Chapter 4 – Differentiation

Subject: *Real Analysis* **Level:** *M.Sc.*
Source: *Syed Gul Shah (Chairman, Department of Mathematics, UoS Sargodha)*
Collected & Composed by: *Atiq ur Rehman (mathcity@gmail.com), http://www.mathcity.org/msc*

❖ *Derivative of a function:*

Let f be defined and real valued on $[a,b]$. For any point $c\in[a,b]$, form the quotient

$$\frac{f(x)-f(c)}{x-c}$$

and define

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

provided this limit exits.

We thus associate a function f' with the function f, where domain of f' is the set of points at which the above limit exists.

The function f' is so defined is called the derivative of f.

(i) If f' is defined at point x, we say that f is differentiable at x.

(ii) $f'(c)$ exists if and only if for a real number $\varepsilon>0$, $\exists$ a real number $\delta>0$ such that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\varepsilon \qquad \text{whenever} \qquad |x-c|<\delta$$

(iii) If $x-c=h$ then we have

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

(iv) f is differentiable at c if and only if c is a removable discontinuity of the function $\varphi(x)=\dfrac{f(x)-f(c)}{x-c}$.

❖ *Example*

(i) A function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ;\ x\neq 0\\ 0 & ;\ x=0\end{cases}$$

This function is differentiable at $x=0$ because

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$
$$=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x}=\lim_{x\to 0}x\sin\tfrac{1}{x}=0$$

(ii) Let $f(x)=x^n$; $n\geq 0$ (n is integer), $x\in\mathbb{R}$.

Then

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\frac{x^n-c^n}{x-c}$$
$$=\lim_{x\to c}\frac{(x-c)(x^{n-1}+cx^{n-2}+\dots\dots\dots+c^{n-2}x+c^{n-1})}{x-c}$$
$$=\lim_{x\to c}(x^{n-1}+cx^{n-2}+\dots\dots\dots+c^{n-2}x+c^{n-1})$$
$$=nc^{n-1}$$

implies that f is differentiable every where and $f'(x)=nx^{n-1}$.

❖ *Theorem*

Let f be defined on $[a,b]$, if f is differentiable at a point $x \in [a,b]$, then f is continuous at x. (Differentiability implies continuity)

Proof

We know that

$$\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x) \qquad \text{where } t \neq x \quad \text{and} \quad a<t<b$$

Now

$$\lim_{t\to x}\left(f(t)-f(x)\right) = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)\lim_{t\to x}\left(t-x\right)$$

$$= f'(x)\cdot 0$$

$$= 0$$

$$\Rightarrow \lim_{t\to x} f(t) = f(x).$$

Which show that f is continuous at x.

NOTE

(i) The converse of the above theorem does not hold.

Consider $f(x) = |x| = \begin{cases} x & if \;\; x \geq 0 \\ -x & if \;\; x < 0 \end{cases}$

$f'(0)$ does not exists but $f(x)$ is continuous at $x=0$

(ii) If f is discontinuous at $c \in \mathbf{D}_f$ then $f'(c)$ does not exists.

e.g.

$$f(x) = \begin{cases} 1 & if \;\; x>0 \\ 0 & if \;\; x \leq 0 \end{cases}$$

is discontinuous at $x=0$ therefore it is not differentiable at $x=0$.

(iii) f is differentiable at a point c if and only if $D_+ f(c)$ (right derivative) and $D_- f(c)$ (left derivative) exists and equal.

i.e. $D_+ f(c) = D_- f(c) = Df(c)$

❖ *Example*

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & if \;\; x>1 \\ x^3 & if \;\; x \leq 1 \end{cases}$$

then

$$D_+ f(1) = \lim_{\substack{x\to 1+h \\ h\to 0}} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{1+h-1} = \lim_{h\to 0} \frac{(1+h)^2-1}{h}$$

$$= \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} (2+h) = 2$$

and

$$D_- f(1) = \lim_{\substack{x\to 1-h \\ h\to 0}} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{h\to 0} \frac{f(1-h)-f(1)}{1-h-1} = \lim_{h\to 0} \frac{(1-h)^3-1}{-h}$$

$$= \lim_{h\to 0} \frac{1-3h+3h^2-h^3-1}{-h} = \lim_{h\to 0}\left(3-3h+h^2\right) = 3$$

Since $D_+ f(1) \neq D_- f(1) \Rightarrow f'(1)$ does not exist even though f is continuous at $x=1$. $f'(x)$ exist for all other values of x.

❖ *Theorem*

Suppose f and g are defined on $[a,b]$ and are differentiable at a point $x \in [a,b]$, then $f+g$, fg and $\dfrac{f}{g}$ are differentiable at x and

(i) $(f+g)'(x) = f'(x) + g'(x)$

(ii) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(iii) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

The proof of this theorem can be get from any F.Sc or B.Sc text book.

Note

The derivative of any constant is zero.

And if f is defined by $f(x)=x$ then $f'(x)=1$

And for $f(x)=x^n$ then $f'(x)=nx^{n-1}$ where n is positive integer, if $n<0$ we have to restrict ourselves to $x=0$.

Thus every polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is differentiable every where and so every rational function except at the point where denominator is zero.

❖ *Theorem (Chain Rule)*

Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$. A function g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$.

If $h(t) = g\left(f(t)\right)$; $a \leq t \leq b$

Then h is differentiable at x and $h'(x) = g'\left(f(x)\right) \cdot f'(x)$.

Proof

Let $y = f(x)$

By the definition of the derivative we have

$$f(t) - f(x) = (t-x)\left[f'(x) + u(t)\right] \quad .... (i)$$

and $$g(s) - g(y) = (s-y)\left[g'(y) + v(s)\right] \quad .... (ii)$$

where $t \in [a,b]$, $s \in I$ and $u(t) \to 0$ as $t \to x$ and $v(s) \to 0$ as $s \to y$.

Let us suppose $s = f(t)$ then

$$\begin{aligned} h(t) - h(x) &= g\left(f(t)\right) - g\left(f(x)\right) \\ &= \left[f(t) - f(x)\right]\left[g'(y) + v(s)\right] && \text{by } (ii) \\ &= (t-x)\left[f'(x) + u(t)\right]\left[g'(y) + v(s)\right] && \text{by } (i) \end{aligned}$$

or if $t \neq x$

$$\frac{h(t) - h(x)}{t-x} = \left[f'(x) + u(t)\right]\left[g'(y) + v(s)\right]$$

taking the limit as $t \to x$ we have

$$\begin{aligned} h'(x) &= \left[f'(x) + 0\right]\left[g'(y) + 0\right] \\ &= g'\left(f(x)\right) \cdot f'(x) && \because\ y = f(x) \end{aligned}$$

which is the required result.

It is known as *chain rule*.

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❖ Example

Let f be defined by

$$f(x)=\begin{cases} x\sin\frac{1}{x} & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$$

$$\Rightarrow \quad f'(x)=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x} \qquad \text{where } x\neq 0.$$

$\because$ at $x=0$, $\frac{1}{x}$ is not defined.

$\therefore$ Applying the definition of the derivative we have

$$f'(0)=\lim_{t\to 0}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0}\frac{t\sin\frac{1}{t}}{t} = \lim_{t\to 0}\sin\frac{1}{t}$$

which does not exit.

The derivative of the function $f(x)$ does not exist at $x=0$ but it is continuous at $x=0$ (i.e. it is not differentiable although it is continuous at $x=0$)

Same the case with absolute value function.

❖ Example

Let f be defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$$

We have $f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$ where $x\neq 0$.

$\because$ at $x=0$, $\frac{1}{x}$ is not defined.

$\therefore$ Applying the definition of the derivative we have

$$\left|\frac{f(t)-f(0)}{t-0}\right| = \left|t\sin\frac{1}{t}\right| \le t \qquad , (t\neq 0)$$

Taking limit as $t\to 0$ we see that $f'(0)=0$

Thus f is differentiable at points x but f' is not a continuous function, since $\cos\frac{1}{x}$ does not tend to a limit as $x\to 0$.

❖ Local Maximum

Let f be a real valued function defined on a metric space X , we say that f has a local maximum at a point $p\in X$ if there exist $\delta>0$ such that $f(q)\le f(p)$ $\forall$ $q\in X$ with $d(p,q)<\delta$.

Local minimum is defined likewise.

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❖ Theorem

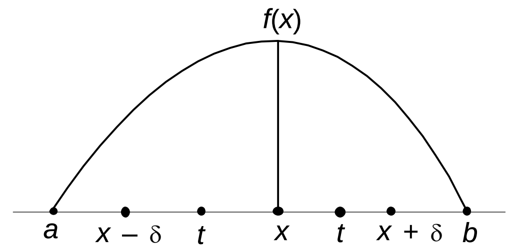
Let f be defined on $[a,b]$, if f has a local maximum at a point $x\in[a,b]$ and if $f'(x)$ exist then $f'(x)=0$.

(The analogous for local minimum is of course also true)

Proof

Choose δ such that

$$a<x-\delta<x<x+\delta<b$$



Now if $x-\delta<t<x$ then

$$\frac{f(t)-f(x)}{t-x}\geq 0$$

Taking limit as $t\to x$ we get

$$f'(x)\geq 0 \quad \ldots\ldots\ldots\ldots (i)$$

If $x<t<x+\delta$

Then

$$\frac{f(t)-f(x)}{t-x}\leq 0$$

Again taking limit when $t\to x$ we get

$$f'(x)\leq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Combining (i) and (ii) we have

$$f'(x)=0$$

❖ Generalized Mean Value Theorem

If f and g are continuous real valued functions on closed interval $[a,b]$, then there is a point $x\in(a,b)$ at which

$$[f(b)-f(a)]g'(x)=[g(b)-g(a)]f'(x)$$

The differentiability is not required at the end point.

Proof

Let

$$h(t)=[f(b)-f(a)]g(t)-[g(b)-g(a)]f(t) \qquad (a\leq t\leq b)$$

$\because$ h involves f and g therefore h is

i) Continuous on close interval $[a,b]$.

ii) Differentiable on open interval (a,b).

iii) and $h(a)=h(b)$.

To prove the theorem we have to show that $h'(x)=0$ for some $x\in(a,b)$

There are two cases to be discussed

(i) h is constant function.

$\Rightarrow$ $h'(x)=0 \quad \forall \quad x\in(a,b)$

(ii) If h is not constant.

then $h(t)>h(a)$ for some $t\in(a,b)$

Let x be the point in the interval (a,b) at which h attain its maximum,

then $h'(x)=0$

Similarly,

if $h(t)<h(a)$ for some $t\in(a,b)$ then $\exists$ a point $x\in(a,b)$ at which the function h attain its minimum and since the derivative at a local minimum is zero therefore we get $h'(x)=0$

Hence

$$h'(x)=[f(b)-f(a)]g'(x)-[g(b)-g(a)]f'(x)=0$$

This gives the desire result.

❖ *Geometric Interpretation of M.V.T.*

Consider a plane curve C represented by $x = f(t)$, $y = g(t)$ then theorem states that there is a point S on C between two points $P\big(f(a), g(a)\big)$ and $Q\big(f(b), g(b)\big)$ of C such that the tangent at S to the curve C is parallel to the chord PQ.

❖ *Lagrange's M.V.T.*

Let f be

i) continuous on $[a,b]$
ii) differentiable on (a,b)

then $\exists$ a point $x \in (a,b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(x)$.

Proof

Let us design a new function

$$h(t) = \big[f(b) - f(a)\big]t - (b-a)f(t) \qquad ,(a \le t \le b)$$

then clearly $h(a) = h(b)$

Since $h(t)$ depends upon t and $f(t)$ therefore it possess all the properties of f.

Now there are two cases

i) h is a constant.
implies that $h'(x) = 0 \qquad \forall \; x \in (a,b)$

ii) h is not a constant, then
if $h(t) > h(a)$ for some $t \in (a,b)$
then $\exists$ a point $x \in (a,b)$ at which h attains its maximum
implies that $h'(x) = 0$
and if $h(t) < h(a)$
then $\exists$ a point $x \in (a,b)$ at which h attain its minimum
implies that $h'(x) = 0$
$\because \; h(t) = \big[f(b) - f(a)\big]t - (b-a)f(t)$
$\therefore \; h'(x) = \big[f(b) - f(a)\big] - (b-a)f'(x)$

Which gives

$$\frac{f(b) - f(a)}{b - a} = f'(x) \quad \text{as desired.}$$

❖ *Theorem (Intermediate Value Theorem or Darboux,s Theorem)*

Suppose f is a real differentiable function on some interval $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$ then there exist a point $x \in (a,b)$ such that $f'(x) = \lambda$.

A similar result holds if $f'(a) > f'(b)$.

Proof

Put $g(t) = f(t) - \lambda t$

Then $g'(t) = f'(t) - \lambda$

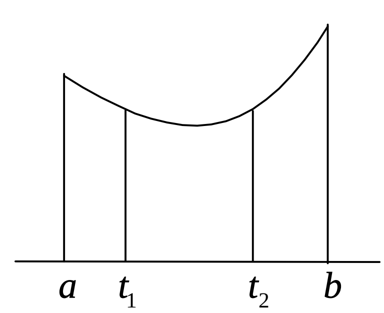
If $t = a$ we have

$$g'(a) = f'(a) - \lambda$$

$\because \; f'(a) - \lambda < 0 \qquad \therefore \; g'(a) < 0$

implies that g is monotonically decreasing at a.

$\Rightarrow \; \exists$ a point $t_1 \in (a,b)$ such that $g(a) > g(t_1)$.



Similarly,

$$g'(b) = f'(b) - \lambda$$

$\because \; f'(b) - \lambda > 0 \qquad \therefore \; g'(b) > 0$

implies that g is monotonically increasing at b.

$\Rightarrow$ $\exists$ a point $t_2 \in (a,b)$ such that $g(t_2) < g(b)$

$\Rightarrow$ the function attain its minimum on (a,b) at a point x (say)

such that $g'(x) = 0 \quad \Rightarrow \quad f'(x) - \lambda = 0$

$$\Rightarrow \quad f'(x) = \lambda .$$

Note

We know that a function f may have a derivative f' which exist at every point but is discontinuous at some point however not every function is a derivative. In particular derivatives which exist at every point on the interval have one important property in common with function which are continuous on an interval is that intermediate value are assumed.

The above theorem relates to this fact.

❖ Question

If a and c are real numbers, $c > 0$ and f is defined on $[-1,1]$ by

$$f(x) = \begin{cases} x^a \sin x^{-c} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

then discuss the differentiability as well as continuity at $x = 0$.

Solution

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$$= \lim_{t \to x} \frac{t^a \sin t^{-c} - x^a \sin x^{-c}}{t - x}$$

$$\Rightarrow \quad f'(0) = \lim_{t \to 0} \frac{t^a \sin t^{-c}}{t}$$

$$= \lim_{t \to 0} t^{a-1} \sin t^{-c}$$

If $a - 1 > 0$, then $\lim\limits_{t \to 0} t^{a-1} \sin t^{-c} = 0 \quad \Rightarrow \quad f'(0) = 0$ when $a > 0$.

If $a - 1 < 0$ i.e. when $a < 1$ we have $t^{a-1} = t^{-b}$ where $b > 0$

And $\lim\limits_{t \to 0} t^{a-1} \sin t^{-c} = \lim\limits_{t \to 0} t^{-b} \sin t^{-c}$

Which does not exist.

If $a - 1 = 0$, we get $\lim\limits_{t \to \infty} \sin t^{-c}$

Which also does not exist.

Hence $f'(0)$ exists if and only if $a > 1$.

Also $\lim\limits_{x \to 0} x^a \sin x^{-c}$ exist and zero when $a > 0$, which equals the actual value of the function $f(x)$ at zero.

Hence the function is continuous at $x = 0$.

❖ Question

Let f be defined for all real x and suppose that

$\left| f(x) - f(y) \right| \leq (x - y)^2 \quad \forall$ real x & y. Prove that f is constant.

Solution

Since $\quad \left| f(x) - f(y) \right| \leq (x - y)^2$

Therefore

$$-(x - y)^2 \leq f(x) - f(y) \leq (x - y)^2$$

Dividing throughout by $x - y$, we get

$$-(x-y) \le \frac{f(x)-f(y)}{x-y} \le (x-y) \quad \text{when} \quad x>y$$

and

$$-(x-y) \ge \frac{f(x)-f(y)}{x-y} \ge (x-y) \quad \text{when} \quad x<y$$

Taking limit as $x \to y$, we get

$$\left.\begin{array}{l} 0 \le f'(y) \le 0 \\ 0 \ge f'(y) \ge 0 \end{array}\right] \Rightarrow f'(y) = 0$$

which shows that function is constant.

❖ *Question*

If $f'(x) > 0$ in (a,b) then prove that f is strictly increasing in (a,b) and let g be its inverse function, prove that the function g is differentiable and that

$$g'\big(f(x)\big) = \frac{1}{f(x)} \quad ; \quad a<x<b$$

Solution

Let $y \in \big(f(a), f(b)\big)$

$\Rightarrow y = f(x)$ for some $x \in (a,b)$

$$\Rightarrow g'(y) = \lim_{z\to y} \frac{g(z)-g(y)}{z-y}$$

$$= \lim_{x_z \to x} g\big(f(x_z)\big) = \frac{g\big(f(x_z)\big) - g\big(f(x)\big)}{f(x_z)-f(x)}$$

$$= \lim_{x_z\to x} \frac{f^{-1}\big(f(x_z)\big) - f^{-1}\big(f(x)\big)}{f(x_z)-f(x)}$$

$$= \lim_{x_z\to x} \frac{x_z - x}{f(x_z)-f(x)} = \frac{1}{\lim\limits_{x_z\to x} \dfrac{f(x_z)-f(x)}{x_z-x}} = \frac{1}{f'(x)}$$

❖ *Question*

Suppose f is defined and differentiable for every $x>0$ and $f'(x) \to 0$ as $x \to +\infty$ put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Solution

Since f is defined and differentiable for $x>0$ therefore we can apply the Lagrange's M.V. T. to have

$$f(x+1) - f(x) = (x+1-x)\, f'(x_1) \quad \text{where} \quad x < x_1.$$

$\because f'(x) \to 0$ as $x \to \infty$

$\therefore f'(x_1) \to 0$ as $x \to \infty$

$\Rightarrow f(x+1) - f(x) \to 0$ as $x \to 0$

$\Rightarrow g(x) \to 0$ as $x \to 0$

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❖ Question (L Hospital Rule)

Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$ and $f(x) = g(x) = 0$.

Prove that $\lim\limits_{t\to x} \dfrac{f(t)}{g(t)} = \dfrac{f'(x)}{g'(x)}$

Proof

$$\lim_{t\to x} \frac{f(t)}{g(t)} = \lim_{t\to x} \frac{f(t)-0}{g(t)-0} = \lim_{t\to x} \frac{f(t)-f(x)}{g(t)-(x)} \qquad \because\ f(x) = g(x) = 0$$

$$= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \cdot \frac{t-x}{g(t)-(x)}$$

$$= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \cdot \lim_{t\to x} \frac{1}{\dfrac{g(t)-(x)}{t-x}}$$

$$= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \cdot \frac{1}{\lim\limits_{t\to x} \dfrac{g(t)-(x)}{t-x}} = f'(x) \cdot \frac{1}{g'(x)} = \frac{f'(x)}{g'(x)}$$

Q.E.D.

❖ Question

Suppose f is defined in the neighborhood of a point x and $f''(x)$ exists.

Show that $\lim\limits_{h\to 0} \dfrac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$

Solution

By use of Lagrange's Mean Value Theorem

$$f(x+h)+f(x) = hf'(x_1) \quad \text{where} \quad x < x_1 < x+h \ \ldots\ldots\ldots\ldots\ldots\ (i)$$

and

$$-[f(x-h)-f(x)] = hf'(x_2) \quad \text{where} \quad x-h < x_2 < x \ \ldots\ldots\ldots\ldots\ldots\ (ii)$$

Subtract (ii) from (i) to get

$$f(x+h)+f(x-h)-2f(x) = h[f'(x_1)-f'(x_2)]$$

$$\Rightarrow \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \frac{f'(x_1)-f'(x_2)}{h}$$

$\because\ x_2 - x_1 \to 0$ as $h \to 0$

therefore

$$\therefore \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{x_1\to x_2} \frac{f'(x_1)-f'(x_2)}{x_1-x_2}$$

$$= f''(x_2)$$

❖ Question

If $c_0 + \dfrac{c_1}{2} + \dfrac{c_2}{3} + \ldots\ldots\ldots + \dfrac{c_{n-1}}{n} + \dfrac{c_n}{n+1} = 0$

Where $c_0, c_1, c_2, \ldots\ldots, c_n$ are real constants.

Prove that $c_0 + c_1x + c_2x^2 + \ldots\ldots\ldots + c_nx^n = 0$ has at least one real root between 0 and 1.

Solution

Suppose $f(x) = c_0x + \dfrac{c_1}{2}x^2 + \ldots\ldots\ldots\ldots + \dfrac{c_n}{n+1}x^{n+1}$

Then $f(0) = 0$ and $f(1) = c_0 + \dfrac{c_1}{2} + \dfrac{c_2}{3} + \ldots\ldots\ldots\ldots + \dfrac{c_n}{n+1} = 0$

$\Rightarrow\ f(0) = f(1) = 0$

$\because\ f(x)$ is a polynomial therefore we have

i) It is continuous on $[0,1]$
ii) It is differentiable on $(0,1)$
iii) And $f(a)=0=f(b)$

$\Rightarrow$ the function f has local maximum or a local minimum at some point $x\in(0,1)$

$\Rightarrow$ $f'(x)=c_0+c_1x+c_2x^2+.........+c_nx^n=0$ for some $x\in(0,1)$

$\Rightarrow$ the given equation has real root between 0 and 1.

❖ Riemann Differentiation of Vector valued function

If $f(t)=f_1(t)+i\,f_2(t)$

$f'(t)=f_1'(t)+i\,f_2'(t)$

where $f_1(t)$ and $f_2(t)$ are the real and imaginary part of $f(t)$.

The Rule of differentiation of real valued functions are valid in case of vector valued function but the situation changes in the case of Mean Value Theorem.

❖ Example

Take $f(x)=e^{ix}=\cos x+i\sin x$ in $(0,2\pi)$.

Then $f(2\pi)=\cos 2\pi+i\sin 2\pi=1$

$f(0)=\cos(0)+i\sin(0)=1$

$\Rightarrow$ $f(2\pi)-f(0)=0$ but $f'(x)=i\,e^{ix}$

$\Rightarrow$ $\dfrac{f(2\pi)-f(0)}{2\pi-0}\neq i\,e^{ix}$ (there is no such x)

$\Rightarrow$ the M.V.T. fails.

In case of vector valued functions, the M.V.T. is not of the form as in the case of real valued function.

❖ Theorem

Let f be a continuous mapping of the interval $[a,b]$ into a space $\mathbb{R}^k$ and $\underline{f}$ be differentiable in (a,b) then $\exists$ $x\in(a,b)$ such that $\left|\underline{f}(b)-\underline{f}(a)\right|\le(b-a)\left|\underline{f}'(x)\right|$.

Proof

Put $\underline{z}=\underline{f}(b)-\underline{f}(a)$

And suppose $\varphi(t)=\underline{z}\cdot\underline{f}(t)$ $(a\le t\le b)$

$\varphi(t)$ so defined is a real valued function and it possess the properties of $f(t)$.

$\Rightarrow$ M.V.T. is applicable to $\varphi(t)$.

We have $\varphi(b)-\varphi(a)=(b-a)\varphi'(x)$

i.e. $\varphi(b)-\varphi(a)=(b-a)\underline{z}\cdot\underline{f}'(x)$ for some $x\in(a,b)$ ……….. (i)

Also $\varphi(b)=\underline{z}\cdot\underline{f}(b)$ and $\varphi(a)=\underline{z}\cdot\underline{f}(a)$

$\Rightarrow$ $\varphi(b)-\varphi(a)=\underline{z}\cdot\left(\underline{f}(b)-\underline{f}(a)\right)$ …………. (ii)

from (i) and (ii)

$$\underline{z}\cdot\underline{z}=(b-a)\underline{z}\cdot f'(x)$$
$$\le(b-a)\left|\underline{z}\right|\left|\underline{f}'(x)\right|$$

$\Rightarrow$ $\left|\underline{z}\right|^2\le(b-a)\left|\underline{z}\right|\left|\underline{f}'(x)\right|$

$\Rightarrow$ $\left|\underline{z}\right|\le(b-a)\left|\underline{f}'(x)\right|$

i.e. $\left|\underline{f}(b)-\underline{f}(a)\right|\le(b-a)\left|\underline{f}'(x)\right|$ $\because$ $\underline{z}=\underline{f}(b)-\underline{f}(a)$

which is the required result.

❖ *Question*

If $f(x)=\left|x^3\right|$, then compute $f'(x),\ f''(x)$ and $f'''(x)$, and show that $f'''(0)$ does not exist.

Solution

$$f(x)=\left|x^3\right|=\begin{cases} x^3 & if\quad x\geq 0 \\ -x^3 & if\quad x<0 \end{cases}$$

Now $D_+f(0)=\lim\limits_{x\to 0+0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0+0}\dfrac{x^3-0}{x-0}=\lim\limits_{x\to 0+0}x^2=0$

& $D_-f(0)=\lim\limits_{x\to 0-0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0-0}\dfrac{-x^3-0}{x-0}=\lim\limits_{x\to 0-0}(-x^2)=0$

$\because\quad D_+f(x)=D_-f(x)$

$\therefore\quad f'(x)$ exists at $x=0$ & $f'(0)=0$.

Now if $x\neq 0$ and $x>0$ then

$f(x)=x^3 \quad\Rightarrow\quad f'(x)=3x^2$

and if $x\neq 0$ and $x<0$ then

$f(x)=-x^3 \quad\Rightarrow\quad f'(x)=-3x^2$

i.e. $$f'(x)=\begin{cases} 3x^2 & if\quad x>0 \\ 0 & if\quad x=0 \\ -3x^2 & if\quad x<0 \end{cases}$$

Now $D_+f'(0)=\lim\limits_{x\to 0+0}\dfrac{f'(x)-f'(0)}{x-0}=\lim\limits_{x\to 0+0}\dfrac{3x^2-0}{x-0}$

$=\lim\limits_{x\to 0+0}3x=0$

And Now $D_-f'(0)=\lim\limits_{x\to 0-0}\dfrac{f'(x)-f'(0)}{x-0}=\lim\limits_{x\to 0-0}\dfrac{-3x^2-0}{x-0}$

$=\lim\limits_{x\to 0+0}(-3x)=0$

$\because\quad D_+f'(x)=D_-f'(x)$

$\therefore\quad f''(x)$ exists at $x=0$ & $f''(0)=0$.

Now if $x\neq 0$ and $x>0$ then

$f'(x)=3x^2 \quad\Rightarrow\quad f''(x)=6x$

and if $x\neq 0$ and $x<0$ then

$f'(x)=-3x^2 \quad\Rightarrow\quad f''(x)=-6x$

i.e. $$f''(x)=\begin{cases} 6x & if\quad x>0 \\ 0 & if\quad x=0 \\ -6x & if\quad x<0 \end{cases}$$

Now $D_+f''(0)=\lim\limits_{x\to 0+0}\dfrac{f''(x)-f''(0)}{x-0}=\lim\limits_{x\to 0+0}\dfrac{6x-0}{x-0}=6$

And $D_-f''(0)=\lim\limits_{x\to 0-0}\dfrac{f''(x)-f''(0)}{x-0}=\lim\limits_{x\to 0-0}\dfrac{-6x-0}{x-0}=-6$

$\because\quad D_+f''(0)\neq D_-f''(0)$

$\therefore\quad f'''(0)$ doest not exist.

But $f'''(0)$ exist if $x\neq 0$, and equal to 6 if $x>0$ and equal to -6 if $x<0$.

………………………………….

ENOUGH