We show now how these measures are derived (directly or indirectly) from the steady-state probability of n in the system, p_n as

$$L_s = \sum_{n=1}^{\infty} np_n$$

$$L_q = \sum_{n=c+1}^{\infty} (n - c)p_n$$

The relationship between L_s and W_s (also L_q and W_q) is known as **Little's formula**, and is given as

$$L_s = \lambda_{\text{eff}} W_s$$

$$L_q = \lambda_{\text{eff}} W_q$$

These relationships are valid under rather general conditions. The parameter λ_{eff} is the *effective* arrival rate at the system. It equals the (nominal) arrival rate λ when all arriving customers can join the system. Otherwise, if some customers cannot join because the system is full (e.g., a parking lot), then $\lambda_{\text{eff}} < \lambda$. We will show later how λ_{eff} is determined.

A direct relationship also exists between W_s and W_q. By definition,

$$\begin{pmatrix}\text{Expected waiting} \\ \text{time in system}\end{pmatrix} = \begin{pmatrix}\text{Expected waiting} \\ \text{time in queue}\end{pmatrix} + \begin{pmatrix}\text{Expected service} \\ \text{time}\end{pmatrix}$$

This translates to

$$W_s = W_q + \frac{1}{\mu}$$

Next, we can relate L_s to L_q by multiplying both sides of the last formula by λ_{eff}, which together with Little's formula gives

$$L_s = L_q + \frac{\lambda_{\text{eff}}}{\mu}$$

By definition, the difference between the average number in the system, L_s, and the average number in the queue, L_q, must equal the average number of *busy* servers, $\bar{c}$. We thus have,

$$\bar{c} = L_s - L_q = \frac{\lambda_{\text{eff}}}{\mu}$$

It follows that

$$\begin{pmatrix}\text{Facility} \\ \text{utilization}\end{pmatrix} = \frac{\bar{c}}{c}$$

Example 15.6-1

Visitors' parking at Ozark College is limited to five spaces only. Cars making use of this space arrive according to a Poisson distribution at the rate of six cars per hour. Parking time is exponentially distributed with a mean of 30 minutes. Visitors who cannot find an empty space on arrival may temporarily wait inside the lot until a parked car leaves. That temporary space can hold only three cars. Other cars that cannot park or find a temporary waiting space must go elsewhere. Determine the following:

(a) The probability, p_n, of n cars in the system.
(b) The effective arrival rate for cars that actually use the lot.
(c) The average number of cars in the lot.
(d) The average time a car waits for a parking space inside the lot.
(e) The average number of *occupied* parking spaces.
(f) The average utilization of the parking lot.

We note first that a parking space acts as a server, so that the system has a total of $c = 5$ parallel servers. Also, the maximum capacity of the system is $5 + 3 = 8$ cars.

The probability p_n can be determined as a special case of the generalized model in Section 15.5 using

$$\lambda_n = 6 \text{ cars/hour}, n = 0, 1, 2, \ldots, 8$$

$$\mu_n = \begin{cases} n\left(\frac{60}{30}\right) = 2n \text{ cars/hour}, n = 1, 2, 3, 4, 5 \\ 5\left(\frac{60}{30}\right) = 10 \text{ cars/hour}, n = 6, 7, 8 \end{cases}$$

From Section 15.5, we get

$$p_n = \begin{cases} \dfrac{3^n}{n!} p_0, & n = 1, 2, 3, 4, 5 \\ \dfrac{3^n}{5!\, 5^{n-5}} p_0, & n = 6, 7, 8 \end{cases}$$

The value of p_0 is computed by substituting p_n, $n = 1, 2, \ldots, 8$, in the following equation

$$p_0 + p_1 + \ldots + p_8 = 1$$

or

$$p_0 + p_0\left(\frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!} + \frac{3^6}{5!\,5} + \frac{3^7}{5!\,5^2} + \frac{3^8}{5!\,5^3}\right) = 1$$

This yields $p_0 = .04812$ (verify!). From p_0, we can now compute p_1 through p_8 as

n	1	2	3	4	5	6	7	8
p_n	.14436	.21654	.21654	.16240	.09744	.05847	.03508	.02105

The effective arrival rate λ_{eff} can be computed by observing the schematic diagram in Figure 15.4, where customers arrive from the source at the rate λ cars per hour. An arriving car may enter the parking lot or go elsewhere with rates λ_{eff} or λ_{lost}, which means that $\lambda = \lambda_{eff} + \lambda_{lost}$. A car will not be able to enter the parking lot if 8 cars are already in. This means that the proportion of cars that will *not* be able to enter the lot is p_8. Thus,

$$\lambda_{lost} = \lambda p_8 = 6 \times .02105 = .1263 \text{ cars per hour}$$

$$\lambda_{eff} = \lambda - \lambda_{lost} = 6 - .1263 = 5.8737 \text{ cars per hour}$$

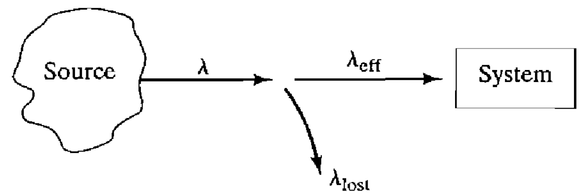

FIGURE 15.4
Relationship between λ, λ_{eff}, and λ_{lost}

The average number of cars in the lot (those waiting for or occupying a space) equals L_s, the average number in the system. We can compute L_s from p_n as

$$L_s = 0p_0 + 1p_1 + \ldots + 8p_8 = 3.1286 \text{ cars}$$

A car waiting in the temporary space is actually a car in queue. Thus, its waiting time until a space is found is W_q. To determine W_q we use

$$W_q = W_s - \frac{1}{\mu}$$

Thus,

$$W_s = \frac{L_s}{\lambda_{\text{eff}}} = \frac{3.1286}{5.8737} = .53265 \text{ hour}$$

$$W_q = .53265 - \frac{1}{2} = .03265 \text{ hour}$$

The average number of occupied parking spaces is the same as the average number of busy servers,

$$\bar{c} = L_s - L_q = \frac{\lambda_{\text{eff}}}{\mu} = \frac{5.8737}{2} = 2.9368 \text{ spaces}$$

From $\bar{c}$, we get

$$\text{Parking lot utilization} = \frac{\bar{c}}{c} = \frac{2.9368}{5} = .58736$$

PROBLEM SET 15.6A

1. In Example 15.6-1, do the following:
   *(a) Compute L_q directly using the formula $\sum_{n=c+1}^{\infty}(n - c)p_n$.
   (b) Compute W_s from L_q.
   *(c) Compute the average number of cars that will not be able to enter the parking lot during an 8-hour period.
   *(d) Show that $c - (L_s - L_q)$, the average number of empty spaces, equals $\sum_{n=0}^{c-1}(c - n)p_n$.
2. Solve Example 15.6-1 using the following data: number of parking spaces = 6, number of temporary spaces = 4, λ = 10 cars per hour, and average parking time = 45 minutes.

15.6.2 Single-Server Models

This section presents two models for the single server case ($c = 1$). The first model sets no limit on the maximum number in the system, and the second model assumes a finite system limit. Both models assume an infinite-capacity source. Arrivals occur at the rate λ customers per unit time and the service rate is μ customers per unit time.

The results of the two models (and indeed of all the remaining models in Section 15.6) are derived as special cases of the results of the generalized model of Section 15.5.

The Kendall notation will be used to summarize the characteristics of each situation. Because the derivations of p_n in Section 15.5 and of all the measures of performance in Section 15.6.1 are totally independent of a specific queue discipline, the symbol GD (general discipline) will be used with the notation.

$(M/M/1)$:$(GD/\infty/\infty)$. Using the notation of the generalized model, we have

$$\left.\begin{array}{l}\lambda_n = \lambda \\ \mu_n = \mu\end{array}\right\}, n = 0, 1, 2, \ldots$$

Also, $\lambda_{\text{eff}} = \lambda$ and $\lambda_{\text{lost}} = 0$, because all arriving customers can join the system.

Letting $\rho = \frac{\lambda}{\mu}$, the expression for p_n in the generalized model then reduces to

$$p_n = \rho^n p_0, n = 0, 1, 2, \ldots$$

To determine the value of p_0, we use the identity

$$p_0(1 + \rho + \rho^2 + \ldots) = 1$$

Assuming $\rho < 1$, the geometric series will have the finite sum $\left(\frac{1}{1-\rho}\right)$, thus

$$p_0 = 1 - \rho, \text{ provided } \rho < 1.$$

The general formula for p_n is thus given by the following geometric distribution

$$p_n = (1 - \rho)\rho^n, n = 1, 2, \ldots (\rho < 1)$$

The mathematical derivation of p_n imposes the condition $\rho < 1$, or $\lambda < \mu$. If $\lambda \geq \mu$, the geometric series will not converge, and the steady-state probabilities p_n will not exist. This result makes intuitive sense, because unless the service rate is larger than the arrival rate, queue length will continually increase and no steady state can be reached.

The measure of performance L_q can be derived in the following manner:

$$\begin{aligned}L_s = \sum_{n=0}^{\infty} n p_n &= \sum_{n=0}^{\infty} n(1 - \rho)\rho^n \\ &= (1 - \rho)\rho \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n \\ &= (1 - \rho)\rho \frac{d}{d\rho}\left(\frac{1}{1 - \rho}\right) = \frac{\rho}{1 - \rho}\end{aligned}$$

Because $\lambda_{\text{eff}} = \lambda$ for the present situation, the remaining measures of performance are computed using the relationships in Section 15.6.1. Thus,

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu - \lambda}$$

$$W_q = W_s - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)}$$

$$L_q = \lambda W_q = \frac{\rho^2}{1-\rho}$$

$$\bar{c} = L_s - L_q = \rho$$

Example 15.6-2

Automata car wash facility operates with only one bay. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour, and may wait in the facility's parking lot if the bay is busy. The time for washing and cleaning a car is exponential, with a mean of 10 minutes. Cars that cannot park in the lot can wait in the street bordering the wash facility. This means that, for all practical purposes, there is no limit on the size of the system. The manager of the facility wants to determine the size of the parking lot.

For this situation, we have $\lambda = 4$ cars per hour, and $\mu = \frac{60}{10} = 6$ cars per hour. Because $\rho = \frac{\lambda}{\mu} < 1$, the system can operate under steady-state conditions.

The TORA or excelPoissonQ.xls input for this model is

Lambda	Mu	c	System limit	Source limit
4	6	1	infinity	infinity

The output of the model is shown in Figure 15.5. The average number of cars waiting in the queue, L_q, is 1.33 cars.

Generally, using L_q as the sole basis for the determination of the number of parking spaces is not advisable, because the design should, in some sense, account for the maximum possible length of the queue. For example, it may be more plausible to design the parking lot such that an arriving car will find a parking space at least 90% of the time. To do this, let S represent the number of parking spaces. Having S parking spaces is equivalent to having $S + 1$ spaces in the *system* (queue plus wash bay). An arriving car will find a space 90% of the time if there are *at most* S cars in the system. This condition is equivalent to the following probability statement:

$$p_0 + p_1 + \dots + p_s \geq .9$$

From Figure 15.5, *cumulative* p_n for $n = 5$ is .91221. This means that the condition is satisfied for $S \geq 5$ parking spaces.

The number of spaces S can be determined also by using the mathematical definition of p_n—that is,

$$(1-\rho)(1 + \rho + \rho^2 + \dots + \rho^S) \geq .9$$

```
Scenario1:  (M/M/1):(GD/infinity/infinity)

Lambda =        4.00000                    Mu =           6.00000
Lambda eff = 4.00000                       Rho/c =        0.66667

Ls =            2.00000                    Lq =           1.33333
Ws =         0.50000                       Wq =         0.33333
```

n	Probability pn	Cumulative Pn	n	Probability pn	Cumulative Pn
0	0.33333	0.33333	13	0.00171	0.99657
1	0.22222	0.55556	14	0.00114	0.99772
2	0.14815	0.70370	15	0.00076	0.99848
3	0.09877	0.80247	16	0.00051	0.99899
4	0.06584	0.86831	17	0.00034	0.99932
5	0.04390	0.91221	18	0.00023	0.99955
6	0.02926	0.94147	19	0.00015	0.99970
7	0.01951	0.96098	20	0.00010	0.99980
8	0.01301	0.97399	21	0.00007	0.99987
9	0.00867	0.98266	22	0.00004	0.99991
10	0.00578	0.98844	23	0.00003	0.99994
11	0.00385	0.99229	24	0.00002	0.99996
12	0.00257	0.99486	25	0.00001	0.99997

FIGURE 15.5

TORA output of Example 15.6-2 (file toraEx15.6-2.txt)

The sum of the truncated geometric series equals $\frac{1-\rho^{S+1}}{1-\rho}$. Thus the condition reduces to

$$(1-\rho^{S+1}) \geq .9$$

Simplification of the inequality yields

$$\rho^{S+1} \leq .1$$

Taking the logarithms on both sides (and noting that $\log(x) < 0$ for $0 < x < 1$, which reverses the direction of the inequality), we get

$$S \geq \frac{\ln(.1)}{\ln\left(\frac{4}{6}\right)} - 1 = 4.679 \approx 5$$

PROBLEM SET 15.6B

1. In Example 15.6-2, do the following.
   - **(a)** Determine the percent utilization of the wash bay.
   - **(b)** Determine the probability that an arriving car must wait in the parking lot prior to entering the wash bay.
   - **(c)** If there are seven parking spaces, determine the probability that an arriving car will find an empty parking space.
   - **(d)** How many parking spaces should be provided so that an arriving car may find a parking space 99% of the time?