f(x) = 1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \frac{1}{40,320}x^4 - \cdots.$$

In this case, equation (5) becomes

$$\begin{aligned} \left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \frac{1}{40,320}x^4 - \cdots \right) (1 + q_1x + q_2x^2) - p_0 - p_1x - p_2x^2 \\ = 0 + 0x + 0x^2 + 0x^3 + 0x^4 + c_5x^5 + c_6x^6 + \cdots. \end{aligned}$$

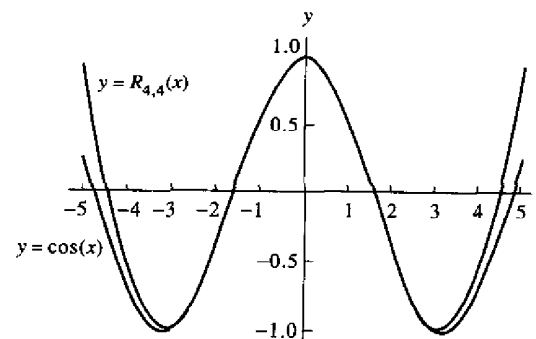


Figure 4.18 The graph of $y = \cos(x)$ and its Padé approximation $R_{4,4}(x)$.

When the coefficients of the first five powers of x are compared, we get the following system of linear equations:

$$(10) \quad \begin{aligned} 1 - p_0 &= 0 \\ -\frac{1}{2} + q_1 - p_1 &= 0 \\ \frac{1}{24} - \frac{1}{2}q_1 + q_2 - p_2 &= 0 \\ -\frac{1}{720} + \frac{1}{24}q_1 - \frac{1}{2}q_2 &= 0 \\ \frac{1}{40,320} - \frac{1}{720}q_1 + \frac{1}{24}q_2 &= 0. \end{aligned}$$

The last two equations in (10) must be solved first. They can be rewritten in a form that is easy to solve:

$$q_1 - 12q_2 = \frac{1}{30} \quad \text{and} \quad -q_1 + 30q_2 = -\frac{1}{56}.$$

First find q_2 by adding the equations; then find q_1

$$(11) \quad \begin{aligned} q_2 &= \frac{1}{18} \left(\frac{1}{30} - \frac{1}{56} \right) = \frac{13}{15,120}, \\ q_1 &= \frac{1}{30} + \frac{156}{15,120} = \frac{11}{252}. \end{aligned}$$

Now the first three equations of (10) are used. It is obvious that $p_0 = 1$, and we can

use q_1 and q_2 in (11) to solve for p_1 and p_2 :

$$(12) \quad \begin{aligned} p_1 &= -\frac{1}{2} + \frac{11}{252} = -\frac{115}{252}, \\ p_2 &= \frac{1}{24} - \frac{11}{504} + \frac{13}{15,120} = \frac{313}{15,120}. \end{aligned}$$

Now use the coefficients in (11) and (12) to form the rational approximation to $f(x)$:

$$(13) \quad f(x) \approx \frac{1 - 115x/252 + 313x^2/15,120}{1 + 11x/252 + 13x^2/15,120}.$$

Since $\cos(x) = f(x^2)$, we can substitute x^2 for x in equation (13) and the result is the formula for $R_{4,4}(x)$ in (8). ■

Continued Fraction Form

The Padé approximation $R_{4,4}(x)$ in Example 4.17 requires a minimum of 12 arithmetic operations to perform an evaluation. It is possible to reduce this number to seven by the use of continued fractions. This is accomplished by starting with (8) and finding the quotient and its polynomial remainder.

$$\begin{aligned} R_{4,4}(x) &= \frac{15,120/313 - (6900/313)x^2 + x^4}{15,120/13 + (660/13)x^2 + x^4} \\ &= \frac{313}{13} - \left(\frac{296,280}{169} \right) \left(\frac{12,600/823 + x^2}{15,120/13 + (600/13)x^2 + x^4} \right). \end{aligned}$$

The process is carried out once more using the term in the previous remainder. The result is

$$\begin{aligned} R_{4,4}(x) &= \frac{313}{13} - \frac{296,280/169}{\frac{15,120/13 + (660/13)x^2 + x^4}{12,600/823 + x^2}} \\ &= \frac{313}{13} - \frac{296,280/169}{\frac{379,380}{10,699} + x^2 + \frac{420,078,960/677,329}{12,600/823 + x^2}}. \end{aligned}$$

The fractions are converted to decimal form for computational purposes and we obtain

$$(14) \quad \begin{aligned} R_{4,4}(x) &= 24.07692308 \\ &\sim \frac{1753.13609467}{35.45938873 + x^2 + 620.19928277/(15.30984204 + x^2)}. \end{aligned}$$

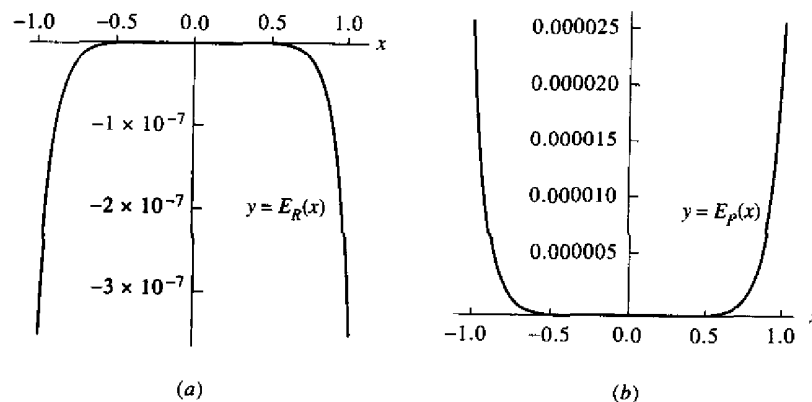


Figure 4.19 (a) Graph of the error $E_R(x) = \cos(x) - R_{4,4}(x)$ for the Padé approximation $R_{4,4}(x)$. (b) Graph of the error $E_P(x) = \cos(x) - P_6(x)$ for the Taylor approximation $P_6(x)$.

To evaluate (14), first compute and store x^2 , then proceed from the bottom right term in the denominator and tally the operations: addition, division, addition, addition, division, and subtraction. Hence it takes a total of seven arithmetic operations to evaluate $R_{4,4}(x)$ in continued fraction form in (14).

We can compare $R_{4,4}(x)$ with the Taylor polynomial $P_6(x)$ of degree $N = 6$, which requires seven arithmetic operations to evaluate when it is written in the nested form

$$(15) \quad \begin{aligned} P_6(x) &= 1 + x^2 \left(-\frac{1}{2} + x^2 \left(\frac{1}{24} - \frac{1}{720}x^2 \right) \right) \\ &= 1 + x^2(-0.5 + x^2(0.0416666667 - 0.0013888889x^2)). \end{aligned}$$

The graphs of $E_R(x) = \cos(x) - R_{4,4}(x)$ and $E_P(x) = \cos(x) - P_6(x)$ over $[-1, 1]$ are shown in Figure 4.19(a) and (b), respectively. The largest errors occur at the end points and are $E_R(1) = -0.0000003599$ and $E_P(1) = 0.0000245281$, respectively. The magnitude of the largest error for $R_{4,4}(x)$ is about 1.467% of the error for $P_6(x)$. The Padé approximation outperforms the Taylor approximation better on smaller intervals, and over $[-0.1, 0.1]$ we find that $E_R(0.1) = -0.0000000004$ and $E_P(0.1) = 0.0000000966$, so the magnitude of the error for $R_{4,4}(x)$ is about 0.384% of the magnitude of the error for $P_6(x)$.

Exercises for Padé Approximations

1. Establish the Padé approximation:

$$e^x \approx R_{1,1}(x) = \frac{2+x}{2-x}.$$

2. (a) Find the Padé approximation $R_{1,1}(x)$ for $f(x) = \ln(1+x)/x$. *Hint.* Start with the Maclaurin expansion:

$$f(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \dots$$

- (b) Use the result in part (a) to establish the approximation

$$\ln(1+x) \approx R_{2,1}(x) = \frac{6x+x^2}{6+4x}.$$

3. (a) Find $R_{1,1}(x)$ for $f(x) = \tan(x^{1/2})/x^{1/2}$. *Hint.* Start with the Maclaurin expansion:

$$f(x) = 1 + \frac{x}{3} + \frac{2x^2}{15} + \dots$$

- (b) Use the result in part (a) to establish the approximation

$$\tan(x) \approx R_{3,2}(x) = \frac{15x-x^3}{15-6x^2}.$$

4. (a) Find $R_{1,1}(x)$ for $f(x) = \arctan(x^{1/2})/x^{1/2}$. *Hint.* Start with the Maclaurin expansion:

$$f(x) = 1 - \frac{x}{3} + \frac{x^2}{5} - \dots$$

- (b) Use the result in part (a) to establish the approximation

$$\arctan(x) \approx R_{3,2}(x) = \frac{15x+4x^3}{15+9x^2}.$$

- (c) Express the rational function $R_{3,2}(x)$ in part (b) in continued fraction form.

5. (a) Establish the Padé approximation:

$$e^x \approx R_{2,2}(x) = \frac{12+6x+x^2}{12-6x+x^2}.$$

- (b) Express the rational function $R_{2,2}(x)$ in part (a) in continued fraction form.

6. (a) Find the Padé approximation $R_{2,2}(x)$ for $f(x) = \ln(1+x)/x$. *Hint.* Start with the Maclaurin expansion:

$$f(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \dots$$

- (b) Use the result in part (a) to establish

$$\ln(1+x) \approx R_{3,2}(x) = \frac{30x+21x^2+x^3}{30+36x+9x^2}.$$

- (c) Express the rational function $R_{3,2}(x)$ in part (b) in continued fraction form.

- (a) Find $R_{2,2}(x)$ for $f(x) = \tan(x^{1/2})/x^{1/2}$. *Hint.* Start with the Maclaurin expansion:

$$f(x) = 1 + \frac{x}{3} + \frac{2x^2}{15} + \frac{17x^3}{315} + \frac{62x^4}{2835} + \dots$$

- (b) Use the result in part (a) to establish

$$\tan(x) \approx R_{5,4}(x) = \frac{945x-105x^3+x^5}{945-420x^2+15x^4}.$$

- (c) Express the rational function $R_{5,4}(x)$ in part (b) in continued fraction form.

7. (a) Find $R_{2,2}(x)$ for $f(x) = \arctan(x^{1/2})/x^{1/2}$. *Hint.* Start with the Maclaurin expansion:

$$f(x) = 1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \dots$$

- (b) Use the result in part (a) to establish

$$\arctan(x) \approx R_{5,4}(x) = \frac{945x+735x^3+64x^5}{945+1050x^2+225x^4}.$$

- (c) Express the rational function $R_{5,4}(x)$ in part (b) in continued fraction form.

8. Establish the Padé approximation:

$$e^x \approx R_{3,3}(x) = \frac{120+60x+12x^2+x^3}{120-60x+12x^2+x^3}.$$

9. Establish the Padé approximation:

$$e^x \approx R_{4,4}(x) = \frac{1680+840x+180x^2+20x^3+x^4}{1680-840x+180x^2-20x^3+x^4}.$$

Algorithms and Programs

1. Compare the following approximations to $f(x) = e^x$.

$$\text{Taylor: } T_6(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$\text{Padé: } R_{2,2}(x) = \frac{12 + 6x + x^2}{12 - 6x + x^2}$$

- (a) Plot $f(x)$, $T_6(x)$, and $R_{2,2}(x)$ on the same coordinate system.
 (b) Determine the maximum error that occurs when $f(x)$ is approximated with $T_6(x)$ and $R_{2,2}(x)$, respectively, over the interval $[-1, 1]$.
2. Compare the following approximations to $f(x) = \ln(1+x)$.

$$\text{Taylor: } T_5(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

$$\text{Padé: } R_{3,2}(x) = \frac{30x + 21x^2 + x^3}{30 + 36x + 9x^2}$$

- (a) Plot $f(x)$, $T_5(x)$, and $R_{3,2}(x)$ on the same coordinate system.
 (b) Determine the maximum error that occurs when $f(x)$ is approximated with $T_5(x)$ and $R_{3,2}(x)$, respectively, over the interval $[-1, 1]$.
3. Compare the following approximations to $f(x) = \tan(x)$.

$$\text{Taylor: } T_9(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835}$$

$$\text{Padé: } R_{5,4}(x) = \frac{945x - 105x^3 + x^5}{945 - 420x^2 + 15x^4}$$

- (a) Plot $f(x)$, $T_9(x)$, and $R_{5,4}(x)$ on the same coordinate system.
 (b) Determine the maximum error that occurs when $f(x)$ is approximated with $T_9(x)$ and $R_{5,4}(x)$, respectively, over the interval $[-1, 1]$.
4. Compare the following Padé approximations to $f(x) = \sin(x)$ over the interval $[-1.2, 1.2]$.

$$R_{5,4}(x) = \frac{166,320x - 22,260x^3 + 551x^5}{15(11,088 + 364x^2 + 5x^4)}$$

$$R_{7,6}(x) = \frac{11,511,339,840x - 1,640,635,920x^2 + 52,785,432x^5 - 479,249x^7}{7(1,644,477,120 + 39,702,960x^2 + 453,960x^4 + 2,623x^6)}$$

- (a) Plot $f(x)$, $R_{5,4}(x)$, and $R_{7,6}(x)$ on the same coordinate system.
 (b) Determine the maximum error that occurs when $f(x)$ is approximated with $R_{5,4}(x)$ and $R_{7,6}(x)$, respectively, over the interval $[-1.2, 1.2]$.

5. (a) Use equations (6) and (7) to derive $R_{6,6}(x)$ and $R_{8,8}(x)$ for $f(x) = \cos(x)$ over the interval $[-1.2, 1.2]$.
 (b) Plot $f(x)$, $R_{6,6}(x)$, and $R_{8,8}(x)$ on the same coordinate system.
 (c) Determine the maximum error that occurs when $f(x)$ is approximated with $R_{6,6}(x)$ and $R_{8,8}(x)$, respectively, over the interval $[-1.2, 1.2]$.