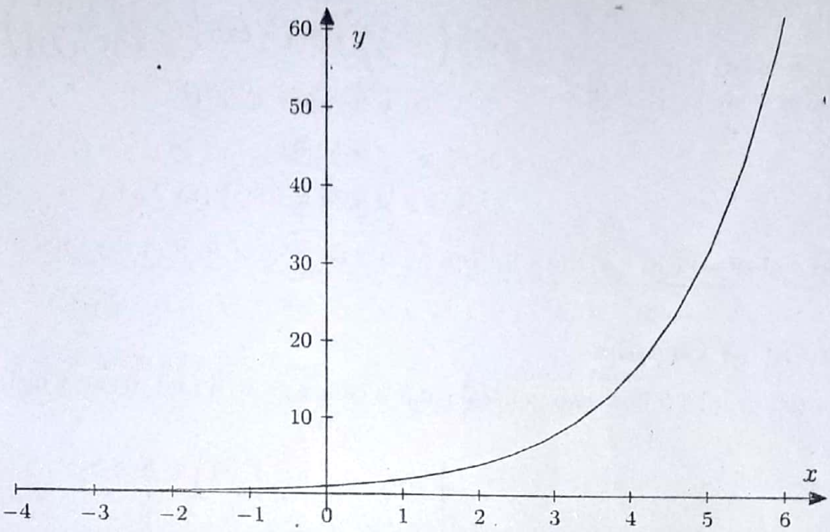
2.4 Exponentials

We may use the term 'exponential growth' as a way to describe the rate of a reaction as the temperature increases. In using this phrase we are describing a very fast rate of increase. We can represent this idea mathematically with one of the simplest exponential equations:

$$\boxed{y = a^x}$$

where a is a constant, x is the controlled variable (in the above case temperature) and y is the observed variable (in the above case rate of reaction).



Above we have the graph of $y = 2^x$ and we can see the graph increases at very fast rate which is what we would expect with exponential growth. Other mathematical things to note are:

- The graph passes through the point $(0, 1)$ as any variable raised to the power of 0 is equal to 1.
- The values of y are small and positive when x is negative.
- The values of y are large and positive when x is positive.

The Exponential Function

So far we have used the word exponential to describe equations in the form $y = a^x$ however the word is usually reserved to describe a special type of function. It is quite likely we will have seen the symbol π before and know it represents the infinite number 3.14159265... In this section we will be working with the number:

$$e = 2.7182818\ldots$$

Like π, e is another number that goes on forever! When we refer to *the* exponential function we mean $y = e^x$. This can be seen as a special case of the previous section when the constant is $a = e$ in the equation $y = a^x$. When we refer to e^x we say 'e to the x'.

Note: Another notation for e^x is $\exp(x)$. They mean the same thing.

Chemistry Example: The Arrhenius equation below describes the exponential relationship between the rate of a reaction k and the temperature T

$$k = A\exp\left(-\frac{E_a}{RT}\right)$$

where R, E_a and A are all constants. Suppose for a reaction that the activation energy is $E_a =$ 52.0 kJ mol^{-1}, the gas constant $R = 8.31$ J K^{-1} mol^{-1} and $A = 1.00$. What is the rate of the reaction k when the temperature $T = 241$ K?

Solution: We substitute our given values from the question into the Arrhenius equation.

$$k = A\exp\left(-\frac{E_a}{RT}\right) = 1 \times \exp\left(-\frac{52 \times 10^3}{8.31 \times 241}\right)$$

$$\Longrightarrow k = \exp(-25.9648\ldots)$$

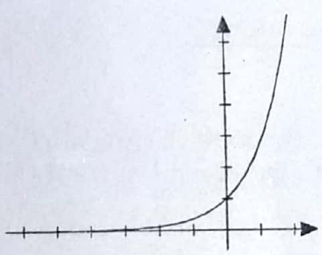
$$\Longrightarrow k = 5.2920\ldots \times 10^{-12}$$

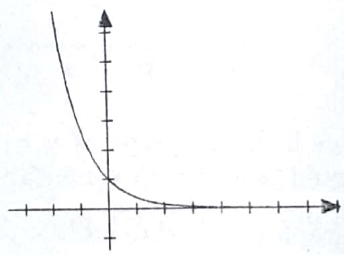
$$\Longrightarrow k = 5.29 \times 10^{-12} \text{ (to 3 s.f.)}$$

Remember we can use the e button on our calculator to find the final answer.

Exponential Graphs

Exponential graphs follow two general shapes called growth and decay which are shown below.





- On the left exponential growth which is represented by the equation $y = e^x$ note that the graph gets steeper from left to right.
- On the right exponential decay which is represented by the equation $y = e^{-x}$ note that the graph gets shallower from left to right.

Algebraic Rules for Exponentials

We use very similar rules as the ones we had for powers. The only difference is that the base in now constant and the power is a variable.

1. $a^x \times a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{x \times y}$

Note: Above we have that x and y are variables and a is a constant for example $a = e$ or $a = 3$. And order to use the rules a must be the same for all terms. For example we can not simplify $e^x \times 3^x \times$

Example: Simplify the following equations:

1. $y = 3^x \times 3^{2x}$
2. $y = \frac{2^x}{4}$
3. $y = \left(e^{-x+1}\right)^x$

Solution:

1. Since the bases are the same we use the first rule.

$$y = 3^x \times 3^{2x} = 3^{x+2x} = 3^{3x}$$

2. The first thing we notice is that 4 can be written as 2^2 so:

$$y = \frac{2^x}{2^2}.$$

Now are we have powers to the same base we can apply the rules. Using the second rule we have:

$$y = 2^{x-2}$$

3. Using the third rule we get that:

$$y = e^{(-x+1)\times x} = e^{-x^2+x}$$

2.5 Logarithms

Just as addition and multiplication have inverses (subtraction and division) so we have inverses for exponentials. These are called logarithms. A logarithm answers the question 'how many of one number do we multiply to get another number?'

Example: How many 2's do we multiply to get 8?

Solution: $2 \times 2 \times 2 = 8$ so we need to multiply 3 of the 2's to get 8. So the logarithm is 3.

We have a notation for logarithms for example the last example would be written as:

$$\log_2(8) = 3$$

where this tells us we need multiply 2 by itself 3 times to get 8.

So more generally we can write:

$$\boxed{\log_a(x) = y}$$

- This tells us we need multiply a by itself y times to get x. (In other words $x = a^y$).
- We say that a is the base of the logarithm.

We can see clearly that as logarithms are the inverse of exponents there exists a relationship between the two given by:

$$\boxed{\text{If } y = a^x \text{ then } x = \log_a(y)}$$

Example: Write $81^{0.5} = 9$ as a logarithm.

Solution: So we need to multiply 81 by itself 0.5 times to get 9 so we have $\log_{81}(9) = 0.5$

Example: If $10^x = 3$ then find x.

Solution: We need to multiply 10 by itself x times to get 3 so we have:

$$x = \log_{10}(3) = 0.477 \text{ to 3 s.f.}$$

Logarithms are useful for expressing quantities that span several orders of magnitude. For example the pH equation $\text{pH} = -\log_{10}[H^+]$ as small change in pH results in a very large change in $[H^+]$.

Logarithms: The Inverses of Exponentials

As taking a logarithm is the inverse of an exponentials to the same base. We can cancel them using the rules below.

$$\boxed{\log_a(a^x) = x}$$

$$\boxed{a^{\log_a(x)} = x}$$

IMPORTANT:

1. When cancelling in the first case all the quantities in the logarithm must be contained in the 'power' of the exponential. For example $\log_a(a^x + 3) \neq x + 3$ as the 3 is not part of the power of the exponential.
2. When cancelling in the second case all of the power must be contained in the logarithm. For example $a^{\log_a(x)+4} \neq x + 4$ as the 4 is not in the logarithm.

Example: Simplify the following:

1. $\log_{10}(10^{5x^3+3x})$
2. $3^{\log_3(x^7+1)}$
3. $\log_3(10^{4x})$
4. $a^{\log_a(4x^3)+2x}$
5. $\log_8(8^{2x^4}+7)$

Solution:

1. $\log_{10}(10^{5x^3+3x}) = 5x^3 + 3x$
2. $3^{\log_3(x^7+1)} = x^7 + 1$
3. We cannot simplify this as the logarithm has the base 3 and the exponential is to the base 10.
4. First note that $a^{\log_a(4x^3)+2x} \neq 4x^3 + 2x$. We do this as follows:
$$a^{\log_a(4x^3)+2x} = a^{\log_a(4x^3)}a^{2x} = 4x^3a^{2x}$$
5. We cannot simplify in this case as the 7 is not part of the exponential.

Logarithms to the Base 10

Many equations in chemistry include logarithms and we tend to use only two logarithms to different bases. The first one is to the base 10 such as in the pH equation $\text{pH} = -\log_{10}[\text{H}^+]$. In the usual notation we would write $\log_{10}$ however we tend to write logs to the base 10 as log so $\text{pH} = -\log[\text{H}^+]$. All the rules we have looked at are the same just when the base $a = 10$ so in particular:

$$\boxed{\text{If } y = 10^x \text{ then } x = \log(y)}$$

Logarithms to the Base e

The second common logarithm found in chemistry is the logarithm to the base e known as the 'natural logarithm'. We use the notation $\ln(x)$ but this means the same as $\log_e(x)$. The natural logarithm is the inverse operation for the exponential ($y = e^x$) so we have the relationship as before:

$$\boxed{\text{If } y = e^x \text{ then } x = \ln(y)}$$

Example:

If $3 = e^x$ then $x = \ln 3 \approx 1.09861$

As natural logarithms are the inverse operation to exponentials ($y = e^x$) we have the following rules:

$$\boxed{\ln(e^x) = x}$$

$$\boxed{e^{\ln(x)} = x}$$

Note: This rules are the same as the more general ones earlier in the section however look different due to the ln notation.

Example:

We can simplify $e^{\ln(x^4+2)} = x^4 + 2$.

Example:

We can simplify $\ln\left(e^{(5x^4+2^x)}\right) = 5x^4 + 2^x$.

Laws of Logarithms

There are 3 laws of logarithms that we use to help algebraically manipulate logarithms:

1. $\log_a(x \times y) = \log_a(x) + \log_a(y)$
2. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
3. $\log_a(x^b) = b \times \log_a(x)$

Note: Remember we write $\ln(x)$ for $\log_e(x)$.

Example: Simplify each of the following into a single logarithm:

1. $\log 5 + \log 2$
2. $\log_3 2 + \log_3 2 + \log_3 2$
3. $3\log 3 - 2\log 2$
4. $\ln 8 - \ln 4$

Solution:

1. Using the first law we can see that:

$$\log 5 + \log 2 = \log 5 \times 2 = \log 10.$$

Since log is used to denote the base 10 logarithm, $\log 10 = 1$

2. This can be done using law 1 or by using law 3:

$$\log_3 2 + \log_3 2 + \log_3 2 = 3\log_3 2 = \log_3 2^3 = \log_3 8$$

3. Law 3 needs to be used to remove the co-efficient in front of the logs:

$$3\log 3 - 2\log 2 = \log 3^3 - \log 2^2 = \log 27 - \log 4.$$

Now law 2 is applied to produce, $\log\frac{27}{4}$ which can't be simplified further.

4. Law 2 is used to give:

$$\ln 8 - \ln 4 = \ln\frac{8}{4} = \ln 2$$

Chemistry Example: A general reaction takes place of the form:

$$a\,\mathrm{A} + b\,\mathrm{B} \longrightarrow c\,\mathrm{C} + d\,\mathrm{D}$$

where the equilibrium constant K is defined as $K = [\mathrm{A}]^{-a} \times [\mathrm{B}]^{-b} \times [\mathrm{C}]^{c} \times [\mathrm{D}]^{d}$. Find the logarithm of K as a sum of logarithms.

Solution: First we take the log of both sides of the equation.

$$\log K = \log\left([\mathrm{A}]^{-a} \times [\mathrm{B}]^{-b} \times [\mathrm{C}]^{c} \times [\mathrm{D}]^{d}\right)$$

Then using law 1 we can expand this to be a sum of several logs.

$$\log K = \log\left([\mathrm{A}]^{-a}\right) + \log\left([\mathrm{B}]^{-b}\right) + \log\left([\mathrm{C}]^{c}\right) + \log\left([\mathrm{D}]^{d}\right)$$

Now we can use law 3 to bring the powers down.

$$\log K = -a\log[\mathrm{A}] - b\log[\mathrm{B}] + c\log[\mathrm{C}] + d\log[\mathrm{D}]$$

Converting between Logarithms to Different Bases

We may wish to convert between the different logarithms of different bases so for example we might want to convert $\log(x)$ to $\ln(x)$. We do this using the following formula:

$$\log_a(x) = \frac{1}{\log_b(a)} \times \log_b(x)$$

Example: Convert $\log(x)$ to be in the form $\ln(x)$

Solution: Remember that $\log(x) = \log_{10}(x)$ and $\ln(x) = \log_e(x)$. So using the formula above we have:

$$\log_{10}(x) = \frac{1}{\log_e(10)} \times \log_e(x)$$

$$\Longrightarrow \log_{10}(x) = \frac{\ln(x)}{\ln(10)}$$

$\log_{10} x =$