3/1/05 117

Chapter

# :- Affine Geometry

## Geometry

The study of properties of figures in space, preserved by the group of transformation acting on it.

- length (distance)
- Angles
- parallelism
- Collinearity

Considering the geometry in this way was first proposed by the German Mathematician "Felix klein" and leads to generate different Geometries.

## Euclidian Geometry

In euclidian geometry, the properties concerned with the distance, are preserved by the group of transformates acting on the space of figures.

Remark

It may be noted that the euclidian plane geometry was the space $R^2$.

It may also be noted that group of transformation associated with Euclidian Geometry is the group of isometries of the plane. (An isometry preserves the distance).

Affine Geometry.

In Affine Geometry the group is of transformations of the form

$f(\underline{x}) = A\underline{x} + \underline{a}$, $\forall\ x \in \mathbb{R}^2$, called affine transformations.

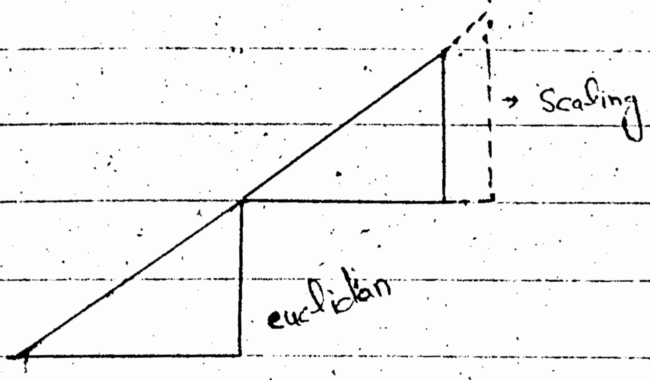
where $\underline{a} \in \mathbb{R}^2$ and $A$ is $2\times2$ non-singular matrix.

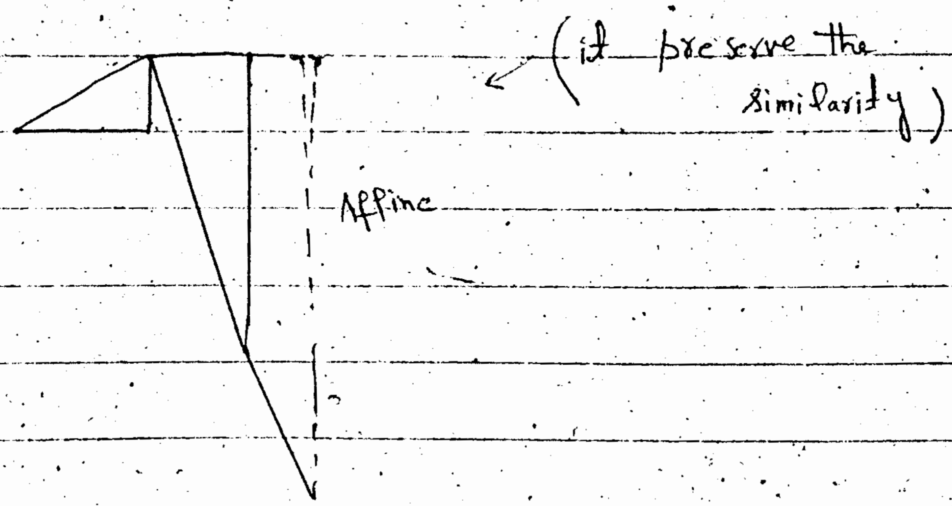
It may be noted that in affine geometry all triangle are congurent, in the sense that any triangle can be mapped onto any other triangle by an affine transformation, which is called to be the fundamental:

Theorem of Affine Geometry:

- Affine geometry has same features common with euclidian Geometry.

(Affine transformation is that transformation which scale it and which also preserve Ratio)



## Euclidian Geometry

Since euclidian plane geometry consists of the space of figures in $\mathbb{R}^2$, the group of isometries acting on it, so it to investigate the transformations which leave the lengths unchanged.

Such transformation (isometries) have one of the following forms.

4/1/05

Such Terms formates (isometries) have one of the following form.

- Translation along a line in $\mathbb{R}^2$.
- Reflection in a line in $\mathbb{R}^2$.
- Rotation about a pt in $\mathbb{R}^2$ (Reflection followed by a translation).

## Euclidian Properties

The properties of figure which are unchanged by the group of isometries, i.e Euclidian properties are the properties that are preserved by a rigid (body) figure as it moves around the plane.

Remark

A triangle is Congurent to another triangle in $\mathbb{R}^2$, if there exists a mapping which is a composition of isometries and mapped a triangle upto another.

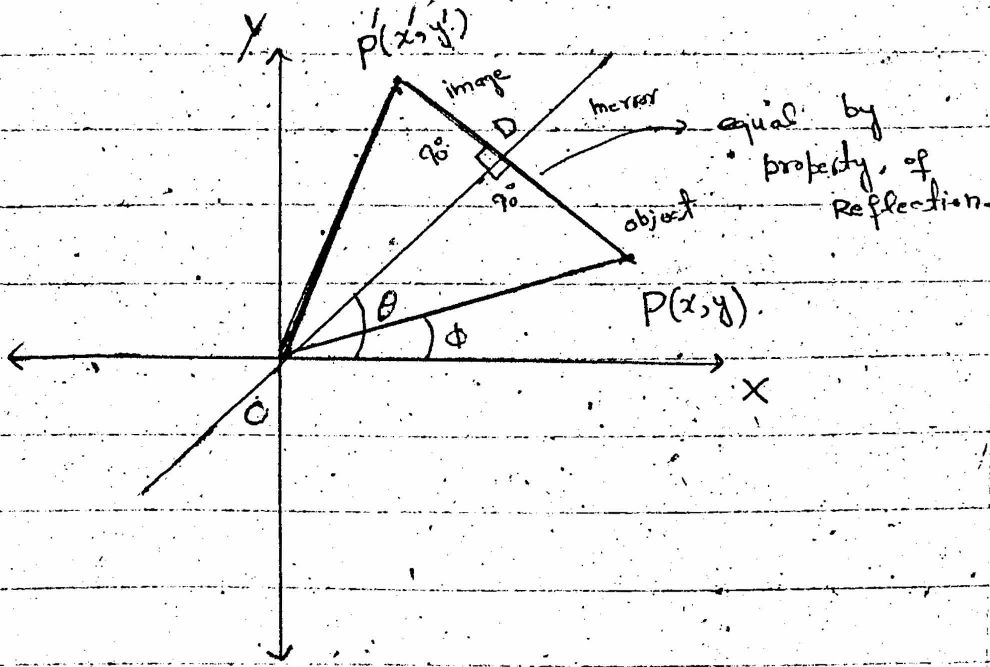
Note

While Considering the geametrys in terms of space and a group of transformations acting on it, It is often Convenient to have algebraic Representations for the corresponding transformations.

# Algebraic Representation of isometries.

## Reflection

The distance image and mirror = mirror and object)




Considering the pt. p(x, y) to be Reflection in a line, OD passing through the origin which makes an angle θ in the anti clock wise direction with x-axis.

let p'(x', y') be the reflection of pt p(x, y) using the fact that

$$\angle ODP \cong \angle ODP'$$

It can be proved that

$$|\overline{OP}| = |\overline{OP'}|$$

Let $P(r,\phi)$ be the polar Coordinates of $P(x,y)$ then $P'(r, 2\theta-\phi)$ will be the polar coordinates of $P'(x',y')$.

Now

$$x = r\cos\phi \quad , \quad y = r\sin\phi$$

and $x' = r\cos(2\theta-\phi)$ , $y' = r\sin(2\theta-\phi)$.

i.e

$$x' = r\cos 2\theta\cos\phi + r\sin 2\theta\sin\phi$$
$$= r\cos\phi\cos 2\theta + r\sin\phi\sin 2\theta$$
$$x' = x\cos 2\theta + y\sin 2\theta$$
$$y' = r\sin 2\theta\cos\phi - r\cos 2\theta\sin\phi$$
$$= r\cos\phi\sin 2\theta - r\sin\phi\cos 2\theta$$

$$y' = x\sin 2\theta - y\cos 2\theta .$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos 2\theta + y\sin 2\theta \\ x\sin 2\theta - y\cos 2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence the above Reflection can be expressed as

$$f(\underline{x}) = A\underline{x} \quad , \forall\ \underline{x} \in \mathbb{R}^2, \text{ where}$$

$$A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

## Orthognal Matrix

A matrix A is said to be orthognal if

$$AA^T = I \quad (A^{-1} = A^T)$$

**Remark** It can be shown that the matrix

$$A = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$ of transformation

of reflection, is orthognal.

$$A^T = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

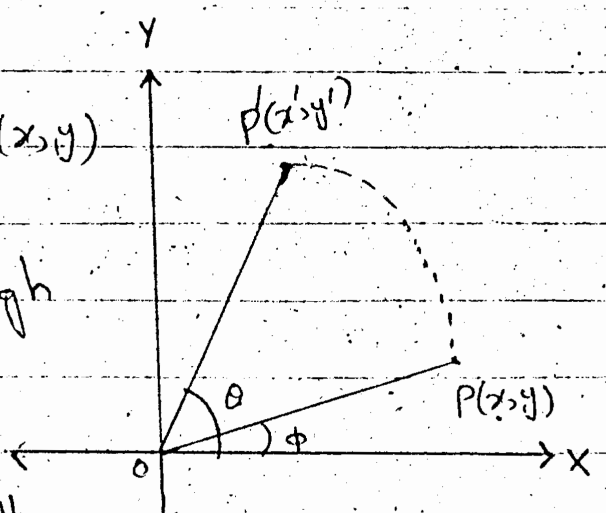
$$AA^T = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 2\theta + \sin^2 2\theta & \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta \\ \sin 2\theta \cos 2\theta - \cos 2\theta \sin 2\theta & \cos^2 2\theta + \sin^2 2\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad AA^T = 0$$

## Rotation




Consider the pt $P(x, y)$ to be rotated about origin through an angle $\theta$ in anti-clock wise direction with x-axis.

Let $P(r, \phi)$ be the polar coordinates of $P(x, y)$ then $P'(r, \theta + \phi)$ will be the polar coordinates of $P'(x', y')$

Now

$$x = r\cos\phi \quad , \quad y = r\sin\phi$$

while $\quad x' = r\cos(\theta + \phi) \; , \; y' = r\sin(\theta + \phi)$

$$x' = r\cos\phi\cos\theta - r\sin\theta\sin\phi$$

$$= r\cos\phi\cos\theta - r\sin\phi\sin\theta$$

$$x' = x\cos\theta - y\sin\theta$$

and

$$y' = r\sin\theta\cos\phi + r\cos\theta\sin\phi$$

$$= r\cos\phi\sin\theta + r\sin\phi\cos\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta & -y\sin\theta \\ x\cos\theta & y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

**Remark.** It can be shown that the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

transformation of Reflection is orthogonal i.e

$$AA^T = I$$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^T = I$$

05|01|05

**Ex:** find the equation of line

$$y = 2x + 5$$

after Reflecting it in a line,

origin which makes the angle $(\frac{\pi}{8})$ in anti-clock wise direction i.e with x-axis

Sol

let $p(x,y)$ be a pt. on the line $y = 2x+5$ in $\mathbb{R}^2$ and $p'(x',y')$ be the pt. after Reflecting $p(x,y)$ in the given line then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ 2x+5 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2(\frac{\pi}{8}) & \sin 2(\frac{\pi}{8}) \\ \sin 2(\frac{\pi}{8}) & -\cos 2(\frac{\pi}{8}) \end{bmatrix} \begin{bmatrix} x \\ 2x+5 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \frac{\pi}{4} & \sin \pi/4 \\ \sin \pi/4 & -\cos \pi/4 \end{bmatrix} \begin{bmatrix} x \\ 2x+5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ 2x+5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}(2x+5) \\ \\ \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}(2x+5) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(3x+5) \\ \\ \frac{1}{\sqrt{2}}(-x-5) \end{bmatrix}$$

$$\Rightarrow \quad x' = \frac{1}{\sqrt{2}}(3x+5)$$

and $\quad y' = \frac{1}{\sqrt{2}}(-x-5)$

Xing $y'$ by 3 and ting with $x'$

$$x' + 3y' = \frac{1}{\sqrt{2}}(\cancel{3x}+5)$$
$$\frac{1}{\sqrt{2}}(\cancel{3x}-15)$$

$$x' + 3y' = \frac{1}{\sqrt{2}}(-10)$$

$$x' + 3y' = \frac{-10}{\sqrt{2}}$$

**Ex**

Find the equation of line $y = 2x+3$ after rotating at about origin, through the angle $\left(\frac{\pi}{4}\right)$ in anti clockwise direction.

**Sol** Let $p(x,y)$ be a pt on the line $y = 2x+3$ in $\mathbb{R}^2$ and $p'(x',y')$ be the pt after rotating $p(x,y)$ (in the given line) about origin then $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ 2x+3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ 2x+3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}(2x+3) \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}(2x+3) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(x-2x-3) \\ \frac{1}{\sqrt{2}}(x+2x+3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(-x-3) \\ \frac{1}{\sqrt{2}}(3x+3) \end{bmatrix}$$

Here $x' = \frac{1}{\sqrt{2}}(-x-3)$ and $y' = \frac{1}{\sqrt{2}}(3x+3)$

xing $x'$ by 3 and +ing with $y'$

$$3x'+y = \frac{1}{\sqrt{2}}(\cancel{-3x}-9)$$
$$+ \frac{1}{\sqrt{2}}(\cancel{3x}+3)$$

$$= \frac{1}{\sqrt{2}}(-6)$$

$$\Rightarrow \quad 3x'+y' = \frac{-6}{\sqrt{2}}$$

Remark

It can be shown that for matrix of rotation about origin through an angle $\theta$, in anti-clockwise direction

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$A(-\theta) = A^T = A^{-1}$$

As

$$A(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= A^T$$

$$\therefore AA^T = I$$

$$\Rightarrow A^T = A^{-1}$$

Remark.

Reflection and Rotation are distance preserving transformations.

Proof

Reflection

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the pt in $\mathbb{R}^2$ after Reflecting the pts $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in a line through the origin, making an angle $\theta$ in anti-clockwise direction with x-axis

then

$$\begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

and $$\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} x_1 \cos 2\theta + y_1 \sin 2\theta \\ x_1 \sin 2\theta - y_1 \cos 2\theta \end{bmatrix}$$

$\Rightarrow x_1' = x_1 \cos 2\theta + y_1 \sin 2\theta$ and

$y' = x_1 \sin 2\theta - y_1 \cos 2\theta$

and

$x_2' = x_2 \cos 2\theta + y_2 \sin 2\theta$ and

$y_2' = x_2 \sin 2\theta - y_2 \cos 2\theta$

Let $d$ and $d'$ be the distance b/w the pts $P_1, P_2$ and $P_1', P_2^*$ respectively

Then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and $$d' = \sqrt{(x_2' - x_1^*)^2 + (y_2' - y_1')^2}$$

putting values

$$d' = \sqrt{(x_2 \cos 2\theta + y_2 \sin^2\theta - x_1 \cos 2\theta - y_1 \sin 2\theta)^2 + (x_2 \sin 2\theta - y_2 \cos\theta - x_1 \sin 2\theta + y_1 \cos 2\theta)^2}$$

$$= \sqrt{\left[(x_2-x_1)\cos 2\theta + (y_2-y_1)\sin 2\theta\right]^2 + \left[(x_2-x_1)\sin 2\theta - (y_2-y_1)\cos 2\theta\right]^2}$$

$$= \sqrt{\begin{aligned}&(x_2-x_1)^2\cos^2 2\theta + (y_2-y_1)^2\sin^2 2\theta + \cancel{2(x_2-x_1)(y_2-y_1)\cos 2\theta\sin 2\theta}\\ &+ (x_2-x_1)^2\sin^2 2\theta + (y_2-y_1)^2\cos^2 2\theta - \cancel{2(x_2-x_1)(y_2-y_1)\cos 2\theta\sin 2\theta}\end{aligned}}$$

$$= \sqrt{(x_2-x_1)^2(\cos^2 2\theta + \sin^2 2\theta) + (y_2-y_1)^2(\sin^2 2\theta + \cos^2 2\theta)}$$

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = d$$

$$\Rightarrow d' = d$$

(ii) <u>Rotation</u>

$$\begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

and
$$\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$x_1' = x_1\cos\theta - y_1\sin\theta, \quad y_1' = x_1\sin\theta + y_1\cos\theta$$

and

$$x_2' = x_2\cos\theta - y_2\sin\theta, \quad y_2' = x_2\sin\theta + y_2\cos\theta$$

Let $d$ and $d'$ be the distance b/w the pts $P_1, P_2$ and $P_1', P_2'$ respectively

then $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

and $d' = \sqrt{(x_2'-x_1')^2 + (y_2'-y_1)^2}$

$= \sqrt{(x_2\cos\theta + y_2\sin\theta - x_1\cos\theta + y_1\sin\theta)^2 + (x_2\sin\theta + y_2\cos\theta - x_1\sin\theta - y_1\cos\theta)^2}$

$= \sqrt{[(x_2-x_1)\cos\theta - (y_2-y_1)\sin\theta]^2 + [(x_2-x_1)\sin\theta + (y_2-y_1)\cos\theta]^2}$

$= \sqrt{(x_2-x_1)^2(\cos^2\theta + \sin^2\theta) + (y_2-y_1)^2(\sin^2\theta + \cos^2\theta)}$

$d' = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = d$

$\Rightarrow \underline{\underline{d'=d}}$

**Ex**

Show that every orthogonal matrix of order 2x2 is of one of the following form

(a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ , (b) $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

Sol

Let $A$ be an orthogonal matrix, where

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

then

$$A^T A = I$$

i.e $$\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = I$$

$$\begin{pmatrix} x_1^2 + x_3^2 & x_1 x_2 + x_3 x_4 \\ x_2 x_1 + x_2 x_4 & x_2^2 + x_4^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e $x_1^2 + x_3^2 = 1$ —— (i)

$x_1 x_2 + x_3 x_4 = 0$ —— (ii)

$x_2^2 + x_4^2 = 1$ —— (iii)

from eq (i), it can be observed that

$|x_1| \leq 1$ or $|x_3| \leq 1$

($x_1$ and $x_3$ can neither be equal to zero nor be equal to one)

$\because x_1^2 + x_3^2 = 1$

$\therefore x_1^2 \leq 1$

& $x_3^2 \leq 1$

i.e

$|x_1| \leq 1$

& $|x_3| \leq 1$

$\Rightarrow |\cos\theta| \leq 1$

Moreover $|\cos\theta| \leq 1$

So let $x_1 = \cos\theta$ for some $\theta$ —— (iv)

from eq (i)

$$\cos^2\theta + x_3^2 = 1$$

$$x_3^2 = 1 - \cos^2\theta = \sin^2\theta$$

$$\Rightarrow \boxed{x_3 = \pm\sin\theta} \quad \text{(v)}$$

Now

considering $x_3 = \sin\theta$.

putting $x_1 = \cos\theta$ and $x_3 = \sin\theta$ in eq (ii)

$$\cos\theta\, x_2 + \sin\theta\, x_4 = 0$$

$$x_2 \cos\theta = -x_4 \sin\theta$$

$$\Rightarrow x_2 = -x_4 \frac{\sin\theta}{\cos\theta} \quad \text{(vii)}$$

putting the value of $x_2$ in eq (iii)

$$\left(-x_4 \frac{\sin\theta}{\cos\theta}\right)^2 + x_4^2 = 1$$

$$x_4^2\left(\frac{\sin^2\theta}{\cos^2\theta}\right) + x_4^2 = 1$$

$$x_4^2(\tan^2\theta + 1) = 1 \Rightarrow x_4^2(\sec^2\theta) = 1$$

$$x_4^2 = \frac{1}{\sec^2\theta}, \quad x_4^2 = \cos^2\theta$$

$$x_4 = \pm\cos\theta$$

For $x_4 = \cos\theta$

put in (vii)

$$x_2 = -\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$\Rightarrow \boxed{x_2 = -\sin\theta}$$

for $x_4 = -\cos\theta$ put in eq (vii)

$$x_2 = \cos\theta \cdot \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \boxed{x_2 = \sin\theta}$$

Hence for $x_3 = \sin\theta$ and $x_4 = \cos\theta$, we have the matrix A can be written as

$$A_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for $x_3 = \sin\theta$ and $x_4 = -\cos\theta$, we have the matrix A can be written as

$$A_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

which show that the matrix $A_1$ and $A_2$ are the forms (a) and (b) respectively.

now, considering $x_3 = -\sin\theta$

putting the value of $x_3 = \sin\theta$ & $x_1 = \cos\theta$ in eq (ii)

$$\cos\theta\, x_2 - x_4 \sin\theta = 0$$

$$x_2 \cos\theta = x_4 \sin\theta$$

$$x_2 = x_4\left(\frac{\sin\theta}{\cos\theta}\right) \Rightarrow x_2 = x_4\tan\theta \quad \text{(viii)}$$

put in eq. (iii)

$$(x_4\tan\theta)^2 + x_4^2 = 1$$

$$x_4^2(\tan^2\theta + 1) = 1 \Rightarrow x_4^2\sec^2\theta = 1$$

$$x_4^2 = \frac{1}{\sec^2\theta} \Rightarrow x_4^2 = \cos^2\theta$$

$$x_4 = \pm\cos\theta$$

for $x_4 = \cos\theta$

put in eq (viii)

$$x_2 = \cancel{\cos\theta}\cdot\frac{\sin\theta}{\cancel{\cos\theta}}$$

$$\Rightarrow \boxed{x_2 = \sin\theta}$$

for $x_4 = -\cos\theta$

put in (viii)

$$x_2 = -\cancel{\cos\theta}\cdot\frac{\sin\theta}{\cancel{\cos\theta}}$$

$$\Rightarrow \boxed{x_2 = -\sin\theta}$$

Hence for $x_3 = -\sin\theta$; & $x_4 = \cos\theta$; we have the matrix A can be written as

$$A_3 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for $x_3 = -\sin\theta$ and $x_4 = -\cos\theta$,
we have the matrix A can be
written as
$$A_4 = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$$

It may be observed that the matrix $A_3$, can be written in the form (a), Replacing $\theta$ by $(-\theta)$.

Similarly, the matrix $A_4$ can be written in the form (b), Replacing $\theta$ by $-\theta$.

## Euclidian Transformation

Any transformation $f$ is said to $\mathbb{R}^2 \to \mathbb{R}^2$ Euclidian transformation, if it can be written as
$$f(\underline{x}) = A\underline{x} + \underline{a}, \quad \forall\ \underline{x} \in \mathbb{R}^2$$
where A is $2\times2$, orthognal matrix and $\underline{a} \in \mathbb{R}^2$

6/1/05

Remark:

Since every isometre can be expressed as $f(\underline{x}) = A\underline{x} + \underline{a}$, $\forall\ \underline{x} \in \mathbb{R}^2$
where A is $2\times2$ orthognial matrix

and $a \in \mathbb{R}^2$, So Euclidian transformation can be defined as under.

if $a=0$ & $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Euclidian Transformation of $\mathbb{R}^2$;
is a function
$$f(\underline{x}) = A\underline{x} + \underline{a} \quad , \forall\ \underline{x} \in \mathbb{R}^2$$
where $A$ is $2\times 2$ orthogonal matrix.

**Theorem:**
The set of all Euclidian transformation in $\mathbb{R}^2$, forms a group under the composition of functions.

**Proof:**
let $t_1$ and $t_2$ be two Euclidian transformation defined as
$$t_1(\underline{x}) = A_1\underline{x} + \underline{a}_1 \quad , \forall\ \underline{x} \in \mathbb{R}^2$$
and
$$t_2(\underline{x}) = A_2\underline{x} + \underline{a}_2 \quad , \forall\ \underline{x} \in \mathbb{R}^2$$
then it can be shown that the composition of euclidian transformations satisfied the following properties:

(i) Clouser property:
$$(t_1 o t_2)(\underline{x}) = t_1(t_2(\underline{x}))$$
$$= t_1(A_2(\underline{x}) + \underline{a}_2)$$
$$= A_1(A_2\underline{x} + \underline{a}_2) + \underline{a}_1$$
$$= A_1(A_2 x) + A_1\underline{a}_2 + \underline{a}_1$$
$$= (A_1A_2)\underline{x} + A_1(\underline{a}_2) + \underline{a}_1$$

It can be shown that, since $A_1$ and $A_2$ are orthognal, so is $A_1A_2$

$$\Big(\text{because } A_1, A_2 \text{ orthogonal}$$

$$(A_1A_2)(A_1A_2)^T = (A_1A_2)(A_2^T A_1^T)$$

$$= A_1(A_2A_2^T)A_1^T = A_1 I A_1^T$$

$$= A_1A_1^T = I\Big)$$

$\because A$ is orthogonal $A^{-1} = A^T$

Moreover, $A_1\underline{a}_2 \in \mathbb{R}^2$,

so $A_1\underline{a}_2 + \underline{a}_1 \in \mathbb{R}^2$

Thus $t_1 o t_2$ is an Euclidian transformation. It may be noted that $t_2 o t_1$ will also be an euclidean transformation.

Hence composition of Euclidian transformation, is closed.

Hence closure property is satisfied.

(ii) Associative property:

Since composition of functions is always associative, so is composition of Euclidian transformations.

(iii) Existence of identity element in $\mathbb{R}^2$

Since an identity function can be expressed as

$$I(\underline{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} + \underline{0} \quad , \forall x \in \mathbb{R}^2$$

Hence $I$ is an Euclidian transformation.

To show $I$ is an identity element under Binary operation of composition of Euclidian transformations,

let $t$ be an Euclidian transformations in $\mathbb{R}^2$, defined as

$$t(\underline{x}) = A\underline{x} + \underline{a} \quad , \forall x \in \mathbb{R}^2$$

then $$(I \circ t)(\underline{x}) = I(t(\underline{x}))$$

$$= I(A\underline{x} + \underline{a}) = I(A\underline{x}) + I(\underline{a})$$

$$= (IA)\underline{x} + I(\underline{a}) = A\underline{x} + \underline{a} = t(\underline{x})$$

$$\Rightarrow (I \circ t) = t$$

Similarly, it can be shown that

$$t \circ I = t$$

$$\therefore \quad t \circ I = I \circ t = t$$

Since $t$ was taken to be arbitrary euclidean transformation, so

$$I \circ t = t \circ I = t \quad , \forall t$$

Hence $I$ is the identity element under composition of transformation.

(iv) <u>Existence of inverse elements.</u>

Consider the euclidean transformation $t$ defined as

$$t(\underline{x}) = A\underline{x} + \underline{a} \quad , \quad \forall\, x \in \mathbb{R}^2 .$$

$$\Rightarrow A\underline{x} = t(\underline{x}) - \underline{a}$$

$$\underline{x} = A^{-1}(t(x)) - A^{-1}(\underline{a})$$

which shows that the inverse function $t^{-1} : R^2 \to R^2$ will be defined as

$$t^{-1}(\underline{x}) = A^{-1}(x) - A^{-1}(\underline{a})$$

$$, \forall\, \underline{x} \in \mathbb{R}^2$$

$f : R^2 \to \mathbb{R}$

$f(x) = 2x + 3$

$f^{-1} : \mathbb{R} \to \mathbb{R}$

$f^{-1}(x) = \left(\frac{x-3}{2}\right)$

$f(x) = 2x + 3$

$2x = f(x) - 3$

$x = \{f(x) - 3\}\frac{1}{2}$

$f^{-1}(x) = f^{-1}\left\{\frac{f(x)-3}{2}\right\}$

$f^{-1}(x) = \frac{1}{2}\{(f^{-1}f)(x) - 3\}$

$= \frac{1}{2}(x-3)$

which shows that the inverse function

$t^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ will be defined as

$$t^{-1}(x) = A^{-1}(x) - A^{-1}\underline{a}$$

$$\forall\, \underline{x} \in \mathbb{R}^2$$

Since $A$ is orthogonal by definition,

so will be $A^{-1}$

i.e $\quad (A^{-1})(A^{-1})^T$

$= (A^{-1})(A^T)^T \quad \because A$ is orthogonal $A^{-1} = A^T$

$= (A^{-1})(A)$

$= I$

Moreover, $\bar{A}^1\underline{a}$ obviously belongs to $\mathbb{R}^2$

Hence $\bar{t}^1: R^2 \to R^2$ is also an euclidean transformation in $\mathbb{R}^2$.

Also

$$(\bar{t}^1 o t)(x) = \bar{t}^1(t(x)), \quad \forall x \in \mathbb{R}^2$$

$$= \bar{t}^1(A\underline{x} + \underline{a})$$

$$= \bar{A}^1(A\underline{x} + \underline{a}) - \bar{A}^1(\underline{a})$$

$$= (\bar{A}^1 A)(\underline{x}) + \cancel{\bar{A}^1(\underline{a})} - \cancel{\bar{A}^1(\underline{a})}$$

$$= I(\underline{x})$$

$$\Rightarrow \bar{t}^1 o t = I$$

Similarly, it can be shown that

$$(t o \bar{t}^1) = I$$

$$\text{i.e } (t o \bar{t}^1) = (\bar{t}^1 o t) = I$$

Since $t$ was taken to be arbitrary euclidean transformation, so for every euclidean transformation $t \exists$ an euclidean transformation $\bar{t}^1$.

$$\text{S.t } t o \bar{t}^1 = \bar{t}^1 o t = I$$

Hence the set of all euclidean transformations forms of group under binary operation of composition.

11/01/05

**Remark** It can be observed that in every orthogonal matrix A, the columns are orthogonal. In particular since every column of matrix A has unit length. So the first column may be taken in the form $\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$ for some real number $\theta$. So the 2nd column may be written as

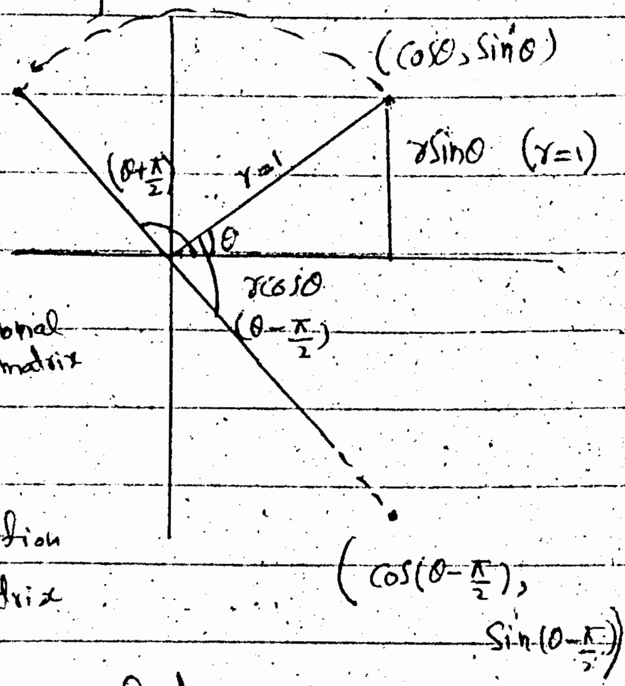
$$\begin{bmatrix}\cos(\theta+\frac{\pi}{2}) \\ \sin(\theta+\frac{\pi}{2})\end{bmatrix} = \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}$$

or $$\begin{bmatrix}\cos(\theta-\frac{\pi}{2}) \\ \sin(\theta-\frac{\pi}{2})\end{bmatrix} = \begin{bmatrix}\sin\theta \\ -\cos\theta\end{bmatrix}$$

Hence the matrix A will be in the form

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & +\cos\theta\end{bmatrix}$$ Rotational matrix

or $$\begin{bmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{bmatrix}$$ Reflection matrix




**Ex**

If $t_1$ and $t_2$ are euclidean transformation in $\mathbb{R}^2$, where

$$t_1(x) = \begin{bmatrix}3/5 & -4/5 \\ 4/5 & 3/5\end{bmatrix}x + \begin{bmatrix}1 \\ -2\end{bmatrix}$$

and

$$t_2(x) = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} \underline{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

then determine $(t_1 o t_2)$ and $(t_2 o t_1)$.

Sol

$$(t_1 o t_2)(x) = t_1(t_2(x))$$

$$= t_1\left(\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} \underline{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$$

$$= \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}\left(\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} \underline{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}\begin{pmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{pmatrix} \underline{x} + \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{12}{25} - \frac{12}{25} & \frac{9}{25} - \frac{16}{25} \\ -\frac{16}{25} + \frac{9}{25} & \frac{12}{25} + \frac{12}{25} \end{pmatrix} \underline{x} + \begin{pmatrix} -\frac{6}{5} - \frac{4}{5} \\ -\frac{8}{5} + \frac{3}{5} \end{pmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{pmatrix} -24/25 & -7/25 \\ -7/25 & 24/25 \end{pmatrix} \underline{x} + \begin{pmatrix} -10/5 \\ -5/5 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -24/25 & -7/25 \\ -7/25 & 24/25 \end{pmatrix} \underline{x} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -24/25 & -7/25 \\ -7/25 & 24/25 \end{bmatrix} \underline{x} + \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

Similarly,

$$(t_2 \circ t_1)\underline{x} = t_2(t_1(\underline{x}))$$

$$= t_2\left(\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} \left(\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \underline{x} + \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12/25 + 12/25 & \frac{16}{25} + \frac{9}{25} \\ 9/25 + \frac{16}{25} & -\frac{12}{25} + \frac{12}{25} \end{bmatrix} \underline{x} + \begin{bmatrix} -\frac{4}{5} - \frac{6}{5} \\ \frac{3}{5} - 8/5 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

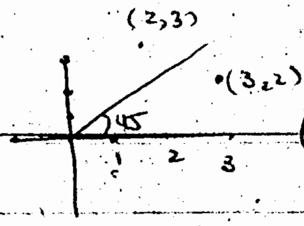
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow (t_2 \circ t_1)\underline{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

This matrix is along the reflection line $y = \underline{x}$

Reflection on i.e

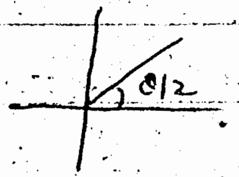
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \Rightarrow \begin{bmatrix} \cos 2 \cdot \frac{\pi}{4} & \sin 2\frac{\pi}{4} \\ \sin 2\pi/4 & -\cos 2\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$




**Ex**

Determine the inverse ($t^{-1}$) function if $t$ is an euclidean transformation defined as

$$t(\underline{x}) = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} \underline{x} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$




**Sol**

Let $A = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$

then and $\underline{a} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$t(\underline{x}) = A\underline{x} + \underline{a}$

$\underline{x} = A^{-1}t(\underline{x}) - A^{-1}\underline{a}$ ; $\frac{1}{A} = A^{-1}$

$t^{-1}(\underline{x}) = A^{-1}(x) - A^{-1}\underline{a}$
$(A^T)$

$A^{-1} = A^T$ because $A$ is orthogonal

$$t^{-1}(\underline{x}) = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} \underline{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{bmatrix} 3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/5 - 8/5 \\ -\frac{4}{5} - \frac{6}{5} \end{bmatrix} = \begin{bmatrix} -5/5 \\ -10/5 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

**Ex**

Find the translation of the line $y = 2x - 1$ by the pt $\underline{a} = (3,5)$.

**Sol**

Let $p(x,y)$ be a point on the line $y = 2x - 1$, — (*)

and $p'(x',y')$ be the corresponding pt after translation, then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$y' = 2x' - 1$ it is // to $y = 2x - 1$ is this case the slope of this line same but intersection different

$x' = x + 3$ —(1), $y' = y + 5$

$= 2x - 1 + 5$ by ⊛

$= 2x + 4$ —(2)

xing eq (1) by 2

and then sub. then

$2x' = \cancel{2x} + 6$

$y' = \cancel{2x} + 4$

$2x' - y' = 2 \Rightarrow y' = 2x' - 2$

$y' = 2(x' - 1)$ Ans

12/1/05

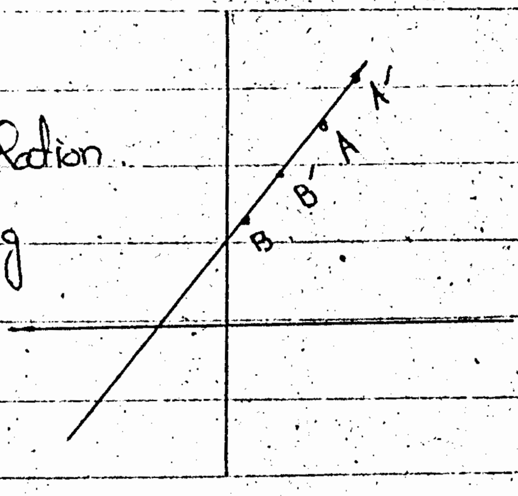
**Remark**

It may be observed that the line obtained after translation has the same slope i.e the translation does not change the slope of a st. line

It may also be noted that

if a st. line is translated along the line, with the same slope then the to be translated remains unchanged.

Ex show that a translation is a distance preserving transformation.




Sol

let $t$ be a translation defined in $\mathbb{R}^2$, as

$$t(\underline{x}) = \underline{x} + \underline{a}, \quad \forall\, x \in \mathbb{R}^2, \text{ for some } a \in \mathbb{R}^2$$

where $\underline{a} = \begin{bmatrix} p \\ q \end{bmatrix}$

and let $P_1(x_1, y_1)$; $P_2(x_2, y_2)$ be two pts to be translated. let $P_1'(x_1', y_1')$ and $P_2'(x_2', y_2')$ be the two pts corresponding to $P_1$ and $P_2$ respectively after the given translation.

Moreover, let $d$ and $d'$ be the distance b/w $P_1$ and $P_2$ and $P_1'$ and $P_2'$ respectively.

then $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ —— (i)

and

$d' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}$ —— (ii)

From the translation $t$, it can be written as

$$\begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} x_1 + p \\ y_1 + q \end{bmatrix}$$

when $\underline{a} = \begin{bmatrix} p \\ q \end{bmatrix}$

$\Rightarrow x_1' = x_1 + p$

$y_1' = y_1 + q$

Similarly,

$$\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} x_2 + p \\ y_2 + q \end{bmatrix}$$

$\Rightarrow x_2' = x_2 + p$

$y_2' = y_2 + q$

put these value in eq (ii)

$\Rightarrow d' = \sqrt{(x_2 + \cancel{p} - x_1 - \cancel{p})^2 + (y_2 + \cancel{q} - y_1 - \cancel{q})^2}$

$d' = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d$

$\Rightarrow d' = d$

Ex

Show that Composition of reflection and translation, is not Commutative.

Sol

Let $t_1$ and $t_2$ be the reflection in line making an angle $\left(\frac{\theta}{2}\right)$ in anticlockwise direction with x-axis and the translation by vector $\underline{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ respectively.

i.e $t_1(\underline{x}) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \underline{x}, \quad \forall\, \underline{x} \in \mathbb{R}^2$

where $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

and $t_2(\underline{x}) = \left( \underline{x} + u, \quad \forall\, \underline{x} \in \mathbb{R} \right)$.

Then $(t_1 \circ t_2)(\underline{x}) = t_1(t_2(x))$

$= t_1(\underline{x} + \underline{u})$

$= A(\underline{x} + \underline{u})$

$= A\underline{x} + A\underline{u} \quad \forall\, \underline{x} \in \mathbb{R}^2$ ——— (i)

Now

$(t_2 \circ t_1)(\underline{x}) = t_2(t_1(\underline{x}))$

$= t_1(\underline{x}) + \underline{u}, \quad \forall\, \underline{x} \in \mathbb{R}^2$

$= A\underline{x} + \underline{u}$ ——— (ii) from (i) & (ii) it can be written as

$$\boxed{t_1 o t_2 \neq t_2 o t_1}$$

Remark:

Keeping in view the above example to see whether $\exists$ a matrix $A$, s.t

$(t_1 o t_2)(x) = (t_2 o t_1)(x)$, $\forall x \in \mathbb{R}^2$

Consider

$$A\underline{x} + A\underline{y} = A\underline{x} + \underline{y}$$

$$\Rightarrow \quad A\underline{y} = \underline{y},$$

or $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e

$$\cos\theta = 1 = \cos\theta \quad \text{— (iii)}$$

and

$$\sin\theta = 0 \quad \text{— (iv)}$$

It is easy to see that $\exists$ any real number $\theta$ which satisfies both (iii) and (iv)

Thus the composition of reflection and translation can never be Commutative for any angle $\theta$.

Exercise

Show that every isometry is one one (obviously onto)

Solution

Let $t: \mathbb{R}^2 \to \mathbb{R}^2$ be any isometry and $P_1$ and $P_2$ be any two disting point in $\mathbb{R}^2$ (domain) to show $t$, one one, let

$$t(P_1) = t(P_2) \quad \text{--- (1)}$$

Since '$t$' is distance preserving, So

$$|t(P_1) - t(P_2)| = |P_1 - P_2|$$

So, $|t(P_1) - t(P_2)| = 0 = |P_1 - P_2|$ ∴ by (1)

$$\Rightarrow |P_1 - P_2| = 0$$

$$\Rightarrow P_1 - P_2 = 0$$

$\Rightarrow P_1 = P_2$, a contradiction to the supposition that $P_1 \neq P_2$

Thus if $P_1 \neq P_2$ then $t(P_1) \neq t(P_2)$

Since, $P_1, P_2$ were taken to be arbitrary

Hence '$t$' is one-one.

Exercise

Show that if $t_1$ is a Euclidean translation of $\mathbb{R}^2$ defined as

$$t_1(\underline{x}) = A\underline{x} + \underline{a}, \quad \forall \underline{x} \in \mathbb{R}^2$$

(where $A$ is orthogonal in Euclidean transformation)

Then

(i) The translation $t_2$ in $\mathbb{R}^2$ given by $t_2(x) = A^{-1}x - A^{-1}\underline{a}$, $\forall \underline{x} \in \mathbb{R}^2$ is also a Euclidean translation.

(ii) $t_2$ is the inverse of $t_1$.

Solution

(i) Since $A$ is an orthogonal matrix so considering

$$(A^{-1})(A^{-1})^T = (A^{-1})(A^T)^T \qquad \therefore A^{-1} = A^T$$

$A$ is orthogonal.

$$= A^{-1}A$$

$$= I \quad \text{i.e. } (A^{-1})^{-1} = (A^{-1})^T$$

Show that $A^{-1}$ is orthogonal

(ii)

Consider

$$(t_1 \circ t_2)(\underline{x}) = t_1(t_2(\underline{x}))$$

$$= t_1(A^{-1}\underline{x} - A^{-1}\underline{a})$$

$$= A(A^{-1}\underline{x} - A^{-1}\underline{a}) + \underline{a}$$

$$= (AA^{-1})(\underline{x}) \quad (AA^{-1})(\underline{a}) + \underline{a}$$

$= I(\underline{x}) - I(\underline{a}) + \underline{a}$

$= \underline{x} - \underline{a} + \underline{a}$

$= \underline{x}$

or $I(\underline{x}) \quad \forall\ \underline{x} \in \mathbb{R}^2$

$\Rightarrow (t_1 \circ t_2) = I$

Similarly, it can be show that

$t_2 \circ t_1 = I$

Hence $t_2 \circ t_1 = I = t_1 \circ t_2$

Thus $t_2$ is inverse of $t_1$.

Exercise

Show that the composition of the reflection, is a rotation.

Solution

Let $t_1$ and $t_2$ be two reflections defined in $\mathbb{R}^2$ as $t_1(x) = A\underline{x}$ and $t_2(\underline{x}) = A_2\underline{x}$, $\forall\ \underline{x} \in \mathbb{R}^2$

where

$$A_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & -\cos\theta_2 \end{bmatrix}$$

Consider,

$$(t_1 o t_2)(x) = A_1(t_2(x))$$

$$= t_1(A_2 x) = (A_1 A_2)(x)$$

$$= \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & -\cos\theta_2 \end{bmatrix} x$$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 & \sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix} x$$

$$= \begin{bmatrix} \cos(\theta_1-\theta_2) & -\sin(\theta_1-\theta_2) \\ \sin(\theta_1-\theta_2) & \cos(\theta_1-\theta_2) \end{bmatrix} x$$

$$\Rightarrow (t_1 o t_2)(x) = \begin{bmatrix} \cos(\theta_1-\theta_2) & -\sin(\theta_1-\theta_2) \\ \sin(\theta_1-\theta_2) & \cos(\theta_1-\theta_2) \end{bmatrix} x$$

which shows that $t_1 o t_2$ is a rotation about origion through an angle $(\theta_1-\theta_2)$ i.e

Composition of two reflection, is a rotation through angle of twice the difference of reflection make in anti-clockwise direction with x-axis.

<u>Question</u> show that the composition of reflection and rotation, is a reflection.

<u>Solution</u> Let $t_1$ and $t_2$ be reflection and rotation in $\mathbb{R}^2$ respectively defined as

$$t_1(\underline{x}) = A_1\underline{x} \quad , \forall \ \underline{x} \in \mathbb{R}^2$$

$$t_2(\underline{x}) = A_2\underline{x} \quad , \forall \ \underline{x} \in \mathbb{R}^2$$

where $A_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix}$, $A_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

Consider.

$$(t_1 o t_2)(\underline{x}) = t_1(t_2(\underline{x}))$$

$$= t_1(A_2\underline{x}) = A_1A_2\underline{x}$$

$$= \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix}\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}\underline{x}$$

$$= \begin{bmatrix} \cos(\theta_1-\theta_2) & \sin(\theta_1-\theta_2) \\ \sin(\theta_1-\theta_2) & -\cos(\theta_1-\theta_2) \end{bmatrix}\underline{x}$$

which shows that $(t_1 o t_2)$ is the reflection in the line passing through origin making an angle $\left(\frac{\theta_1-\theta_2}{2}\right)$ in anti-clockwise direction with $x$-axis.

## Remark

It may be noted from the above example that composition of reflection in the line passing through origin making angle $\left(\frac{\theta}{2}\right)$ in anti-clockwise with x-axis, and rotation about the origin through an angle $\theta_2$ in anti-clockwise direction with $x$-axis, is a reflection in the line passing through origin making an angle $\left(\frac{\theta_1-\theta_2}{2}\right)$ in anti-clockwise direction with x-axis.

## Observation (in $\mathbb{R}^2$)

(1) Composition of even number of reflection, is a rotation. (?)

(2) Composition of odd number of reflections, is a reflections

e.g

$$\underbrace{\underbrace{t_1 \circ t_2}_{\text{(rotation)}} \circ \underbrace{t_3 \circ t_4}_{\text{(rotation)}}}_{\text{rotation}}$$

$$\underbrace{\underbrace{t_1}_{\text{reflection}} \circ \underbrace{t_2 \circ t_3}_{\text{reflection}}}_{\text{reflection.}}$$

Exercise

Show that a rotation is an isometry.

Solution

Let 't' be a rotation defined in $\mathbb{R}^2$ as $t(\underline{x}) = A\underline{x} \ \forall \ \underline{x} \in \mathbb{R}^2$

where

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta$.

To show 't' to be an isometry; it is to be shown that 't' is a function from $\mathbb{R}^2$ onto $\mathbb{R}^2$ which is distance preserving.

Since a rotation has already been proved to be distance preserving, so to show 't' onto, let $P \in \mathbb{R}^2$ (co-domain) be any point then $\exists \ A^T(P) \in \mathbb{R}^2$ (domain) s.t

$$t(A^T(P)) = P$$

As $t(A^t(P))$
$= A(A^T(P))$
$= (AA^T)(P)$
$= I(P)$

($\because$ A is orthogonal)

Since, p was taken to be arbitrary, so every element of $\mathbb{R}^2$ (co-domain).

will be some element of $\mathbb{R}^2$ (domain).
Hence, 'T' is an onto function.
Thus, 'T' is an isometry.

Exercise
Show that reflection is an isometry.

Solution
Let 't' be a reflection defined in $\mathbb{R}^2$, as $t(\underline{x}) = A\underline{x} \ \forall \ \underline{x} \in \mathbb{R}^2$
where

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \quad \text{for some } \theta.$$

To show 't' to be an isometry, it is to be shown that 't' is a function from $\mathbb{R}^2$ onto $\mathbb{R}^2$ which is distance pressersving.

Since a reflection has already been proved to be distance preserving, so to show 't' is onto.
Let $P \in \mathbb{R}^2$ (co-domain) be any point;
then $\exists \ A^t(P) \in \mathbb{R}^2$ (domain)
s.that

$$t(A^T(P)) = P. \quad \text{As } t(A^t(P)) = A(A^t(P))$$
$$= (AA^t)(P)$$
$$= I(P)$$
$$= P$$

Since 'P' was taken to

be arbitrary, so every element of $\mathbb{R}^2$ (co-domain) will be image of some element of $\mathbb{R}^2$ (domain).
Hence 't' is an onto function.
Thus 't' is an isometry.

Exercise
show that a translation in $\mathbb{R}^2$ is an isometry.

Solution
Let 't' be a translation defined in $\mathbb{R}$, as $t(\underline{x}) = \underline{x} + \underline{a}$, $\forall\ \underline{x} \in \mathbb{R}^2$ and for some $\underline{a} \in \mathbb{R}^2$.
To show 't' an ismoetry, it is to be shown that t is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is distance preserving has already been proved to be distance preserving, so to show t an onto function,
let $P \in \mathbb{R}^2$ (co-domain), Then $\exists$ an element $(P-a) \in \mathbb{R}^2$ (domain)
s.t $t(\underline{P}-\underline{a}) = \underline{P}$

As $t(\underline{P}-\underline{a}) = \underline{P} - \underline{a} + \underline{a}$

$= \underline{P}$

Since $P$ was taken to be arbitrary element of $\mathbb{R}^2$ (domain) of $\mathbb{R}^2$ (co-domain), so every element of $\mathbb{R}^2$ (co-domain) will be image of some element of $\mathbb{R}^2$ (domain).

Hence, 't' is an onto function

Thus, $t$ is an isometry.

## Euclidean Congruence

A figure $F_1$ in $\mathbb{R}^2$ is said to be Euclidean congurent to a figure $F_2 \in \mathbb{R}^2$ if $\exists$ a Euclidean transformation which maps the figure $F_1$ onto $F_2$.

It may be noted that each figure is the set of all circles of radius '2' in $\mathbb{R}^2$ is Euclidean congruent to each other.

Exercise

For the following sets determine which sets consists of the figures that are Euclidean congruent to each other.

No 1) the set of all ellipses.

Yes 2) The set of all line

segments of length '1'.

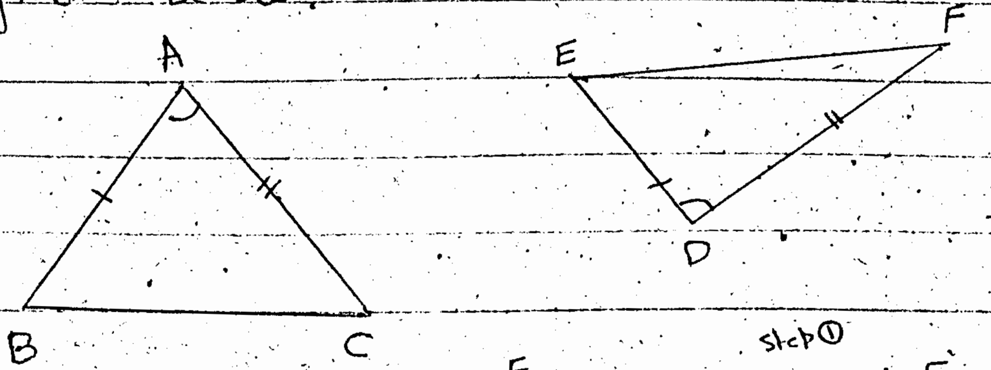
3) The set of all triangles.

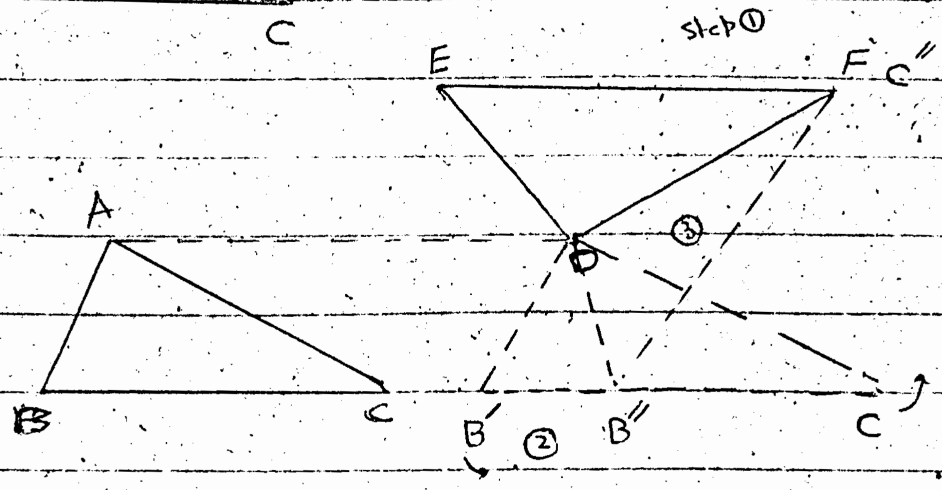
4) The set of all requares of sides of length 2.

18/01/05

Example

show that if for $\Delta ABC$ and $\Delta DEF$ given below.




$\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$ and $\angle BAC \cong \angle EDF$

then

$\Delta ABC \cong \Delta DEF$ (Euclidean Congruent)

To show $\Delta ABC$, Euclidean congurent to the $\Delta DEF$, it is to be shown that $\exists$ a Euclidean transformation

that maps $\Delta ABC$ onto the $\Delta DEF$
It may be noted that the required Euclidean transformation may be determined in stages i.e it may be the composite of Euclidean transformations.

The first Euclidean transformation is a translation which maps the point A to the point D i.e it maps the $\Delta ABC$ onto $\Delta DB'C'$ (as shown in fig)

Since it is given that $\overline{AC} \cong \overline{DF}$ but the translation preserves the distance so $\overline{AC} \cong \overline{DC'}$, which shows that $\overline{DC'} \cong \overline{DF}$.

The second Euclidean transformation is the rotation about the point D through an (angle) $\angle C'BE$.
i.e the $\Delta DB'C'$ will be the mapped onto the $\Delta DB''F$.

Since the translation and rotation preserves the distances, so

$\overline{AB} \cong \overline{DB}$ (translation)

and $\overline{DB'} \cong \overline{DB''}$ (due to rotation) but $\overline{AB} \cong \overline{DE}$, so $\overline{DB''} \cong \overline{DE}$

the third Euclidean transformation is reflection in the line DF which mapps the points B'' to the point E.

i.e the $\Delta DB''F$ will be mapped onto the $\Delta DEF$.

which shows that the composition of translation, rotation and reflection, discussed above, is a Euclidean transformation which maps $\Delta ABC$ onto $\Delta DEF$.

Hence $\Delta ABC$ is Euclidean Congurent to $\Delta DEF$.

**Theorem**

Euclidean Congurence, is an equivalance relation.

**Proof**

Let $\cong$ be the Euclidean Congurence defined in $\mathbb{R}^2$.

i.e (figure)

$F_1 \cong F_2$ if

$\exists$ a Euclidean transformation say, $t$ which maps $F_1$ onto $F_2$. Then it can be shown that $\cong$ is

(i) Reflexive

Let the figure $F \in \mathbb{R}^2$ then the identity function is a Euclidean transformation, which maps $F$ onto itself.

Hence $F \cong F$

Since $F$ was taken to be arbitrary so, $F \cong F \ \forall \ F \in \mathbb{R}^2$

Symmetric

$(x,y) \in R$ then $(y,x) \in R \ \forall \ x,y \in S$

$A = \{1,2,3,\dots 10\}$

$R = \{(2,3), \dots$

? $(3,2) \in R$

Let $F_1 \cong F_2$ for some $F_1, F_2 \in \mathbb{R}^2$, then $\exists$ a Euclidean transformation, say $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which maps $F_1$ onto $F_2$. Then $t^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ will also be Euclidean transformation (?) which maps the $F_2$ onto $F_1$,

$t(\underline{x}) = A\underline{x} + \underline{a}$

$t^{-1}(\underline{x}) = A^T\underline{x} - A^T\underline{a}$

$A$ is orthogonal

$A^t = A^{-1}$

Hence $\therefore F_2 \cong F_1$

Since $F_1$ and $F_2$ were taken to be arbitrary figures from $\mathbb{R}^2$, so

$F_1 \cong F_2 \Rightarrow F_2 \cong F_1 \ \forall \ F_1, F_2 \in \mathbb{R}^2$

## Transitive

Let $F_1 \cong F_2$ and $F_2 \cong F_3$ for $F_1, F_2, F_3 \in \mathbb{R}^2$ then $\exists$ euclidean transformation $t_1$ and $t_2$ which maps figure $F_1$ onto figure $F_2$ onto $F_3$ (figure) respectively.

which shows that $t_2 o t_1$ will also be a Euclidean transformation, which maps $F_1$ onto $F_3$ (?)

i.e $F_1 \cong F_3$.

Since $F_1, F_2$ and $F_3$ were taken to be arbitrary. so

$$F_1 \cong F_2 \text{ and } F_2 \cong F_3$$

$$\Rightarrow F_2 \cong F_3 \quad \forall \; F_1, F_2, F_3 \in \mathbb{R}^2$$

26/01/05

## Exercise

Show that if the isometry $t$ is defined as $t(x) = Ax$, $\forall\, x \in \mathbb{R}^2$ then $A$ is orthogonal.

## Solution

Let $A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

to show $A$, orthogonal, it is

to be shown that

$$A^tA = I$$

i.e $\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e $x_1^2 + x_3^2 = 1$ ——— (i)

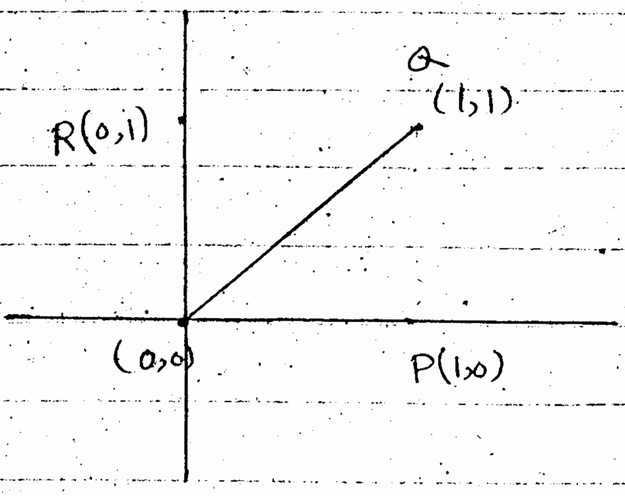
$x_1x_2 + x_3x_4 = 0$ ——— (ii)

$x_2^2 + x_4^2 = 1$ ——— (iii)

Since the above isometry fixes the origion, so considering the point

Reflection (not always) fixes the origion (line must be pass origion)

i.e $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$




Since, the above isometry fixes the origin so, considering the point O(0,0), P(1,0), Q(1,1) and R(0,1), in $\mathbb{R}^2$ (domain).

along with the corresponding images i.e $P', Q', R'$ in $\mathbb{R}^2$ (Co-domain)

Using the fact that every isometry preserves the distance, it can be written as

$$|\overline{OP}| = |\overline{O'P'}|, \quad |\overline{OQ}| = |\overline{O'Q'}|$$

$$|\overline{OR}| = |\overline{O'P'}|$$

where $P' = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$

$$Q' = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_3 + x_4 \end{pmatrix}$$

$$R' = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Now

$$|\overline{OP}| = |\overline{O'P'}| = \sqrt{x_1^2 + x_3^2}$$

$$\Rightarrow \quad x_1^2 + x_3^2 = 1 \quad \text{—— (a)}$$

$$|\overline{OQ}| = \sqrt{2} = |\overline{O'Q'}|$$

$$= \sqrt{(x_1+x_2)^2 + (x_3+x_4)^2}$$

$$\Rightarrow 2 = (x_1+x_2)^2 + (x_3+x_4)^2$$

$$(x_1^2 + x_3^2) + (x_2^2 + x_4^2) + 2(x_1x_2 + x_3x_4) \quad \text{—— (1)}$$

$$|\overline{OR}| = |\overline{O'R'}| = \sqrt{x_2^2 + x_4^2}$$

$$\Rightarrow \quad x_2^2 + x_4^2 = 1 \quad \text{— (b)}$$

putting in ①

$$2 = 1 + 1 + 2(x_1 x_2 + x_3 x_4)$$

$$\Rightarrow \quad x_1 x_2 + x_3 x_4 = 0 \quad \text{— (c)}$$

It may be noted that the results a, b, & c can also be obtained corresponding to any four points in $\mathbb{R}^2$ (domain) along with their images in $\mathbb{R}^2$ (co-domain).

Hence, $A$ is orthogonal.

**Exercise**

Show that for the isometry $t$ defined as

$$t(x) = Ax + a \quad \forall \; x \in \mathbb{R}^2 \text{ and for some } a \in \mathbb{R}^2$$

$A$ is orthogonal.

**Solution**

Consider the isometry $\tilde{t}$ defined as $\tilde{t}(x) = x - a$, $\forall \; x \in \mathbb{R}^2$

$$(\tilde{t} \circ t)(x) = \tilde{t}(t(x)), \quad \forall \; x \in \mathbb{R}^2$$

$= \tilde{t}(A\underline{x}+\underline{a})$

$= A\underline{x}+\underline{a}-\underline{a}$

$= A\underline{x}$

It may be noted that $\tilde{t}$ ~~[illegible]~~, being translation, is an isometry.

Hence, $\tilde{t} o t$ will also be an isometry (being composition of two isometry)

Thus using the previous exercise A is orthogonal.

**Remark**

It may be noted from above exercise that every isometry is a Euclidean transformation. (?)

**Exercise**

Show that an isometry preserves the angles.

**Solution**

Exercise. Show that an isometry preserves the collinearity.

Sol Let P, Q and R be the collinear points in $\mathbb{R}^2$ (domain) where the point Q lies between the point P and R. Let $P'$, $Q'$ and $R'$ be the images of P, Q and R respectively under the given isometry.
Since the every isometry preserve the distance,

so $|\overline{PQ}| = |\overline{P'Q'}|$ OR, $\overline{PQ} \cong \overline{P'Q'}$

$|\overline{QR}| = |\overline{Q'R'}|$

and $|\overline{PR}| = |\overline{P'R'}|$

Since Q is supposed to be between P and R. So

$|\overline{PQ}| + |\overline{QR}| = |\overline{PR}|$

i.e $|\overline{P'Q'}| + |\overline{Q'R'}| = |\overline{P'R'}|$

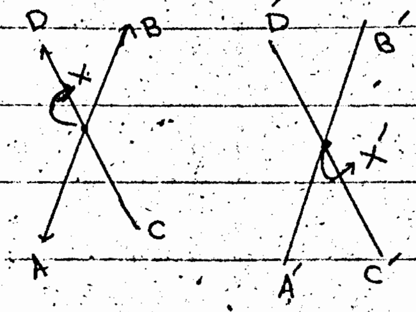
$\Rightarrow$ $Q'$ lies between the point $P'$ and $R'$. Hence $P'$, $Q'$ and $R'$ are collinear.

Thus an isometry preserve the collinearity.

**Exercise**

Show that the concurrent lines, remain concurrent under an isometry.

**Solution**




Let the st. lines $\overline{AB}$ and $\overline{CD}$ intersect at point $X$ in $\mathbb{R}^2$ (domain)

Let $\overline{A'B'}$ and $\overline{C'D'}$ be the corresponding images under the given isometry.

Let $X'$ be the image of $X$.

Since every isometry preserve the collinearity, so the point $A', X', B'$ and $C', X', D'$ must be collinear.

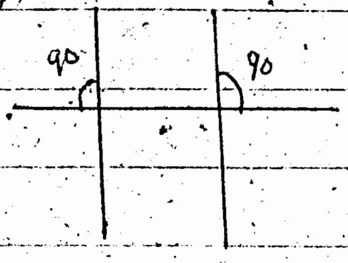
Hence the lines $\overline{A'B'}$ and $\overline{C'D'}$ intersect at point $X'$ i-e The lines $\overline{A'B'}$ and $\overline{C'D'}$ are concurrent.

**Remark**

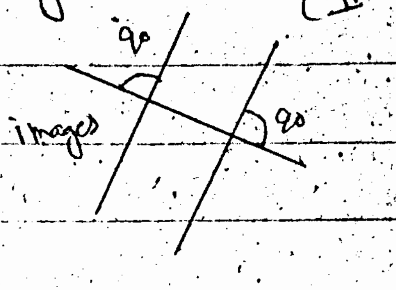
It may be noted that the parallelism and orthogonality of lines can be proved preserve under any isometry.

(Hint)

(I)




(II)



Exercise Describe the following transformation completely,

$$t(\underline{x}) = A\underline{x} + \underline{a}, \quad \forall\, \underline{x} \in \mathbb{R}^2$$

for some $\underline{a} \in \mathbb{R}^2$

where
$$A = \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}, \quad \underline{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Solution

Since the matrix of reflection in the line passing through origion making an angle $(\frac{\theta}{2})$, in anti-clockwise direction with x axis is

$$A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$\Rightarrow A(\pi-\theta) = \begin{pmatrix} \cos(\pi-\theta) & \sin(\pi-\theta) \\ \sin(\pi-\theta) & -\cos(\pi-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Comparing the matrix $A(\pi-\theta)$ with A

$$\cos\theta = \frac{3}{5}, \quad \sin\theta = \frac{4}{5}$$

$$\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53^\circ$$

$\Rightarrow \left(\frac{\pi-\theta}{2}\right) = \left(\frac{180-53}{2}\right)^{\circ} = \left(\frac{127}{2}\right)^{\circ} = 63.5^{\circ}$

Hence the given transformation is the reflection in the line passing through origion making an angle $\left(\frac{\pi-\theta}{2}\right) = (63.5)^{\circ}$ in anti-clockwise direction with X-axis, followed by the translation by the point $(-1, 2)$.