CHAPTER 3

Probability and Probability Distributions

3.1 PROBABILITY OF A SINGLE EVENT

If event A can occur in n_A ways out of a total of N possible and equally likely outcomes, the probability that event A will occur is given by

$$P(A) = \frac{n_A}{N} \tag{3.1}$$

where $P(A)$ = probability that event A will occur
n_A = number of ways that event A can occur
N = total number of equally possible outcomes

Probability can be visualized with a *Venn diagram*. In Fig. 3-1, the circle represents event A, and the *total* area of the rectangle represents all possible outcomes.

$P(A)$ ranges between 0 and 1:

$$0 \le P(A) \le 1 \tag{3.2}$$

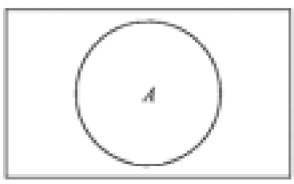

Fig. 3-1

If $P(A) = 0$, event A cannot occur. If $P(A) = 1$, event A will occur with certainty.

If $P(A')$ represents the probability of *nonoccurrence* of event A, then

$$P(A) + P(A') = 1 \tag{3.3}$$

EXAMPLE 1. A head (H) and a tail (T) are the two equally possible outcomes in tossing a balanced coin. Thus

$$P(\mathrm{H}) = \frac{n_\mathrm{H}}{N} = \frac{1}{2}$$

$$P(\mathrm{T}) = \frac{n_\mathrm{T}}{N} = \frac{1}{2}$$

and

$$P(\mathrm{H}) + P(\mathrm{T}) = 1$$

EXAMPLE 2. In rolling a fair die once, there are six possible and equally likely outcomes: 1, 2, 3, 4, 5, and 6. Thus

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

The probability of not rolling a 1 is

$$P(1') = 1 - P(1) = 1 - \frac{1}{6} = \frac{5}{6}$$

and

$$P(1) + P(1') = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

EXAMPLE 3. A card deck has 52 cards divided into 4 suits (diamonds, hearts, clubs, and spades) with 13 cards in each suit (1, 2, 3, . . . , 10, jack, queen, king). If the deck is well-shuffled, each of the 52 cards is equally likely to be picked. Since there are 4 jacks, the probability of picking a jack, J, on a single pick is

$$\mathrm{J} = \frac{n_\mathrm{J}}{N} = \frac{4}{52} = \frac{1}{13}$$

Since there are 13 diamonds, D

$$P(\mathrm{D}') = 1 - P(\mathrm{D}) = 1 - \frac{13}{52} = 1 - \frac{1}{4} = \frac{3}{4}$$

and

$$P(\mathrm{D}) + P(\mathrm{D}') = \frac{1}{4} + \frac{3}{4} = 1$$

EXAMPLE 4. Suppose that in 100 tosses of a balanced coin, we get 53 heads and 47 tails. The relative frequency of heads is 53/100, or 0.53. This is the *relative frequency* or *empirical probability*, which is to be distinguished from the *a priori* or *classical probability* of $P(\mathrm{H}) = 0.5$. As the number of tosses increases and approaches infinity in the limit, the relative frequency or empirical probability approaches the a priori or classical probability. For example, the relative frequency or empirical probability might be 0.517 or 1000 tosses, 0.508 for 10,000 tosses, and so on.

3.2 PROBABILITY OF MULTIPLE EVENTS

1. *Rule of addition for nonmutually exclusive events*. Two events, A and B, are *not mutually exclusive* if the occurrence of A does not preclude the occurrence of B, or vice versa. Then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \tag{3.4}$$

   $P(A \text{ and } B)$ is subtracted to avoid double counting. This can be seen with the Venn diagram in Fig. 3.2.

2. *Rule of addition for mutually exclusive events.* Two events, A and B, are *mutually exclusive* if the occurrence of A precludes the occurrence of B, or vice versa [$P(A \text{ and } B) = 0$]. Then

$$P(A \text{ and } B) = P(A) + P(B) \tag{3.5}$$

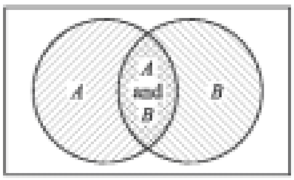

Fig. 3-2

3. *Rule of multiplication for dependent events.* Two events are *dependent* if the occurrence of one is connected in some way with the occurrence of the other. Then the *joint probability* of A and B is

$$P(A \text{ and } B) = P(A) \cdot P(B/A) \tag{3.6}$$

This reads: "The probability that *both* events A and B will take place equals the probability of event A times the probability of event B, given that event A has already occurred."

$$P(B/A) = \textit{conditional probability of } B\text{, given that } A \text{ has already occurred} \tag{3.7}$$

and $$P(A \text{ and } B) = P(B \text{ and } A) \tag{3.8}$$

See Prob. 3.15(c) and (d).

4. *Rule of multiplication for independent events.* Two events, A and B, are *independent* if the occurrence of A is not connected in any way to the occurrence of B. $[P(B/A) = P(B)]$. Then

$$P(A \text{ and } B) = P(A) \cdot P(B) \tag{3.9}$$

EXAMPLE 5. On a single toss of a die, we can get only one of six possible outcomes: 1, 2, 3, 4, 5, or 6. These are *mutually exclusive* events. If the die is fair, $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$. The probability of getting a 2 *or* a 3 on a single toss of the die is

$$P(2 \text{ or } 3) = P(2) + P(3) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Similarly $$P(2 \text{ or } 3 \text{ or } 4) = P(2) + P(3) + P(4) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

EXAMPLE 6. Picking at random a spade or a king on a single pick from a well-shuffled card deck does *not* constitute two *mutually exclusive* events because we could pick the king of spades. Thus

$$P(\text{S or K}) = P(\text{S}) + P(\text{K}) - P(\text{S and K}) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Using *set theory*, the preceding statement can be rewritten in an equivalent way as

$$P(\text{S} \cup \text{K}) = P(\text{S}) + P(\text{K}) - P(\text{S} \cap \text{K}) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

where the symbol $\cup$ (read "union") replaces *or* and $\cap$ (read "intersection") replaces *and*.

EXAMPLE 7. The outcomes of two successive tosses of a balanced coin are *independent* events. The outcome of the first toss in no way affects the outcome on the second toss. Thus

$$P(\text{H and H}) = P(\text{H} \cap \text{H}) = P(\text{H}) \cdot P(\text{H}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}, \text{ or } 0.25$$

Similarly, $$P(\text{H and H and H}) = P(\text{H} \cap \text{H} \cap \text{H}) = P(\text{H}) \cdot P(\text{H}) \cdot P(\text{H}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}, \text{ or } 0.125$$

EXAMPLE 8. The probability that on the first pick from a deck we get the king of diamonds is

$$P(\text{K}_\text{D}) = \frac{1}{52}$$

If the first card picked was indeed the king of diamonds and if the first card was not replaced, the probability of getting another king on the second pick is *dependent* on the first pick because there are now only 3 kings and 51 cards left in the deck. The *conditional probability* of picking another king, given that the king of dimaonds was already picked and not replaced, is

$$P(\mathrm{K}/\mathrm{K_D}) = \frac{3}{51}$$

Thus the probability of picking the king of diamonds on the first pick and, without replacement, picking another king on the second pick is

$$P(\mathrm{K_D} \text{ and } \mathrm{K}) = P(\mathrm{K_D}) \cdot P(\mathrm{K}/\mathrm{K_D}) = \frac{1}{52} \cdot \frac{3}{51} = \frac{3}{2652}$$

or about 1 in 1000. Related to conditional probability is Bayes' theorem (see Prob. 3.17). Problem 3.18 reviews combinations and permutations, or "counting techniques."

3.3 DISCRETE PROBABILITY DISTRIBUTIONS: THE BINOMIAL DISTRIBUTION

A *random variable* is a variable whose values are associated with some probability of being observed. A *discrete* (as opposed to *continuous*) random variable is one that can assume only finite and distinct values. The set of all possible values of a random variable and its associated probabilities is called a *probability distribution*. The sum of all probabilities equals 1 (see Example 9).

One discrete probability distribution is the *binomial distribution*. This is used to find the probability of X number of occurrences or successes of an event, $P(X)$, in n trials of the same experiment when (1) there are only *two* possible and mutually *exclusive outcomes*, (2) the n trials are *independent*, and (3) the probability of occurrence or success, p, remains *constant* in each trial. Then

$$P(X) = \frac{n!}{X!(n-X)!} p^X (1-p)^{n-X} \tag{3.10}$$

where $n!$ (read "n factorial") $= n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$, and $0! = 1$ by definition (see Prob. 3.18).

The mean of the binomial distribution is

$$\mu = np \tag{3.11}$$

The standard deviation is

$$\sigma = \sqrt{np(1-p)} \tag{3.12}$$

If $p = 1 - p = 0.5$, the binomial distribution is symmetrical; if $p < 0.5$, it is skewed to the right; and if $p > 0.5$, it is skewed to the left.

EXAMPLE 9. The possible outcomes in 2 tosses of a balanced coin are TT, TH, HT, and HH. Thus

$$P(0\mathrm{H}) = \frac{1}{4} \qquad P(1\mathrm{H}) = \frac{1}{2} \qquad \text{and} \qquad P(2\mathrm{H}) = \frac{1}{4}$$

The number of heads is therefore a discrete random variable, and the set of all possible outcomes with their associated probabilities is a discrete probability distribution (see Table 3.1 and Fig. 3-3).

Table 3.1 Probability Distribution of Heads in Two Tosses of a Balanced Coin

Number of Heads	Possible Outcomes	Probability
0	TT	0.25
1	TH, HT	0.50
2	HH	0.25
		1.00

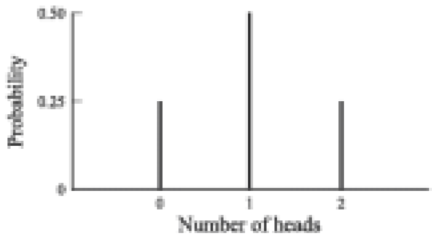

Fig. 3-3 Probability Distribution of Heads in Two Tosses of a Balanced Coin

EXAMPLE 10. Using the binomial distribution, we can find the probability of 4 heads in 6 flips of a balanced coin as follows:

$$P(4) = \frac{6!}{4!(6-4)!}(1/2)^4(1/2)^2 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}(1/16)(1/4) = 15(1/64) = \frac{15}{64} \cong 0.23$$

When n and X are large numbers, lengthy calculations to find probabilities can be avoided by using App. 1. The *expected* number of heads in 6 flips $= \mu = np = (6)(1/2) = 3$ heads. The standard deviation of the probability distribution of 6 flips is

$$\sigma = \sqrt{np(1-p)} = \sqrt{(6)(1/2)(1/2)} = \sqrt{6/4} = \sqrt{1.5} \cong 1.22 \text{ heads}$$

Because $p = 0.5$, this probability distribution is symmetrical. If we were not dealing with a coin and the trials were not dependent (as in sampling without replacement), we would have had to use the *hypergeometric distribution* (see Prob. 3.27).

3.4 THE POISSON DISTRIBUTION

The *Poisson distribution* is another discrete probability distribution. It is used to determine the probability of a designated number of successes *per unit of time*, when the events or successes are independent and the average number of successes per unit of time remains constant. Then

$$P(X) = \frac{\lambda^X e^{-\lambda}}{X!} \tag{3.13}$$

where X = designated number of successes
$P(X)$ = probability of X number of successes
λ = (Greek letter lambda) = average number of successes per unit of time
e = base of the natural logarithmic system, or 2.71828

Given the value of λ (the expected value or mean *and* variance of the Poisson distribution), we can find $e^{-\lambda}$ from App. 2, substitute in Eq. (*3.13*), and find $P(X)$.

EXAMPLE 11. A police department receives an average of 5 calls per hour. The probability of receiving 2 calls in a randomly selected hour is

$$P(X) = \frac{\lambda^X e^{-\lambda}}{X!} = \frac{5^2 e^{-5}}{2!} = \frac{(25)(0.00674)}{2} = 0.08425$$

The Poisson distribution can be used as an approximation to the binomial distribution when n is large and p or $1-p$ is small [say, $n \geq 30$ and $np < 5$ or $n(1-p) < 5$]. See Prob. 3.30.

3.5 CONTINUOUS PROBABILITY DISTRIBUTIONS: THE NORMAL DISTRIBUTION

A *continuous random variable* X is one that can assume an infinite number of values within any given interval. The probability that X falls within any interval is given by the area under the probability distribution (or density function) within that interval. The total area (probability) under the curve is 1 (see Prob. 3.31).

The *normal distribution* is a continuous probability distribution and the most commonly used distribution in statistical analysis (see Prob. 3.32). The normal curve is bell-shaped and symmetrical about its mean. It extends indefinitely in both directions, but most of the area (probability) is clustered around the mean (see Fig. 3-4); 68.26% of the area (probability) under the normal curve is included within one standard deviation of the mean (i.e., within $\mu \pm 1\sigma$), 95.44% within $\mu \pm 2\sigma$, and 99.74% within $\mu \pm 3\sigma$.

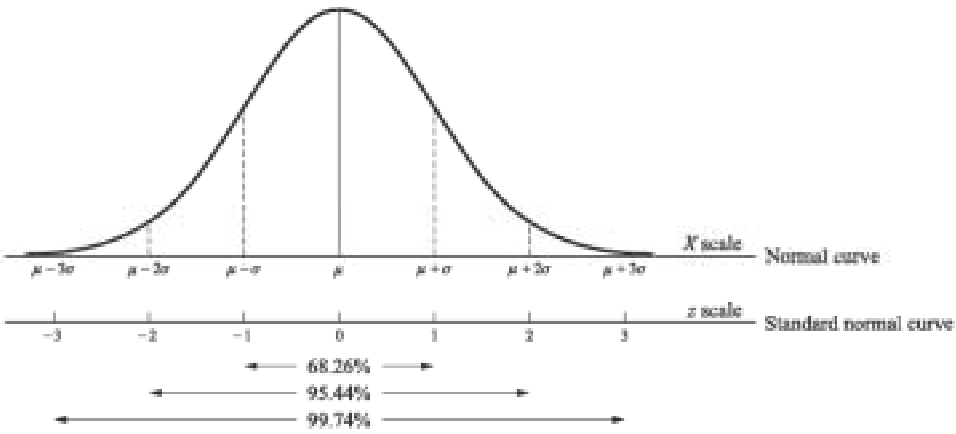

Fig. 3-4

The *standard normal distribution* is a normal distribution with a mean of 0 and a standard deviation of 1 (i.e., $\mu = 0$ and $\sigma = 1$). Any normal distribution (X scale in Fig. 3-4) can be converted into a standard normal distribution by letting $\mu = 0$ and expressing deviations from μ in standard deviation units (z scale).

To find probabilities (areas) for problems involving the normal distribution, we first convert the X value into its corresponding z value, as follows:

$$z = \frac{X - \mu}{\sigma} \tag{3.14}$$

Then we look up the z value in App. 3. This gives the proportion of the area (probability) included under the curve between the mean and that z value.

EXAMPLE 12. The area (probability) under the standard normal curve between $z = 0$ and $z = 1.96$ is obtained by looking up the value of 1.96 in App. 3. We move down the z column in the table to 1.9 and then across unitl we are below the column headed 0.06. The value that we get is 0.4750. This means that 47.50% of the total area (of 1, or 100%) under the curve lies between $z = 0$ and $z = 1.96$ (the shaded area in the figure above the table). Because of symmetry, the area between $z = 0$ and $z = -1.96$ (not given in the table) is also 0.4750, or 47.50%.

EXAMPLE 13. Suppose that X is a normally distributed random variable with $\mu = 10$ and $\sigma^2 = 4$ and we want to find the probability of X assuming a value between 8 and 12. We first calculate the z values corresponding to the X values of 8 and 12 and then look up these z values in App. 3:

$$z_1 = \frac{X_1 - \mu}{\sigma} = \frac{8 - 10}{2} = -1 \qquad \text{and} \qquad z_2 = \frac{X_2 - \mu}{\sigma} = \frac{12 - 10}{2} = +1$$

For $z = 1$, we get 0.3413 from App. 3. Then, $z = \pm 1$ equals 2(0.3413), or 0.6826. This means that the probability of X assuming a value between 8 and 12, or $P(8 < X < 12)$, is 68.26% (see Fig. 3-4).

EXAMPLE 14. Suppose again that X is a normally distributed random variable with $\mu = 10$ and $\sigma^2 = 4$. The probability that X will assume a value between 7 and 14 can be found as follows:

$$z_1 = \frac{X_1 - \mu}{\sigma} = \frac{7 - 10}{2} = -1.5 \qquad \text{and} \qquad z_2 = \frac{X_2 - \mu}{\sigma} = \frac{14 - 10}{2} = 2$$

For $z_1 = -1.5$, we look up 1.50 in App. 3 and get 0.4322. For $z_2 = 2$, we get 0.4772. Therefore, $P(7 < X < 14) = 0.4332 + 0.4772 = 0.9104$, or 91.04% (see Fig. 3-5). Therefore, the probability of X assuming a value *smaller than* 7 or *larger than* 14 (the unshaded tail *areas* in Fig. 3-5) is $1 - 0.9104 = 0.0896$, 8.96%. The normal distribution approximates the binomial distribution when $n \geq 30$ and both $np > 5$ and $n(1 - p) > 5$, and it approximates the Poisson distribution when $\lambda \geq 10$ (see Probs. 3.37 and 3.38). Another continuous probability distribution is the *exponential distribution* (see Prob. 3.39). *Chebyshev's theorem*, or *inequality*, states that regardless of the shape of a distribution, the proportion of the observations or area falling within K standard deviations of the mean is at least $1 - 1/K^2$, for $K \geq 1$ (see Probs. 3.40 and 3.72).

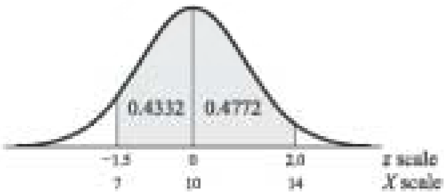

Fig. 3-5

Solved Problems

PROBABILITY OF A SINGLE EVENT

3.1 (*a*) Distinguish among classical or a priori probability, relative frequency or empirical probability, and subjective or personalistic probability. (*b*) What is the disadvantage of each? (*c*) Why do we study probability theory?

(*a*) According to *classical probability*, the probability of an event A is given by

$$P(A) = \frac{n_A}{N}$$

where $P(A)$ = probability that event A will occur
n_A = number of ways event A can occur
N = total number of equally possible outcomes

By the classical approach, we can make probability statements about balanced coins, fair dice, and standard card decks a priori, or without tossing a coin, rolling a die, or drawing a card. *Relative frequency or empirical probability* is given by the ratio of the number of times an event occurs to the total number of actual outcomes or observations. As the number of experiments or trials (such as the tossing of a coin) increases, the relative frequency or empirical probability approaches the classical or a priori probability. *Subjective* or *personalistic probability* refers to the *degree of belief* of an individual that the event will occur, based on whatever evidence is available to the individual.

(*b*) The classical or a priori approach to probability can only be applied to games of chance (such as tossing a balanced coin, rolling a fair die, or picking cards from a standard deck of cards) where we can determine a priori, or without experimentation, the probability that an event will occur. In real-world problems of economics and business, we often cannot assign probabilities a priori and the classical approach cannot be used. The relative-frequency or empirical approach overcomes the disadvantages of the classical approach by using the relative frequencies of past occurrences as probabilities. The difficulty with the relative-frequency or empirical approach is that we get different probabilities (relative frequencies) for different numbers of trials or experiments. These probabilities stabilize, or approach a limit, as the number of trials or experiments increases. Because this may be expensive and time-consuming, people may end up using it without a "sufficient" number of trials or experiments. The disadvantage of the subjective or personalistic approach to probability is that different people faced with the same situation may come up with completely different probabilities.

(*c*) Most of the decisions we face in economics, business, science, and everyday life involve risks and probabilities. These probabilities are easier to understand and illustrate for games of choice because objective probabilities can easily be assigned to various events. However, the primary reason for studying probability theory is to help us make intelligent decisions in economics, business, science, and everyday life when risk and uncertainty are involved.

3.2 What is the probability of (*a*) A head in one toss of a balanced coin? A tail? A head or a tail? (*b*) A 2 in one rolling of a fair die? Not a 2? A 2 or not a 2?

(*a*)

$$P(\mathrm{H}) = \frac{n_{\mathrm{H}}}{N} = \frac{1}{2}$$

$$P(\mathrm{T}) = \frac{n_{\mathrm{T}}}{N} = \frac{1}{2}$$

$$P(\mathrm{H}) + P(\mathrm{T}) = \frac{1}{2} + \frac{1}{2} = 1$$

(*b*) Since each of the 6 sides of a fair die is equally likely to come up and a 2 is one of the possibilities

$$P(2) = \frac{n_2}{N} = \frac{1}{6}$$

The probability of not rolling a 2 [that is, $P(2')$] is given by

$$P(2') = 1 - P(2) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(2) + P(2') = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1, \text{ or certainty}$$

3.3 What is the probability that by picking one card from a well-shuffled deck, the card is (*a*) a king, (*b*) a spade, (*c*) the king of spades, (*d*) not the king of spades, or (*e*) the king of spades or not the king of spades?

(*a*) Since there are 4 kings K in the 52 cards of the standard deck

$$P(\mathrm{K}) = \frac{n_{\mathrm{K}}}{N} = \frac{4}{52} = \frac{1}{13}$$

(*b*) Since there are 13 spades S in the 52 cards, $P(\mathrm{S}) = 13/52 = 1/4$

(*c*) There is only one king of spades in the deck, therefore $P(\mathrm{K_S}) = 1/52$

(*d*) The probability of not picking the king of spades is $P(\mathrm{K_S'}) = 1 - 1/52 = 51/52$

(*e*) $P(\mathrm{K_S}) + P(\mathrm{K_S'}) = 1/52 + 51/52 = 52/52 = 1$, or certainty

3.4 An urn (vase) contains 10 balls that are exactly alike except that 5 are red, 3 are blue, and 2 are green. What is the probability that, in picking up a single ball, the ball is (*a*) Red? (*b*) Blue? (*c*) Green? (*d*) Nonblue? (*e*) Nongreen? (*f*) Green or nongreen? (*g*) What are the odds of picking a blue ball? (*h*) What are the odds of not picking a blue ball?

(*a*) $$P(\mathrm{R}) = \frac{N_\mathrm{R}}{N} = \frac{5}{10} = 0.5$$

(*b*) $$P(\mathrm{B}) = \frac{n_\mathrm{B}}{N} = \frac{3}{10} = 0.3$$

(*c*) $$P(\mathrm{G}) = \frac{n_\mathrm{G}}{N} = \frac{2}{10} = 0.2$$

(*d*) $$P(\mathrm{B}') = 1 - P(\mathrm{B}) = 1 - 0.3 = 0.7$$

(*e*) $$P(\mathrm{G}') = 1 - P(\mathrm{G}) = 1 - 0.2 = 0.8$$

(*f*) $$P(\mathrm{G}) + P(\mathrm{G}') = 0.2 + 0.8 = 1$$

(*g*) The odds of picking a blue ball are given by the ratio of the number of ways of picking a blue ball to the number of ways of not picking a blue ball. Since there are 3 blue balls and 7 nonblue balls, the odds in favor of picking a blue ball are 3 to 7, or 3 : 7.

(*h*) The odds of not (against) picking a blue ball are 7 to 3, or 7 : 3.

3.5 Suppose that a 3 comes up 106 times in 600 tosses of a die. (*a*) What is the relative frequency of the 3? How does this differ from classical or a priori probability? (*b*) What would you expect to be the relative frequency or empirical probability if you increased the number of times the die is rolled?

(*a*) The relative frequency or empirical probability of the 3 is given by the ratio of the number of times 3 comes up (106) out of the total number of times the die is rolled (600). Thus the relative frequency or empirical probability of the 3 is $106/600 \cong 0.177$ in 600 rolls. According to the classical or a priori approach (and without rolling the die at all), $P(3) = 1/6 \cong 0.167$. If the die is fair, we *expect* the 3 to come up 100 times in 600 rolls of the die as compared with the actual, observed, or empirical 106 times.

(*b*) If the number of times the same die is rolled is increased from 600, we expect the relative frequency or empirical probability to approach (i.e., to become less unequal with) the classical or a priori probability.

3.6 The production process results in 27 defective items for each 1000 items produced. (*a*) What is the relative frequency or empirical probability of a defective item? (*b*) How many defective items do you expect out of the 1600 items produced each day?

(*a*) The relative frequency or empirical probability of a defective item is $27/1000 = 0.027$.

(*b*) By multiplying the number of items produced each day (1600) by the relative frequency or empirical probability of a defective item (0.027), we get the number of defective items we expect out of each day's output. This is $(1600)(0.027) = 43$, to the nearest item.

PROBABILITY OF MULTIPLE EVENTS

3.7 Define and give some examples of events that are (*a*) mutually exclusive, (*b*) not mutually exclusive, (*c*) independent, and (*d*) dependent.

(*a*) Two or more events are *mutually exclusive*, or *disjoint*, if the occurrence of one of them precludes or prevents the occurrence of the other(s). When one event takes place, the other(s) will not. For example, in a single flip of a coin, we get either a head or a tail, but not both. Heads and tails are therefore mutually exclusive events. In a simple toss of a die, we get one and only one of six possible outcomes: 1, 2, 3, 4, 5, or 6. The outcomes are therefore mutually exclusive. A card picked at random can be of only one suit: diamonds, hearts, clubs, or spades. A child is born either a boy or a girl. An item produced on an assembly line is either good or defective.

(*b*) Two or more events are *not mutually exclusive* if they may occur at the same time. The occurrence of one does not preclude the occurrence of the other(s). For example, a card picked at random from a deck of cards can be both an ace and a club. Therefore, aces and clubs are not mutually exclusive events, because we could pick the ace of clubs. Because we could have inflation and recession at the same time, inflation and recession are not mutually exclusive events.

(*c*) Two or more events are *independent* if the occurrence of one of them in no way affects the occurrence of the other(s). For example, in two successive flips of a balanced coin, the outcome of the second flip in no way depends on the outcome of the first flip. The same is true for two successive tosses of a pair of dice or picks of two cards from a deck with replacement.

(*d*) Two or more events are dependent if the occurrence of one of them affects the probability of the occurrence of the other(s). For example, if we pick a card from a deck and do not replace it, the probability of picking the same card on the second pick is 0. All other probabilities also are affected, since there are now only 51 cards in the deck. Similarly, if the proportion of defective items is greater for the evening than for the morning shift, the probability that an item picked at random from the evening output is defective is greater than for the morning output.

3.8 Draw a Venn diagram for (*a*) mutually exclusive events and (*b*) not mutually exclusive events. (*c*) Are mutually exclusive events dependent or independent? Why?

(*a*) Figure 3-6 illustrates the Venn diagram for events A and B which are mutually exclusive.

(*b*) Figure 3-7 illustrates the Venn diagram for events A and B which are not mutually exclusive.

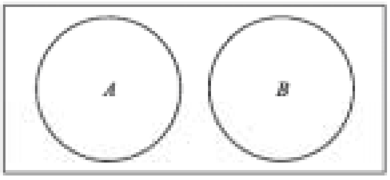

Fig. 3-6

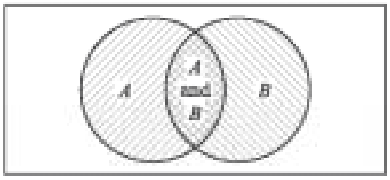

Fig. 3-7

(*c*) Mutually exclusive events are dependent events. When one event occurs, the probability of the other occurring is 0. Thus the occurrence of the first affects (precludes) the occurrence of the other.

3.9 What is the probability of getting (*a*) Less than 3 on a single roll of a fair die? (*b*) Hearts or clubs on a single pick from a well-shuffled standard deck of cards? (*c*) A red or a blue ball from an urn containing 5 red balls, 3 blue balls, and 2 green balls? (*d*) More than 3 on a single roll of a fair die?

(*a*) Getting less than 3 on a single roll of a fair die means getting a 1 or a 2. These are mutually exclusive events. Applying the rule of addition for mutually exclusive events, we get

$$P(1 \text{ or } 2) = P(1) + P(2) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Using set theory, $P(1 \text{ or } 2)$ can be rewritten in an equivalent way as $P(1 \cup 2)$, where $\cup$ is read "union" and stands for *or*.

(*b*) Getting a heart or a club on a single pick from a well-shuffled deck of cards also constitutes two mutually exclusive events. Applying the rule of addition, we get

$$P(\text{H or C}) = P(\text{H} \cup \text{C}) = \frac{13}{52} + \frac{13}{52} = \frac{26}{52} = \frac{1}{2}$$

(*c*) $$P(\text{R or B}) = P(\text{R} \cup \text{B}) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5} = 0.8$$

(*d*) $$P(4 \text{ or } 5 \text{ or } 6) = P(4 \cup 5 \cup 6) = P(4) + P(5) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

3.10 (*a*) What is the probability of getting an ace or a club on a single pick from a well-shuffled standard deck of cards? (In all remaining problems, it will be implicitly assumed that coins are balanced, die are fair, and decks of cards are standard and well shuffled and cards are picked at random without replacement.) (*b*) What is the function of the negative term in the rule of addition for events that are not mutually exclusive?

(*a*) Getting an ace or a club does not constitute two mutually exclusive events because we could get the ace of clubs. Applying the rule of addition for events that are not mutually exclusive, we get

$$P(\text{A or C}) = P(\text{A}) + P(\text{C}) - P(\text{A and C}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

The preceding probability statement can be rewritten in an equivalent way using set theory as

$$P(\text{A} \cup \text{C}) = P(\text{A}) + P(\text{C}) - P(\text{A} \cap \text{C})$$

where $\cap$ is read "intersection" and stands for *and*.

(*b*) The function of the negative term in the rule of addition for events that are not mutually exclusive is to avoid double counting. For example, in calculating $P(\text{A or C})$ in part *a*, the ace of clubs is counted twice, once as an ace and once as a club. Therefore, we subtract the probability of getting the ace of clubs in order to avoid this double counting. If the events are mutually exclusive, the probability that both events will occur simultaneously is 0, and no double counting is involved. This is why the rule of addition for mutually exclusive events does not contain a negative term.

3.11 What is the probability of (*a*) Inflation I or recession R if the probability of inflation is 0.3, the probability of recession is 0.2, and the probability of inflation and recession is 0.06? (*b*) Drawing an ace, a club, or a diamond on a single pick from a deck?

(*a*) Since the probability of inflation *and* recession is not 0, inflation and recession are not mutually exclusive events. Applying the rule of addition, we get

$$P(\text{I or R}) = P(\text{I}) + P(\text{R}) - P(\text{I and R})$$

or

$$P(\text{I} \cup \text{R}) = P(\text{I}) + P(\text{R}) - P(\text{I} \cap \text{R})$$

and

$$P(\text{I or R}) = P(\text{I} \cup \text{R}) = 0.3 + 0.2 - 0.06 = 0.44$$

(*b*) Getting an ace, a club, or a diamond does not constitute mutually exclusive events because we could get the ace of clubs or the ace of diamonds. Applying the rule of addition for events that are not mutualy exclusive, we get

$$P(\text{A or C or D}) = P(\text{A}) + P(\text{C}) + P(\text{D}) - P(\text{A and C}) - P(\text{A and D})$$

$$P(\text{A or C or D}) = \frac{4}{52} + \frac{13}{52} + \frac{13}{52} - \frac{1}{52} - \frac{1}{52} = \frac{28}{52} = \frac{7}{13}$$

3.12 What is the probability of (*a*) Two 6s on 2 rolls of a die? (*b*) A 6 on each die in rolling 2 dice once? (*c*) Two blue balls in 2 successive picks with replacement from the urn in Prob. 3.4? (*d*) Three girls in a family with 3 children?

(*a*) Getting a 6 on each of 2 rolls of a die constitutes independent events. Applying the rule of multiplication for independent events, we get

$$P(6 \text{ and } 6) = P(6 \cap 6) = P(6) \cdot P(6) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

(*b*) Getting a 6 on each die in rolling 2 dice once also constitutes independent events. Therefore

$$P(6 \text{ and } 6) = P(6 \cap 6) = P(6) \cdot P(6) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

(*c*) Since we replace the first ball picked, the probability of getting a blue ball on the second pick is the same as on the first pick. The events are independent. Therefore

$$P(\text{B and B}) = P(\text{B} \cap \text{B}) = P(\text{B}) \cdot P(\text{B}) = \frac{3}{10} \cdot \frac{3}{10} = \frac{9}{100} = 0.09$$

(*d*) The probability of a girl, G, on each birth constitutes independent events, each with a probability of 0.5. Therefore

$$P(\text{G and G and G}) = P(\text{G} \cap \text{G} \cap \text{G}) = P(\text{G}) \cdot P(\text{G}) \cdot P(\text{G}) = (0.5) \cdot (0.5) \cdot (0.5) = 0.125$$

or 1 chance in 8.

3.13 (*a*) List all possible outcomes in rolling 2 dice simultaneously. (*b*) What is the probability of getting a total of 5 in rolling 2 dice simultaneously? (*c*) What is the probability of getting a total of 4 or less in rolling 2 dice simultaneously? More than 4?

(*a*) Each die has 6 possible and equally likely outcomes, and the outcome on each die is independent. Since each of the 6 outcomes on the first die can be associated with each of the 6 outcomes on the second die, there are a total of 36 possible outcomes; that is, the *sample space N* is 36. (In Table 3.2, the first number refers to the outcome on the first die, and the second number refers to the second die. The dice can be distinguished by different colors.) The total of the 36 possible outcomes also can be shown by a *tree* (*or sequential*) *diagram*, as in Fig. 3-8.

Table 3.2 Outcomes in Rolling Two Dice Simultaneously

1, 1	2, 1	3, 1	4, 1	5, 1	6, 1
1, 2	2, 2	3, 2	4, 2	5, 2	6, 2
1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
1, 4	2, 4	3, 4	4, 4	5, 4	6, 4
1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
1, 6	2, 6	3, 6	4, 6	5, 6	6, 6

(*b*) Out of the 36 possible and equally likely outcomes, 4 of them give a total of 5. These are 1, 4; 2, 3; 3, 2; and 4, 1. Thus the probability of a total of 5 (event A) in rolling 2 dice simultaneously is given by

$$P(A) = \frac{n_A}{N} = \frac{4}{36} = \frac{1}{9}$$

(*c*) Rolling a total of 4 or less involves rolling a total of 2, 3, or 4. There are 6 possible and equally likely ways of rolling a total of 4 or less. These are 1, 1; 1, 2; 1, 3; 2, 1; 2, 2; and 3, 1. Thus if event A is defined as rolling a total of 4 or less, $P(A) = 6/36 = 1/6$. The probability of getting a total of more than 4 equals 1 minus the probability of getting a total of 4 or less. This is $1 - 1/6 = 5/6$.

3.14 What is the probability of (*a*) Picking a second red ball from the urn in Prob. 3.4 when a red ball was already obtained on the first pick and not replaced? (*b*) A red ball on the second pick when the first ball picked was not red and was not replaced? (*c*) A red ball on the third pick when a red and a nonred ball were obtained on the first two picks and were not replaced?

(*a*) Picking a second red ball from the urn when a red ball was already picked on the first pick and was not replaced is a *dependent* event, since there are now only 4 red balls and 5 nonred balls remaining in the urn. The *conditional probability* of picking a second red ball when a red ball was already obtained on the first pick and was not replaced is $P(\text{R}/\text{R}) = 4/9$.

(*b*) The conditional probability of obtaining a red ball on the second pick when the first ball picked was not red (R') and was not replaced in the urn before the second ball is picked is $P(\text{R}/\text{R}') = 5/9$.

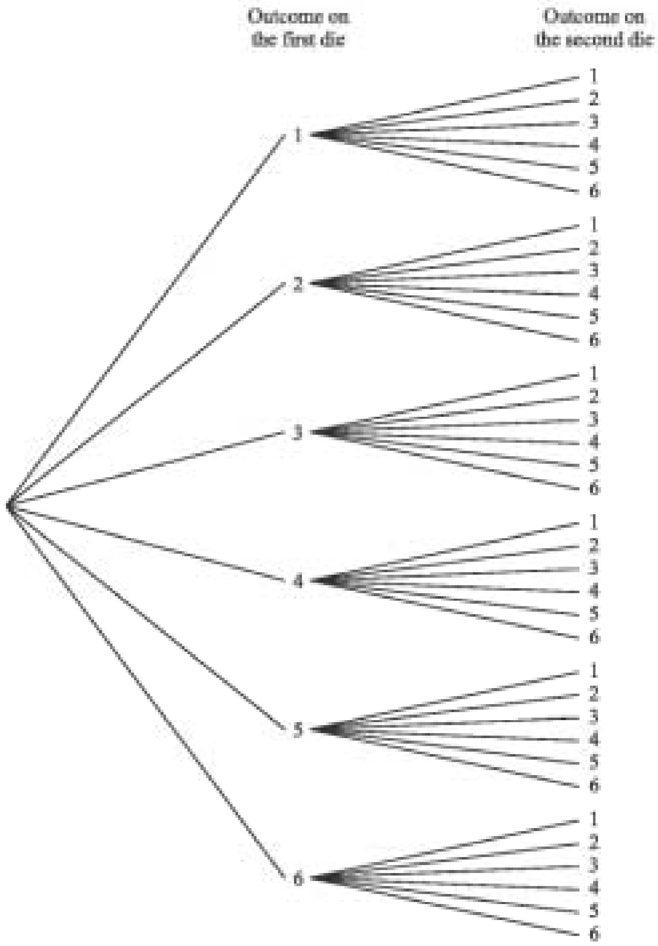

Fig. 3-8 Tree Diagram for Rolling Two Dice Simultaneously

(*c*) Since 2 balls, one of which was red, were already picked and not replaced, there remains a total of 8 balls, of which 4 are red, in the urn. The (conditional) probability of picking another red ball is $P(\text{R}/\text{R and R}') = P(\text{R}/\text{R}' \text{ and R}) = 4/8 = 1/2$.

3.15 What is the probability of obtaining (*a*) Two red balls from the urn in Prob. 3.4 in 2 picks without replacement? (*b*) Two aces from a deck in 2 picks without replacement? (*c*) The ace of clubs and a spade in *that order* in 2 picks from a deck without replacement? (*d*) A spade and the ace of clubs *in that order* in 2 picks from a deck without replacement? (*e*) Three red balls from the urn of Prob. 3.4 in 3 picks without replacement? (*f*) Three red balls from the same urn in 3 picks *with replacement*?

(*a*) Applying the rule of multiplication for dependent events, we get

$$P(\text{R and R}) = P(\text{R} \cap \text{R}) = P(\text{R}) \cdot P(\text{R}/\text{R}) = \frac{5}{10} \cdot \frac{4}{9} = \frac{20}{90} = \frac{2}{9}$$

(*b*)
$$P(\text{A and A}) = P(\text{A} \cap \text{A}) = P(\text{A}) \cdot P(\text{A}/\text{A}) = \frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$$

(*c*)
$$P(\text{A}_\text{C} \text{ and S}) = P(\text{A}_\text{C} \cap \text{S}) = P(\text{A}_\text{C}) \cdot P(\text{S}/\text{A}_\text{C}) = \frac{1}{52} \cdot \frac{13}{51} = \frac{13}{2652}$$

(*d*) $$P(\text{S and A}_C) = P(\text{S} \cap \text{A}_C) = P(\text{S}) \cdot P(\text{A}_C/\text{S}) = \frac{13}{52} \cdot \frac{1}{51} = \frac{13}{2652} = P(\text{A}_C \text{ and S})$$

(*e*) $$P(\text{R and R and R}) = P(\text{R} \cap \text{R} \cap \text{R}) = P(\text{R}) \cdot P(\text{R/R}) \cdot P(\text{R/R and R})$$
$$= \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} = \frac{60}{720} = \frac{1}{12}$$

(*f*) With replacement, picking three balls from an urn constitutes three independent events. Therefore

$$P(\text{R and R and R}) = P(\text{R}) \cdot P(\text{R}) \cdot P(\text{R}) = \frac{5}{10} \cdot \frac{5}{10} \cdot \frac{5}{10} = \frac{125}{1000} = \frac{1}{8} = 0.125$$

3.16 Past experience has shown that for every 100,000 items produced in a plant by the morning shift, 200 are defective, and for every 100,000 items produced by the evening shift, 500 are defective. During a 24-h period, 1000 items are produced by the morning shift and 600 by the evening shift. What is the probability that an item picked at random from the total of 1600 items produced during the 24-h period (*a*) Was produced by the morning shift and is defective? (*b*) Was produced by the evening shift and is defective? (*c*) Was produced by the evening shift and is *not* defective? (*d*) Is defective, whether produced by the morning or the evening shift?

(*a*) The probabilities of picking an item produced by the morning shift M and evening E are

$$P(\text{M}) = \frac{1000}{1600} = 0.625 \qquad \text{and} \qquad P(\text{E}) = \frac{600}{1600} = 0.375$$

The probabilities of picking a defective item D from the morning and evening outputs separately are

$$P(\text{D/M}) = \frac{20}{100{,}000} = 0.002 \qquad \text{and} \qquad P(\text{D/E}) = \frac{500}{100{,}000} = 0.005$$

The probability that an item picked at random from the total of 1600 items produced during the 24-h period was produced by the morning shift and is defective is

$$P(\text{M and D}) = P(\text{M}) \cdot P(\text{D/M}) = (0.625)(0.002) = 0.00125$$

(*b*) $$P(\text{E and D}) = P(\text{E}) \cdot P(\text{D/E}) = (0.375)(0.005) = 0.001875$$

(*c*) $$P(\text{E and D}') = P(\text{E}) \cdot P(\text{D}'/\text{E}) = (0.375)\frac{99{,}500}{100{,}000} = 0.373125$$

(*d*) The expected number of defective items from the morning shift is equal to the probability of a defective item from the morning output times the number of items produced by the morning shift; that is, $(0.002)(1000) = 2$. From the evening shift we expect $(0.005)(600) = 3$ defective items. Thus we expect 5 defective items from the 1600 items produced during the 24-h period. If there are indeed 5 defective items, the probability of picking at random any of the 5 defective items out of a total of 1600 items is 5/1600 or 1/320 or 0.003125.

3.17 (*a*) From the rule of multiplication for dependent events B and A, derive the formula for $P(A/B)$ in terms of $P(B/A)$ and $P(B)$. This is known as *Bayes' theorem* and is used to revise probabilities when additional relevant information becomes available. (*b*) Using Bayes' theorem, find the probability that a defective item picked at random from the 24-h output of 1600 items in Prob. 3.16 was produced by the morning shift; by the evening shift.

(*a*) $$P(B \text{ and } A) = P(B) \cdot P(A/B)$$

By dividing both sides by $P(B)$ and rearranging, we get

$$P(A/B) = \frac{P(B \text{ and } A)}{P(B)}$$

However, $P(B \text{ and } A) = P(A \text{ and } B)$; see Prob. 3.15(*c*) and (*d*). Therefore

$$P(A/B) = \frac{P(A \text{ and } B)}{P(B)} \quad \text{and} \quad P(A/B) = \frac{P(A) \cdot P(B/A)}{P(B)} \quad \textit{Bayes' theorem} \tag{3.15}$$

(*b*) Applying Bayes' theorem to the statement in Prob. 3.16, letting A signify the morning shift M and B signify defective D, and utilizing the results of Prob. 3.16, we get

$$P(\text{M/D}) = \frac{P(\text{M}) \cdot P(\text{D/M})}{P(\text{D})} = \frac{(0.625)(0.002)}{0.003125} = \frac{0.00125}{0.003125} = 0.4$$

That is, the probability that a *defective* item picked at random from the total 24-h output of 1600 items was produced by the morning shift is 40%. Similarly

$$P(\text{E/D}) = P(\text{E}) \cdot P(\text{D/E}) = \frac{(0.375)(0.005)}{0.003125} = \frac{0.001875}{0.003125} = 0.6, \text{ or } 60\%$$

Bayes' theorem can be generalized, for example, to find the probability that a defective item B picked at random was produced by any of n plants ($A_i, i = 1, 2, \ldots, n$), as follows:

$$P(A_i/B) = \frac{P(A_i) \cdot P(B/A_i)}{\sum P(A_i) \cdot P(B/A_i)} \tag{3.16}$$

where $\sum$ refers to the summation over the n plants (the only ones producing the output). Bayes' theorem is applied in business decision theory, but is seldom used in the field of economics. (However, bayesian econometrics is becoming increasingly important.)

3.18 A club has 8 members. (*a*) How many different committees of 3 members each can be formed from the club? (Two committees are different even when only one member is different.) (*b*) How many committees of 3 members each can be formed from the club if each committee is to have a president, a treasurer, and a secretary?

(*a*) We are interested here in finding the number of *combinations* of 8 people taken 3 at a time *without concern for the order*

$${}_8C_3 = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = \frac{336}{6} = 56$$

In general, the number of arrangements of n things taken X at a time without conern for the order is a *combination* given by

$${}_nC_X = \binom{n}{X} = \frac{n!}{X!(n-X)!} \tag{3.17}$$

where $n!$ (read n factorial) $= n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$ and $0! = 1$ by definition.

(*b*) Since each committee of 3 has to have a president, a treasurer, and a secretary, we are now interested in finding the number of *permutations* of 8 people taken 3 at a time, *when the order is important*:

$${}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 = 336$$

In general, the number of arrangements, *in a definite order*, of n things taken X at a time is a permutation given by

$${}_nP_X = \frac{n!}{(n-X)!} \tag{3.18}$$

Permutations and combinations (often referred to as *counting techniques*) are helpful in counting the number of equally likely ways event A can occur in relation to the total of all possible and equally likely outcomes. Combinations and permutations were not used in previous problems because those problems were simple enough without them.

DISCRETE PROBABILITY DISTRIBUTIONS: THE BINOMIAL DISTRIBUTION

3.19 Define what is meant by and give an example of (*a*) a random variable, (*b*) a discrete random variable, and (*c*) a discrete probability distribution. (*d*) What is the distinction between a probability distribution and a relative-frequency distribution?

(*a*) A *random variable* is a variable whose values are associated with some probability of being observed. For example, on 1 roll of a fair die, we have 6 mutually exclusive outcomes (1, 2, 3, 4, 5, or 6), each associated with a probability occurrence of 1/6. Thus the outcome from the roll of a die is a random variable.

(*b*) A *discrete random variable* is one that can assume only finite or distinct values. For example, the outcomes from rolling a die constitute discrete random variables because they are limited to the values 1, 2, 3, 4, 5, and 6. This is to be contrasted with *continuous variables*, which can assume an infinite number of values within any given interval [see Prob. 3.31(*a*)].

(*c*) A *discrete probability distribution* refers to the set of all possible values of a (discrete) random variable and their associated probabilities. The set of the 6 outcomes in rolling a die and their associated probabilities is an example of a discrete probability distribution. The sum of the probabilities associated with all the values that the discrete random variable can assume always equals 1.

(*d*) A *probability distribution* refers to the *classical* or *a priori probabilities* associated with all the values that a random variable can assume. Because those probabilities are assigned a priori and without any experimentation, a probability distribution is often referred to as a *theoretical* (relative) frequency distribution. This differs from an *empirical* (relative) frequency distribution, which refers to the ratio of the number of times each outcome actually occurs to the total number of actual trials or observations. For example, in actually rolling a die a number of times, we are not likely to get each outcome exactly 1/6 of the times. However, as the number of rolls increases, the *empirical* (relative) frequency distribution stabilizes at the (uniform) *probability* or *theoretical* relative-frequency distribution of 1/6.

3.20 Derive the formula for (*a*) the mean μ or expected value $E(X)$ and (*b*) the variance for a *discrete probability distribution*.

(*a*) The formula for the arithmetic mean for grouped population data [Eq. (2.2*a*)] is

$$\mu = \frac{\sum fX}{N}$$

where $\sum fX$ is the sum of the frequency of each class f times the class midpoint X and $N = \sum f$, which is the number of all observations or frequencies. In dealing with probability distributions, the mean μ is often referred to as the "expected value" $E(X)$. The formula for μ or $E(X)$ for a discrete probability distribution can be derived by starting with Eq. (2.2*a*) and letting $f = P(X)$, which is the probability of each of the possible outcomes X. Then, $\sum fX = \sum XP(X)$, which is the sum of the value of each outcome times its probability of occurrence, and $N = \sum f = \sum P(X)$, which is the sum of the probabilities of each outcome, which is 1. Thus

$$E(X) = \mu = \sum XP(X) \tag{3.19}$$

(*b*) The formula for the variance of grouped population data [Eq. (2.9*a*)] is

$$\sigma^2 = \frac{\sum f(X - \mu)^2}{N} \tag{3.20}$$

Once again, letting $f = P(X)$ = probability of each outcome and $N = \sum f = \sum P(X) = 1$, we can get the formula for the variance of a discrete probability distribution:

$$\text{Var } X = \sigma_X^2 = \sum [X - E(X)]^2 \text{P}(X) = \sum X^2 \text{P}(X) - [\text{E}(X)]^2 = \text{E}(X^2) - \mu^2 \tag{3.21}$$

3.21 Table 3.3 gives the number of job applications processed at a small employment agency during the past 100-day period. Determine the expected number of applications processed and the variance and standard deviation.

Table 3.3 Number of Job Applications Processed during the Past 100-Day Period

Number of Job Applicants	Number of Days Achieved
7	10
8	10
10	20
11	30
12	20
14	10
	100

To the extent that we believe that the experience of the past 100 days is typical, we can find the relative-frequency distribution and equate its probability distribution. This and the other calculations to find $E(X)$ and Var X are shown in Table 3.4:

$$\text{Var } X = \sigma_X^2 = \sum X^2 P(X) - [\Sigma XP(X)]^2 = 116 - (10.6)^2 = 116 - 112.36 = 3.64 \text{ applications squared}$$

$$\text{SD } X = \sigma_X = \sqrt{\sigma_X^2} = \sqrt{3.64} \cong 1.91 \text{ applications}$$

Table 3.4 Calculations to Find the Expected Value and Variance

Number, X	Days, f	$P(X)$	$XP(X)$	X^2	$X^2P(X)$
7	10	0.1	0.7	49	4.9
8	10	0.1	0.8	64	6.4
10	20	0.2	2.0	100	20.0
11	30	0.3	3.3	121	36.3
12	20	0.2	2.4	144	28.8
14	10	0.1	1.4	196	19.6
	$N = \sum f = 100$	$\sum P(X) = 1.0$	$\sum XP(X) = 10.6$		$\sum X^2P(X) = 116.0$
$E(X) = \mu = \sum XP(X) = 10.6$ applications					

3.22 (*a*) State the conditions required to apply the binomial distribution. (*b*) What is the probability of 3 heads in 5 flips of a balanced coin? (*c*) What is the probability of less than 3 heads in 5 flips of a balanced coin?

(*a*) The binomial distribution is used to find the probability of X number of occurrences or successes of an event, $P(X)$, in n trials of the same experiment when (1) there are only 2 mutually exclusive outcomes, (2) the n trials are independent, and (3) the probability of occurrence or success, p, remains constant in each trial.

(*b*)
$$P(X) = {}_nC_X p^X (1-p)^{n-X} = \binom{n}{X} p^X (1-p)^{n-X} = \frac{n!}{X!(n-X)!} p^X (1-p)^{n-X}$$

See Eqs. (3.10) and (3.17). In some books, $1 - p$ (the probability of failure) is defined as q. Here $n = 5$, $X = 3$, $p = 1/2$, and $1 - p = 1/2$. Substituting these values into the preceding equation, we get

$$P(3) = \frac{5!}{3!(5-3)!}(1/2)^3(1/2)^{5-3} = \frac{5!}{3!2!}(1/2)^3(1/2)^2 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}(1/2)^5 = 10(1/32) = 0.3125$$

(*c*)
$$P(X < 3) = P(0) + P(1) + P(2)$$
$$P(0) = \frac{5!}{0!5!}(1/2)^0(1/2)^5 = \frac{1}{32} = 0.03125$$
$$P(1) = \frac{5!}{1!(5-1)!}(1/2)^1(1/2)^4 = \frac{5}{32} = 0.15625$$
$$P(2) = \frac{5!}{2!(5-2)!}(1/2)^2(1/2)^3 = \frac{10}{32} = 0.3125$$

Thus $$P(X < 3) = P(0) + P(1) + P(2) = 0.03125 + 0.15625 + 0.3125 = 0.5$$

3.23 (*a*) Suppose that the probability of parents having a child with blond hair is 1/4. If there are 6 children in the family, what is the probability that half of them will have blond hair? (*b*) If the probability of hitting a target on a single shot is 0.3, what is the probability that in 4 shots the target will be hit at least 3 times?

(*a*) Here $n = 6$, $X = 3$, $p = 1/4$, and $1 - p = 3/4$. Substituting these values into the binomial formula, we get

$$P(3) = \frac{6!}{3!(6-3)!}(1/4)^3(3/4)^3 = \frac{6!}{3!3!}(1/64)(27/64) = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}(27/4096)$$
$$= 20\frac{27}{4096} = \frac{540}{4096} \cong 0.13$$

(*b*) Here $n = 4$, $X \geq 3$, $p = 0.3$, and $1 - p = 0.7$:

$$P(X \geq 3) = P(3) + P(4)$$
$$P(3) = \frac{4!}{3!(4-3)!}(0.3)^3(0.7)^1 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 1}(0.027)(0.7) = (4)(0.0189) = 0.0756$$
$$P(4) = \frac{4!}{4!(4-4)!}(0.3)^4(0.7)^0 = (0.3)^4 = 0.0081$$

Thus $$P(X \geq 3) = P(3) + P(4) = 0.0756 + 0.0081 = 0.0837$$

3.24 (*a*) A quality inspector picks a sample of 10 tubes at random from a very large shipment of tubes known to contain 20% defective tubes. What is the probability that no more than 2 of the tubes picked are defective? (*b*) An inspection engineer picks a sample of 15 items at random from a manufacturing process known to produce 85% acceptable items. What is the probability that 10 of the items picked are acceptable?

(*a*) Here $n = 10$, $X \leq 2$, $p = 0.2$, and $1 - p = 0.8$:

$$P(X \leq 2) = P(0) + P(1) + P(2)$$

$$P(0) = \frac{10!}{0!(10-0)!}(0.2)^0(0.8)^{10}$$

$= 0.1074$ (looking up $n = 10$, $X = 0$, and $p = 0.2$ in App. 1)

$P(1) = 0.2684$ (looking up $n = 10$, $X = 1$, and $p = 0.2$ in App. 1)

$P(2) = 0.3020$ (looking up $n = 10$, $X = 2$, and $p = 0.2$ in App. 1)

Thus $P(X \leq 2) = P(0) + P(1) + P(2) = 0.1074 + 0.2684 + 0.3020 = 0.6778$

(*b*) Here $n = 15$, $X = 10$, $p = 0.85$, and $1 - p = 0.15$. Since App. 1 only gives binomial probabilities for up to 0.5, we should transform the problem. The probability of $X = 10$ acceptable items with $p = 0.85$ equals the probability of $X = 5$ defective items with $p = 0.15$. Using $n = 15$, $X = 5$ defective, p (of objective) $= 0.15$, we get 0.0449 (from App. 1).

3.25 (*a*) If 4 balanced coins are tossed simultaneously (or 1 balanced coin is tossed 4 times), compute the entire probability distribution and plot it. (*b*) Compute and plot the probability distribution for a sample of 5 items taken at random from a production process known to produce 30% defective items.

(*a*) Using $n = 4$; $X = $ 0H, 1H, 2H, 3H, or 4H; $P = 1/2$; and App. 1, we get $P(0\text{H}) = 0.0625$, $P(1\text{H}) = 0.2500$, $P(2\text{H}) = 0.3750$, $P(3\text{H}) = 0.2500$, $P(4\text{H}) = 0.0625$, and

thus $$P(0\text{H}) + P(1\text{H}) + P(2\text{H}) + P(3\text{H}) + P(4\text{H})$$
$$= 0.0625 + 0.2500 + 0.3750 + 0.2500 + 0.0625 = 1$$

See Fig. 3-9. Note that $p = 0.5$ and the probability distribution in Fig. 3-9 is symmetrical.

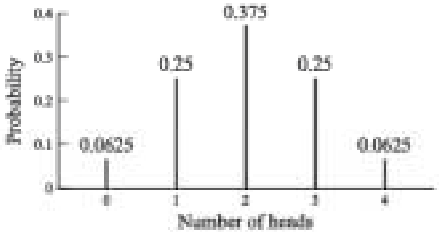

Fig. 3-9 Probability Distribution of Heads in Tossing Four Balanced Coins

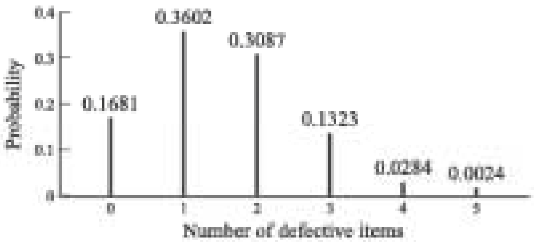

Fig. 3-10 Probability Distribution of Defective Items

(*b*) Using $n = 5$; $X = 0, 1, 2, 3, 4$, or 5 defective; and $p = 0.3$, we get $p(0) = 0.1681$, $P(1) = 0.3602$, $P(2) = 0.3087$, $P(3) = 0.1323$, $P(4) = 0.0284$, $P(5) = 0.0024$. Therefore

$$P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$$
$$= 0.1681 + 0.3602 + 0.3087 + 0.1323 + 0.0284 + 0.0024 = 1$$

See Fig. 3-10. Note that $p < 0.5$ and the probability distribution in Fig. 3-10 is skewed to the right.

3.26 Calculate the expected value and standard deviation and determine the symmetry or asymmetry of the probability distribution of (*a*) Prob. 3.23(*a*), (*b*) Prob. 3.23(*b*), (*c*) Prob. 3.24(*a*), and (*d*) Prob. 3.24(*b*).

(*a*) $E(X) = \mu = np = (6)(1/4) = 3/2 = 1.5$ blond children

$\text{SD}\,X = \sqrt{np(1-p)} = \sqrt{6(1/4)(3/4)} = \sqrt{18/16} = \sqrt{1.125} \cong 1.06$ blond children

Because $p < 0.5$, the probability distribution of blond children is skewed to the right.

(*b*)
$$E(X) = \mu = np = (4)(0.3) = 1.2 \text{ hits}$$
$$\text{SD } X = \sqrt{np(1-p)} = \sqrt{(4)(0.3)(0.7)} = \sqrt{0.84} \cong 0.92 \text{ hits}$$

Because $p < 0.5$, the probability distribution is skewed to the right.

(*c*)
$$E(X) = \mu = np = (10)(0.2) = 2 \text{ defective tubes}$$
$$\text{SD } X = \sqrt{np(1-p)} = \sqrt{(10)(0.2)(0.8)} = \sqrt{1.6} \cong 1.26 \text{ defective tubes}$$

Because $p < 0.5$, the probability distribution is skewed to the right.

(*d*)
$$E(X) = \mu = np = (15)(0.85) = 12.75 \text{ acceptable items}$$
$$\text{SD } X = \sqrt{np(1-p)} = \sqrt{15(0.85)(0.15)} = \sqrt{1.9125} \cong 1.38 \text{ acceptable items}$$

Because $p > 0.5$, the probability distribution is skewed to the left.

3.27 When sampling is done from a finite population *without replacement*, the binomial distribution cannot be used because the events are not independent. Then the *hypergeometric* distribution is used. This is given by

$$P_H = \frac{\binom{N - X_t}{n - X}\binom{X_t}{X}}{\binom{N}{n}} \quad \text{hypergeometric distribution} \tag{3.22}$$

It measures the number of successes X in a sample size n taken at random and without replacement from a population of size N, of which X_t items have the characteristic denoting success. (*a*) Using the formula, determine the probability of picking 2 men in a sample of 6 selected at random without replacement from a group of 10 people, 5 of which are men. (*b*) What would the result have been if we had (incorrectly) used the binomial distribution?

(*a*) Here $X = 2$ men, $n = 6$, $N = 10$, and $X_t = 5$:

$$P_H = \frac{\binom{10-5}{6-2}\binom{5}{2}}{\binom{10}{6}} = \frac{\binom{5}{4}\binom{5}{2}}{\binom{10}{6}} = \frac{\frac{5!}{4!1!}\,\frac{5!}{2!3!}}{\frac{10!}{6!4!}} = \frac{(5)(10)}{210} \cong 0.24$$

(*b*)
$$P(2) = \frac{n!}{X!(n-X)!}p^X(1-p)^{n-X} = \frac{6!}{2!4!}(1/2)^2(1/2)^4 = \frac{15}{64} = 0.23$$

It should be noted that when the sample is very small in relation to the population (say, less than 5% of the population), sampling without replacement has little effect on the probability of success in each trial and the binomial distribution (which is easier to use) is a good approximation for the hypergeometric distribution. This is the reason the binomial distribution was used in Prob. 3.24(*a*).

THE POISSON DISTRIBUTION

3.28 (*a*) What is the difference between the binomial and the Poisson distributions? (*b*) Give some examples of when we can apply the Poisson distribution. (*c*) Give the formula for the Poisson distribution and the meaning of the various symbols. (*d*) Under what conditions can the Poisson distribution be used as an approximation to the binomial distribution? Why can this be useful?

(*a*) Whereas the binomial distribution can be used to find the probability of a designated number of successes in n trials, the Poisson distribution is used to find the probability of a designated number of successes *per unit of time*. The other conditions required to apply the binomial distribution also are required to apply the Poisson distribution; that is (1) there must be only two mutually exclusive outcomes, (2) the events must be independent, and (3) the average number of successes per unit of time must remain constant.

(b) The Poisson distribution is often used in operations research in solving management problems. Some examples are the number of telephone calls to the police per hour, the number of customers arriving at a gasoline pump per hour, and the number of traffic accidents at an intersection per week.

(c) The probability of a designated number of successes per unit of time, $P(X)$, can be found by

$$P(X) = \frac{\lambda^X e^{-\lambda}}{X!}$$

where X = designated number of successes
λ = the average number of successes over a specific time period
e = the base of the natural logarithm system, or 2.71828

Given the value of λ, we can find $e^{-\lambda}$ from App. 2, substitute it into the formula, and find $P(X)$. Note that λ is the mean and variance of the Poisson distribution.

(d) We can use the Poisson distribution as an approximation to the binomial distribution when n, the number of trials, is large and p or $1 - p$ is small (rare events). A good rule of thumb is to use the Poisson distribution when $n \geq 30$ and np or $n(1 - p) < 5$. When n is large, it can be very time-consuming to use the binomial distribution and tables for binomial probabilities, for very small values of p may not be available. If $n(1 - p) < 5$, success and failure should be redefined so that $np < 5$ to make the approximation accurate.

3.29 Past experience indicates that an average number of 6 customers per hour stop for gasoline at a gasoline pump. (a) What is the probability of 3 customers stopping in any hour? (b) What is the probability of 3 customers or less in any hour? (c) What is the expected value, or mean, and standard deviation for this distribution?

(a) $$P(3) = \frac{6^3 e^{-6}}{3!} = \frac{(216)(0.00248)}{3 \cdot 2 \cdot 1} = \frac{0.53568}{6} = 0.08928$$

(b) $$P(X \leq 3) = P(0) + P(1) + P(2) + P(3)$$

$$P(0) = \frac{6^0 e^{-6}}{0!} = \frac{(1)(0.00248)}{1} = 0.00248$$

$$P(1) = \frac{6^1 e^{-6}}{1!} = \frac{(6)(0.00248)}{1} = 0.01488$$

$$P(2) = \frac{6^2 e^{-6}}{2!} = \frac{(36)(0.0248)}{2.1} = 0.04464$$

$$P(3) = 0.08928 \text{ (from part } a\text{)}$$

Thus $$P(X \leq 3) = 0.00248 + 0.01488 + 0.04464 + 0.08928 = 0.15128$$

(c) The expected value, or mean, of this Poisson distribution is $\lambda = 6$ customers, and the standard deviation is $\sqrt{\lambda} = \sqrt{6} \cong 2.45$ customers.

3.30 Past experience shows that 1% of the lightbulbs produced in a plant are defective. Find the probability that more than 1 bulb is defective in a random sample of 30 bulbs, using (a) the binomial distribution and (b) the Poisson distribution.

(a) Here $n = 30$, $p = 0.01$, and we are asked to find $P(X > 1)$. Using App. 1, we get

$$P(2) + P(3) + P(4) + \cdots = 0.0328 + 0.0031 + 0.0002 = 0.0361, \text{ or } 3.61\%$$

(b) Since $n = 30$ and $np = (30)(0.01) = 0.3$, we can use the Poisson approximation of the binomial distribution. Letting $\lambda = np = 0.3$, we have to find $P(X > 1) = 1 - P(X \leq 1)$, where X is the number of defective bulbs. Using Eq. (3.13), we get

$$P(1) = \frac{0.3^1 e^{-0.3}}{1!} = (0.3)(0.74082) = 0.222246$$

$$P(0) = \frac{0.3^0 e^{-0.3}}{0!} = e^{-0.3} = 0.74082$$

$$P(X \leq 1) = P(1) + P(0) = 0.222246 + 0.74082 = 0.963066$$

Thus $$P(X > 1) = 1 - P(X \leq 1) = 1 - 0.963066 = 0.036934, \text{ or } 3.69\%$$

As n becomes larger, the approximation becomes even closer.

CONTINUOUS PROBABILITY DISTRIBUTIONS: THE NORMAL DISTRIBUTION

3.31 (*a*) Define what is meant by a continuous variable and give some examples. (*b*) Define what is meant by a continuous probability distribution. (*c*) Derive the formula for the expected value and variance of a continuous probability distribution.

(*a*) A *continuous variable* is one that can assume any value within any given interval. A continuous variable can be measured with any degree of accuracy simply by using smaller and smaller units of measurement. For example, if we say that a production process takes 10 h, this means anywhere between 9.5 and 10.4 h (10 h rounded to the nearest hour). If we used minutes as the unit of measurement, we could have said that the production process takes 10 h and 20 min. This means anywhere between 10 h and 19.5 min and 10 h and 20.4 min, and so on. Time is thus a continuous variable, and so are weight, distance, and temperature.

(*b*) A *continuous probability distribution* refers to the range of all possible values that a continuous random value can assume, together with the associated probabilities. The probability distribution of a continuous random variable is often called a *probability density function*, or simply a *probability function*. It is given by a smooth curve such that the total area (probability) under the curve is 1. Since a continuous random variable can assume an infinite number of values within any given interval, the probability of a *specific* value is 0. However, we can measure the probability that a continuous random variable X assumes any value within a given interval (say, between X_1 and X_2) by the area under the curve within that interval:

$$P(X_1 < X < X_2) = \int_{X_1}^{X_2} f(X)\, dX \qquad (3.23)$$

where $f(X)$ is the equation of the probability density function, and the integration sign, $\int$, is analogous to the summation sign $\sum$ for discrete variables. Probability tables for some of the most used continuous probability distributions are given in the appendixes, thus eliminating the need to perform the integration ourselves.

(*c*) The expected value, or mean, and variance for continuous probability distributions can be derived by substituting $\int$ for $\sum$ and $f(X)$ for $P(X)$ into the formula for the expected value and variance for discrete probability distributions [Eqs. (*3.19*) and (*3.20*)]:

$$E(X) = \mu = \int X f(X)\, dX \qquad (3.24)$$

$$\text{Var}\, X = \sigma^2 = \int [X - E(X)]^2 f(X)\, dX \qquad (3.25)$$

3.32 (*a*) What is a normal distribution? (*b*) What is its usefulness? (*c*) What is the standard normal distribution? What is its usefulness?

(*a*) The *normal distribution* is a continuous probability function that is bell-shaped, symmetrical about the mean, and mesokurtic (defined in Sec. 2.4). As we move further away from the mean in both directions, the normal curve approaches the horizontal axis (but never quite touches it). The equation of the normal probability function is given by

$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2\right] \qquad (3.26)$$

where $f(X)$ = height of the normal curve
exp = constant 2.7183
π = constant 3.1416
μ = mean of the distribution
σ = standard deviation of the distribution

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2\right] dX = 1$$ (the total area under the normal curve from minus infinity to plus infinity)

(*b*) The normal distribution is the most commonly used of all probability distributions in statistical analysis. Many distributions actually found in nature and industry are normal. Some examples are the IQs (intelligence quotients), weights, and heights of a large number of people and the variations in dimensions of a large number of parts produced by a machine. The normal distribution often can be used to approximate other distributions, such as the binomial and the Poisson distributions (see Probs. 3.37 and 3.38). Distributions of sample means and proportions are often normal, regardless of the distribution of the parent population (see Sec. 4.2).

(*c*) The *standard* normal distribution is a normal distribution with $\mu = 0$ and $\sigma^2 = 1$. Any normal distribution (defined by a particular value for μ and σ^2) can be transformed into a standard normal distribution by letting $\mu = 0$ and expressing deviations from μ in standard deviation units. We often can find areas (probabilities) by converting X values into corresponding z values [that is, $z = (X - \mu)/\sigma$] and looking up these z values in App. 3.

3.33 Find the area under the standard normal curve (*a*) between $z \pm 1$, $z \pm 2$, and $z \pm 3$; (*b*) from $z = 0$ to $z = 0.88$; (*c*) from $z = -1.60$ to $z = 2.55$; (*d*) to the left of $z = -1.60$; (*e*) to the right of $z = 2.55$; (*f*) to the left of $z = -1.60$ and to the right of $z = 2.55$.

(*a*) The area (probability) included under the standard normal curve between $z = 0$ and $z = 1$ is obtained by looking up the value of 1.0 in App. 3. This is accomplished by moving down the z column in the table to 1.0 and then across until we are below the column headed .00. The value that we get is 0.3413. This means that 34.13% of the total area (of 1, or 100%) under the curve lies between $z = 0$ and $z = 1.00$. Because of symmetry, the area between $z = 0$ and $z = -1$ is also 0.3413, or 34.13%. Therefore, the area between $z = -1$ and $z = 1$ is 68.26% (see Fig. 3-4). Similarly, the area between $z = 0$ and $z = 2$ is 0.4772, or 47.72% (by looking up $z = 2.00$ in the table), so that the area between $z = \pm 2$ is 95.44% (see Fig. 3-4). The area between $z \pm 3 = 99.74\%$ (see Fig. 3-4). Note that the table only gives detailed z values for up to 2.99 because the area under the curve outside $z \pm 3$ is negligible.

(*b*) The area between $z = 0$ and $z = 0.88$ is obtained by looking up 0.88 in the table. This is 0.3106.

(*c*) The area between $z = 0$ and $z = -1.60$ is obtained by looking up $z = 1.60$ in the table. This is 0.4452. The area between $z = 0$ and $z = 2.55$ is obtained by looking up $z = 2.55$ in the table. This is 0.4946. Thus the area under the standard normal curve from $z = -1.60$ and $z = 2.55$ equals 0.4452 plus 0.4946. This is 0.9398, or 93.98% (see Fig. 3-11). In all problems of this nature it is helpful to sketch a figure.

(*d*) We know that the total area under the normal curve is equal to 1. Because of symmetry, 0.5 of the area lies on either side of $\mu = 0$. Since 0.4452 extends from $z = 0$ to $z = -1.60$, $0.5 - 0.4452 = 0.0548$, or 5.48%, is the area in the left tail, to the left of -1.60 (see Fig. 3-11).

(*e*) $0.5 - 0.4946 = 0.0054$, or 0.54%, is the area in the right tail, to the right of $z = 2.55$ (see Fig. 3-11).

(*f*) The area to the left of $z = -1.60$ and to the right of $z = 2.55$ is equal to $1 - 0.9398$ (see part *c*). This is 0.0602, or 6.02% of the total.

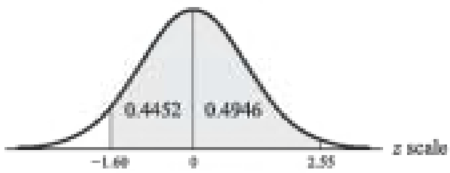

Fig. 3-11

3.34 The lifetime of lightbulbs is known to be normally distributed with $\mu = 100$ h and $\sigma = 8$ h. What is the probability that a bulb picked at random will have a lifetime between 110 and 120 burning hours?

We are asked here to find $P(110 < X < 120)$, where X refers to time measured in hours of burning time. Given $\mu = 100$ h and $\sigma = 8$ h, and letting $X_1 = 110$ h and $X_2 = 120$ h, we get

$$z_1 = \frac{X_1 - \mu}{\sigma} = \frac{110 - 100}{8} = 1.25 \qquad \text{and} \qquad z_2 = \frac{X_2 - \mu}{\sigma} = \frac{120 - 100}{8} = 2.50$$

Thus we want the area (probability) between $z_1 = 1.25$ and $z_2 = 2.50$ (the shaded area in Fig. 3-12). Looking up $z_2 = 2.50$ in App. 3, we get 0.4938. This is the area from $z = 0$ to $z_2 = 2.50$. Looking up $z_1 = 1.25$, we get 0.3944. This is the area from $z = 0$ to $z_1 = 1.25$. Subtracting 0.3944 from 0.4938, we get 0.0994, or 9.94%, for the shaded area that gives $P(110 < X < 120)$.

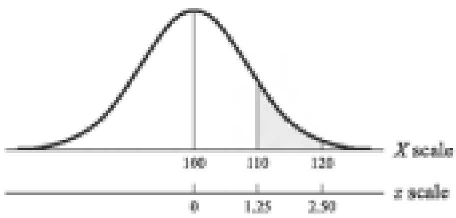

Fig. 3-12

3.35 Assume that family incomes are normally distributed with $\mu = \$16{,}000$, and $\sigma = \$2000$. What is the probability that a family picked at random will have an income: (*a*) Between \$15,000 and \$18,000? (*b*) Below \$15,000? (*c*) Above \$18,000? (*d*) Above \$20,000?

(*a*) We want $P(\$15{,}000 < X < \$18{,}000)$, where X is family income:

$$z_1 = \frac{X_1 - \mu}{\sigma} = \frac{\$15{,}000 - \$16{,}000}{\$2000} = -0.5 \qquad \text{and} \qquad z_2 = \frac{X_2 - \mu}{\sigma} = \frac{\$18{,}000 - \$16{,}000}{\$2000} = 1$$

Thus we want the area (probability) between $z_1 = -0.5$ and $z_2 = 1$ (the shaded area in Fig. 3-13). Looking up $z = 0.5$ in App. 3, we get 0.1915 for the area from $z = 0$ to $z = -0.5$. Looking up $z = 1$, we get 0.3413 for the area from $z = 0$ to $z = 1$. Thus, $P(\$15{,}000 \le X \le \$18{,}000) = 0.1915 + 0.3413 = 0.5328$, or 53.28%.

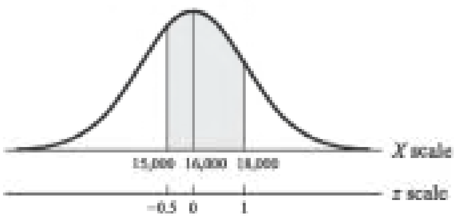

Fig. 3-13

(*b*) $P(X < \$15{,}000) = 0.5 - 0.1915 = 0.3085$, or 30.85% (the unshaded area in the left tail of Fig. 3-13).

(*c*) $P(X > \$18{,}000) = 0.5 - 0.3413 = 0.1587$, or 15.87% (the unshaded area in the right tail of Fig. 3-13).

(*d*) $X = \$20{,}000$ corresponds to $z = (\$20{,}000 - \$16{,}000)/\$2000 = 2$. Therefore, $P(X > \$20{,}000) = 0.5 - 0.4772 = 0.0228$, or 2.28%.

3.36 The grades on the midterm examination in a large statistics section are normally distributed with a mean of 78 and a standard deviation of 8. The professor wants to give the grade of A to 10% of the students. What is the lowest grade point that can be designated an A on the midterm?

In this problem we are asked to find the point grade such that 10% of the students will have higher grades. This involves finding the grade point X such that 10% of the area under the normal curve will be to the right of X (the shaded area in Fig. 3-14). Since the total area under the curve to the right of 78 is 0.5, the *unshaded* area in Fig. 3-14 to the *right* of 78 must be 0.4. We must look *into the body* of App. 3 for the value closest to 0.4. This is 0.3997, which corresponds to the z value of 1.28. The X value (the grade point) that corresponds to the z value of 1.28 is obtained by substituting the known values into $z = (X - \mu)/\sigma$ and solving for X:

$$1.28 = \frac{X - 78}{8}$$

This gives $10.24 = X - 78$. Therefore $X = 78 + 10.24 = 88.24$, or 88 to the nearest whole number.

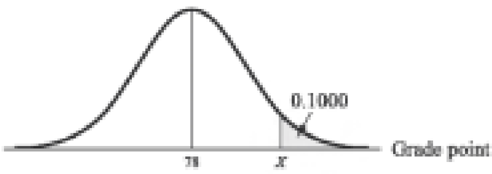

Fig. 3-14

3.37 Experience indicates that 30% of the people entering a store make a purchase. Using (*a*) the binomial distribution and (*b*) the normal approximation to the binomial, find the probability that out of 30 people entering the store, 10 or more will make a purchase.

(*a*) Here $n = 30$, $p = 0.3$, and $1 - p = 0.7$ and we are asked to find $P(X \geq 10)$. Using App. 1 (the table of binomial probabilities),

$$\begin{aligned}(X \geq 10) &= P(10) + P(11) + P(12) + \cdots + P(30) = 0.1416 + 0.1103 + 0.0749 + 0.0444 + 0.0231 \\ &\quad + 0.0106 + 0.0042 + 0.0015 + 0.005 + 0.001 \\ &= 0.4112\end{aligned}$$

(*b*) $\mu = np = (30)(0.3) = 9$ persons, and $\sigma = \sqrt{np(1-p)} = \sqrt{(30)(0.3)(0.7)} = \sqrt{6.3} \cong 2.51$ persons. Since $n = 30$ and both np and $n(1-p) > 5$, we can approximate the binomial probability with the normal. However, the number of people is a discrete variable. In order to use the normal distribution, we must treat the number of people as if it were a continuous variable and find $P(X \geq 9.5)$. Thus

$$z = \frac{X - \mu}{\sigma} = \frac{9.5 - 9}{2.51} = \frac{0.5}{2.51} \cong 0.20$$

From $z = 0.20$, we get 0.0793 (from App. 3). This means that 0.0793 of the area under the standard normal curve lies from $z = 0$ to $z = 0.20$. Therefore, $P(X \geq 9.5) = 0.5 - 0.0793 = 0.4207$ (the normal approximation). As n becomes even larger, the approximation becomes even closer. [If we had not treated the number of people as a continuous variable, we would have found that $P(X \geq 10) = 0.34$, and the approximation would not have been as close.]

3.38 A production process produces 10 defective items per hour. Find the probability that 4 or less items are defective out of the output of a randomly chosen hour using (*a*) the Poisson distribution and (*b*) the normal approximation of the Poisson.

(*a*) Here $\lambda = 10$ and we are asked to find $P(X \le 4)$, where X is the number of defective items from the output of a randomly chosen hour. The value of e^{-10} from App. 2 is 0.00005. Therefore

$$P(0) = \frac{\lambda^0 e^{-10}}{0!} = e^{-10} = 0.00005$$

$$P(1) = \frac{\lambda^1 e^{-10}}{1!} = \frac{10(0.00005)}{1} = 0.0005$$

$$P(2) = \frac{\lambda^2 e^{-10}}{2!} = \frac{10^2(0.00005)}{2} = 0.0025$$

$$P(3) = \frac{\lambda^3 e^{-10}}{3!} = \frac{10^3(0.00005)}{6} = 0.0083335$$

$$P(4) = \frac{\lambda^4 e^{-10}}{4!} = \frac{10^4(0.00005)}{24} = 0.0208335$$

$$\begin{aligned} P(X \le 4) &= P(0) + P(1) + P(2) + P(3) + P(4) \\ &= 0.00005 + 0.0005 + 0.0025 + 0.0083335 + 0.0208335 \\ &= 0.032217, \text{ or about } 3.22\% \end{aligned}$$

(*b*) Treating the items as continuous [see Prob. 3.37(*b*)], we are asked to find $P(X \le 4.5)$, where X is the number of defective items, $\mu = \lambda = 10$, and $\sigma = \sqrt{\lambda} = \sqrt{10} \cong 3.16$. Thus

$$z = \frac{X - \mu}{\sigma} = \frac{4.5 - 10}{3.16} = \frac{-5.5}{3.16} = -1.74$$

For $z = 1.74$ in App. 3, we get 0.4591. This means that $0.5 - 0.4591 = 0.0409$ of the area (probability) under the standard normal curve lies to the left of $z = -1.74$. Thus $P(X \le 4.5) = 0.0409$, or 4.09%. As λ becomes larger, we get a better approximation. [If we had not treated the number of defective items as a continuous variable, we would have found that $P(X \le 4) = 0.287$.]

3.39 If events or successes follow a Poisson distribution, we can determine the probability that the first event occurs within a designated period of time, $P(T \le t)$, by the *exponential probability distribution*. Because we are dealing with time, the exponential is a continuous probability distribution. This is given by

$$P(T \le t) = 1 - e^{-\lambda} \tag{3.27}$$

where λ is the mean number of occurrences for the *interval of interest* and $e^{-\lambda}$ can be obtained from App. 2. The expected value and variance are

$$E(T) = \frac{1}{\lambda} \tag{3.28}$$

$$\text{Var } T = \frac{1}{\lambda^2} \tag{3.29}$$

(*a*) For the statement of Prob. 3.29, find the probability that starting at a random point in time, the first customer stops at the gasoline pump within a half hour. (*b*) What is the probability that no customer stops at the gasoline pump within a half hour? (*c*) What is the expected value and variance of the exponential distribution, where the continuous variable is time T?

(*a*) Since an average of 6 customers stop at the pump per hour, λ = average of 3 customers per half hour. The probability that the first customer will stop within the first half hour is

$$1 - e^{-\lambda} = 1 - e^{-3} = 1 - 0.04979 \text{ (from App. 2)} = 0.9502, \text{ or } 95.02\%$$

(*b*) The probability that no customer stops at the pump within a half hour is

$$e^{-\lambda} = e^{-3} = 0.04979$$

(*c*) $E(T) = 1/\lambda = 1/6 \cong 0.17$ h per car, and var $T = 1/\lambda^2 = 1/36 \cong 0.03$ h per car squared. The exponential distribution also can be used to calculate the time between two successive events.

3.40 The mean level of schooling for a population is 8 years and the standard deviation is 1 year. What is the probability that a randomly selected individual from the population will have had between 6 and 10 years of schooling? Less than 6 years or more than 10 years?

Since we have not been told the form of the distribution, we can use Chebyshev's theorem, which applies to any distribution. With $\mu = 8$ years and $\sigma = 1$ year, 6 years of schooling is 2 standard deviations below μ and 10 years of schooling is 2 standard deviations above μ. Using Chevyshev's theorem or inequality we obtain

$$P(|\bar{X} - \mu| \le K\sigma) \ge 1 - \frac{1}{K^2} \tag{3.30}$$

The probability of an individual picked at random from the population will be within 2 standard deviations from the mean is

$$1 - \frac{1}{K^2} = 1 - \frac{1}{2^2} = \frac{3}{4}, \text{ or } 75\%$$

Therefore, the probability that the individual will have had either less than 6 or more than 10 years of schooling is 25%.

Supplementary Problems

PROBABILITY OF A SINGLE EVENT

3.41 What approach to probability is involved in the following statements? (*a*) The probability of a head in the toss of a balanced coin is 1/2. (*b*) The relative frequency of a head in 100 tosses of a coin is 0.53. (*c*) The probability of rain tomorrow is 20%.
Ans. (*a*) The classical or a priori approach. (*b*) The relative frequency or empirical approach. (*c*) The subjective or personalistic approach.

3.42 What is the probability that in tossing a balanced coin we get (*a*) a tail, (*b*) a head, (*c*) not a tail, or (*d*) a tail or not a tail?
Ans. (*a*) $P(\mathrm{T}) = 1/2$ (*b*) $P(\mathrm{H}) = 1/2$ (*c*) $P(\mathrm{T}') = 1/2$ (*d*) $P(\mathrm{T}) + P(\mathrm{T}') = 1$

3.43 What is the probability that in one roll of a fair die we get (*a*) a 1, (*b*) a 6, (*c*) not a 1, or (*d*) a 1 or not a 1?
Ans. (*a*) $P(1) = 1/6$ (*b*) $P(6) = 1/6$ (*c*) $P(1') = 5/6$ (*d*) $P(1) + P(1') = 1$

3.44 What is the probability that in a single pick from a standard deck of cards we pick (*a*) a club, (*b*) an ace, (*c*) the ace of clubs, (*d*) not a club, or (*e*) a club or not a club?
Ans. (*a*) $P(\mathrm{C}) = 13/52 = 1/4$ (*b*) $P(\mathrm{A}) = 4/52 = 1/13$ (*c*) $P(\mathrm{A_C}) = 1/52$ (*d*) $P(\mathrm{C}') = 3/4$
(*e*) $P(\mathrm{C}) + P(\mathrm{C}') = 1$

3.45 An urn contains 12 balls that are exactly alike except that 4 are blue, 3 are red, 3 are green, and 2 are white. What is the probability that by picking a single ball we pick (*a*) A blue ball? (*b*) A red ball? (*c*) A green ball? (*d*) A white ball? (*e*) A nonred ball? (*f*) A nonwhite ball? (*g*) A white or nonwhite ball? Also