The discussion so far has dealt with the maximization case. The only difference in the minimization case is that the reduced costs (z-equations coefficients) must be ≤ 0 to maintain optimality.

The general optimality conditions can be used to determine the special case where the changes d_j occur *one at a time* instead of simultaneously. This analysis is equivalent to considering the following three cases:

1. Maximize $z = (3 + d_1)x_1 + 2x_2 + 5x_3$
2. Maximize $z = 3x_1 + (2 + d_2)x_2 + 5x_3$
3. Maximize $z = 3x_1 + 2x_2 + (5 + d_3)x_3$

The individual conditions can be accounted for as special cases of the simultaneous case.[5]

Case 1. Set $d_2 = d_3 = 0$ in the simultaneous conditions, which gives

$$4 - d_1 \geq 0 \Rightarrow -\infty < d_1 \leq 4$$

Case 2. Set $d_1 = d_3 = 0$ in the simultaneous conditions, which gives

$$\left.\begin{array}{l} 4 - \frac{1}{4}d_2 \geq 0 \Rightarrow d_2 \leq 16 \\ 1 + \frac{1}{2}d_2 \geq 0 \Rightarrow d_2 \geq -2 \\ 2 - \frac{1}{4}d_2 \geq 0 \Rightarrow d_2 \leq 8 \end{array}\right\} \Rightarrow -2 \leq d_2 \leq 8$$

Case 3. Set $d_1 = d_2 = 0$ in the simultaneous conditions, which gives

$$\left.\begin{array}{l} 4 + \frac{3}{2}d_3 \geq 0 \Rightarrow d_3 \geq -\frac{8}{3} \\ 2 + \frac{1}{2}d_3 \geq 0 \Rightarrow d_3 \geq -4 \end{array}\right\} \Rightarrow -\frac{8}{3} \leq d_3 < \infty$$

The given individual conditions can be translated in terms of the total unit revenue. For example, for toy trucks (variable x_2), the total unit revenue is $2 + d_2$ and the associated condition $-2 \leq d_2 \leq 8$ translates to

$$2 + (-2) \leq 2 + d_2 \leq 2 + 8$$

or

$$\$0 \leq (\text{Unit revenue of toy truck}) \leq \$10$$

This condition assumes that the unit revenues for toy trains and toy cars remain fixed at \$3 and \$5, respectively.

The allowable range (\$0, \$10) indicates that the unit revenue of toy trucks (variable x_2) can be as low as \$0 or as high as \$10 without changing the current optimum, $x_1 = 0, x_2 = 100, x_3 = 230$. The total revenue will change to $1350 + 100d_2$, however.

[5]The individual ranges are standard outputs in all LP software. Simultaneous conditions usually are not part of the output, presumably because they are cumbersome for large problems.

It is important to notice that the changes d_1, d_2, and d_3 may be within their allowable individual ranges without satisfying the simultaneous conditions, and vice versa. For example, consider

$$\text{Maximize } z = 6x_1 + 8x_2 + 3x_3$$

Here $d_1 = 6 - 3 = \$3, d_2 = 8 - 2 = \$6$, and $d_3 = 3 - 5 = -\$2$, which are all within the permissible individual ranges ($-\infty < d_1 \leq 4, -2 \leq d_2 \leq 8$, and $-\frac{8}{3} \leq d_3 < \infty$). However, the corresponding simultaneous conditions yield

$$4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1 = 4 - \tfrac{1}{4}(6) + \tfrac{3}{2}(-2) - 3 = -3.5 < 0 \text{ (not satisfied)}$$

$$1 + \tfrac{1}{2}d_2 = 1 + \tfrac{1}{2}(6) = 4 > 0 \qquad \text{(satisfied)}$$

$$2 - \tfrac{1}{4}d_2 + \tfrac{1}{2}d_3 = 2 - \tfrac{1}{4}(6) + \tfrac{1}{2}(-2) = -.5 < 0 \qquad \text{(not satisfied)}$$

The results above can be summarized as follows:

---

1. The optimal values of the variables remain unchanged so long as the changes $d_j, j = 1, 2, \ldots, n$, in the objective function coefficients satisfy all the optimality conditions when the changes are simultaneous or fall within the optimality ranges when a change is made individually.
2. For other situations where the simultaneous optimality conditions are not satisfied or the individual feasibility ranges are violated, the recourse is to either resolve the problem with the new values of d_j or apply the post-optimal analysis presented in Chapter 4.

PROBLEM SET 3.6D[6]

1. In the TOYCO model, determine if the current solution will change in each of the following cases:
   (i) $z = 2x_1 + x_2 + 4x_3$
   (ii) $z = 3x_1 + 6x_2 + x_3$
   (iii) $z = 8x_1 + 3x_2 + 9x_3$

*2. B&K grocery store sells three types of soft drinks: the brand names A1 Cola and A2 Cola and the cheaper store brand BK Cola. The price per can for A1, A2, and BK are 80, 70, and 60 cents, respectively. On the average, the store sells no more than 500 cans of all colas a day. Although A1 is a recognized brand name, customers tend to buy more A2 and BK because they are cheaper. It is estimated that at least 100 cans of A1 are sold daily and that A2 and BK combined outsell A1 by a margin of at least 4:2.
   (a) Show that the optimum solution does not call for selling the A3 brand.
   (b) By how much should the price per can of A3 be increased to be sold by B&K?
   (c) To be competitive with other stores, B&K decided to lower the price on all three types of cola by 5 cents per can. Recompute the reduced costs to determine if this promotion will change the current optimum solution.

[6]In this problem set, you may find it convenient to generate the optimal simplex tableau with TORA.

3. Baba Furniture Company employs four carpenters for 10 days to assemble tables and chairs. It takes 2 person-hours to assemble a table and .5 person-hour to assemble a chair. Customers usually buy one table and four to six chairs. The prices are $135 per table and $50 per chair. The company operates one 8-hour shift a day.
   (a) Determine the 10-day optimal production mix.
   (b) If the present unit prices per table and chair are each reduced by 10%, use sensitivity analysis to determine if the optimum solution obtained in (a) will change.
   (c) If the present unit prices per table and chair are changed to $120 and $25, will the solution in (a) change?
4. The Bank of Elkins is allocating a maximum of $200,000 for personal and car loans during the next month. The bank charges 14% for personal loans and 12% for car loans. Both types of loans are repaid at the end of a 1-year period. Experience shows that about 3% of personal loans and 2% of car loans are not repaid. The bank usually allocates at least twice as much to car loans as to personal loans.
   (a) Determine the optimal allocation of funds between the two loans and the net rate of return on all the loans.
   (b) If the percentages of personal and car loans are changed to 4% and 3%, respectively, use sensitivity analysis to determine if the optimum solution in (a) will change.

*5. Electra produces four types of electric motors, each on a separate assembly line. The respective capacities of the lines are 500, 500, 800, and 750 motors per day. Type 1 motor uses 8 units of a certain electronic component, type 2 motor uses 5 units, type 3 motor uses 4 units, and type 4 motor uses 6 units. The supplier of the component can provide 8000 pieces a day. The prices per motor for the respective types are $60, $40, $25, $30.
   (a) Determine the optimum daily production mix.
   (b) The present production schedule meets Electra's needs. However, because of competition, Electra may need to lower the price of type 2 motor. What is the most reduction that can be effected without changing the present production schedule?
   (c) Electra has decided to slash the price of all motor types by 25%. Use sensitivity analysis to determine if the optimum solution remains unchanged.
   (d) Currently, type 4 motor is not produced. By how much should its price be increased to be included in the production schedule?

6. Popeye Canning is contracted to receive daily 60,000 lb of ripe tomatoes at 7 cents per pound from which it produces canned tomato juice, tomato sauce, and tomato paste. The canned products are packaged in 24-can cases. A can of juice uses 1 lb of fresh tomatoes, a can of sauce uses $\frac{1}{2}$ lb, and a can of paste uses $\frac{3}{4}$ lb. The company's daily share of the market is limited to 2000 cases of juice, 5000 cases of sauce, and 6000 cases of paste. The wholesale prices per case of juice, sauce, and paste are $21, $9, and $12, respectively.
   (a) Develop an optimum daily production program for Popeye.
   (b) If the price per case for juice and paste remains fixed as given in the problem, use sensitivity analysis to determine the unit price range Popeye should charge for a case of sauce to keep the optimum product mix unchanged.
7. Dean's Furniture Company assembles regular and deluxe kitchen cabinets from precut lumber. The regular cabinets are painted white, and the deluxe are varnished. Both painting and varnishing are carried out in one department. The daily capacity of the assembly department is 200 regular cabinets and 150 deluxe. Varnishing a deluxe unit takes twice as much time as painting a regular one. If the painting/varnishing department is dedicated to the deluxe units only, it can complete 180 units daily. The company estimates that the revenues per unit for the regular and deluxe cabinets are $100 and $140, respectively.

(a) Formulate the problem as a linear program and find the optimal production schedule per day.

(b) Suppose that competition dictates that the price per unit of each of regular and deluxe cabinets be reduced to $80. Use sensitivity analysis to determine whether or not the optimum solution in (a) remains unchanged.

8. *The 100% Optimality Rule.* A rule similar to the *100% feasibility rule* outlined in Problem 12, Set 3.6c, can also be developed for testing the effect of simultaneously changing all c_j to $c_j + d_j$, $j = 1, 2, \ldots, n$, on the optimality of the current solution. Suppose that $u_j \le d_j \le v_j$ is the optimality range obtained as a result of changing each c_j to $c_j + d_j$ one at a time, using the procedure in Section 3.6.3. In this case, $u_j \le 0$ ($v_j \ge 0$), because it represents the maximum allowable decrease (increase) in c_j that will keep the current solution optimal. For the cases where $u_j \le d_j \le v_j$, define r_j equal to $\frac{d_j}{v_j}$ if d_j is positive and $\frac{d_j}{u_j}$ if d_j is negative. By definition, $0 \le r_j \le 1$. The 100% rule says that a sufficient (but not necessary) condition for the current solution to remain optimal is that $r_1 + r_2 + \cdots + r_n \le 1$. If the condition is not satisfied, the current solution may or may not remain optimal. The rule does not apply if d_j falls outside the specified ranges.

Demonstrate that the 100% optimality rule is too weak to be consistently reliable as a decision-making tool by applying it to the following cases:

(a) Parts (ii) and (iii) of Problem 1.

(b) Part (b) of Problem 7.

3.6.4 Sensitivity Analysis with TORA, Solver, and AMPL

We now have all the tools needed to decipher the output provided by LP software, particularly with regard to sensitivity analysis. We will use the TOYCO example to demonstrate the TORA, Solver, and AMPL output.

TORA's LP output report provides the sensitivity analysis data automatically as shown in Figure 3.14 (file toraTOYCO.txt). The output includes the reduced costs and the dual prices as well as their allowable optimality and feasibility ranges.

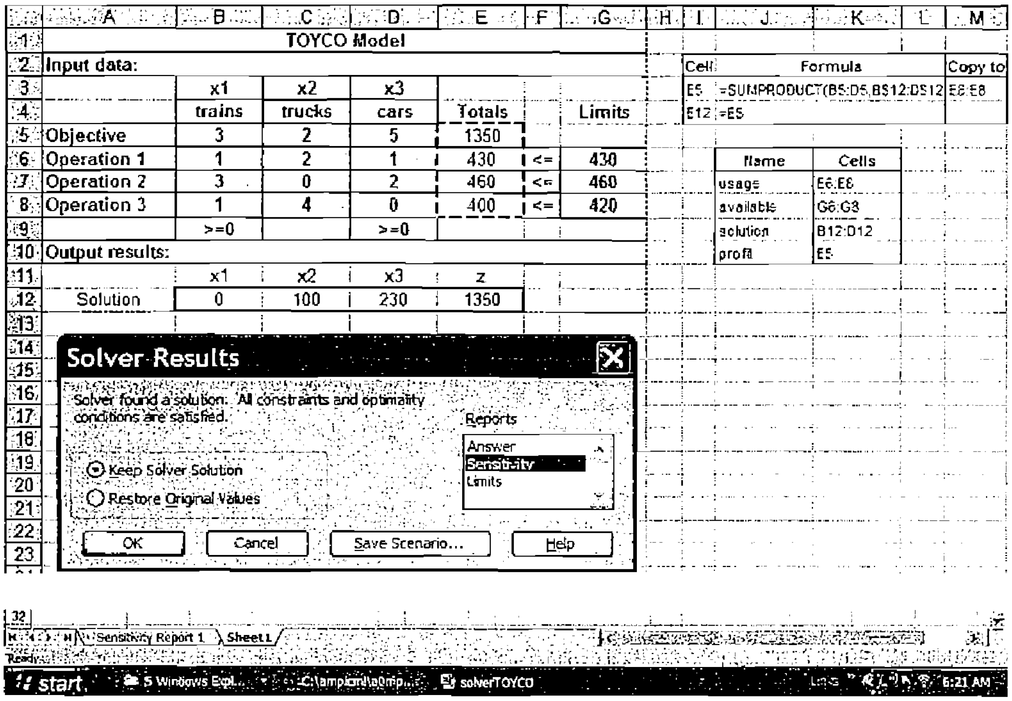
FIGURE 3.14

TORA sensitivity analysis for the TOYCO model

Sensitivity Analysis

Variable	CurrObjCoeff	MinObjCoeff	MaxObjCoeff	Reduced Cost
x1:	3.00	-infinity	7.00	4.00
x2:	2.00	0.00	10.00	0.00
x3:	5.00	2.33	infinity	0.00

Constraint	Curr RHS	Min RHS	Max RHS	Dual Price
1(<):	430.00	230.00	440.00	1.00
2(<):	460.00	440.00	860.00	2.00
3(<):	420.00	400.00	infinity	0.00



Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B12	Solution x1	0	-4	3	4	1E+30
C12	Solution x2	100	0	2	8	2
D12	Solution x3	230	0	5	1E+30	2.666666667

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E6	Operation 1 Totals	430	1	430	10	200
E7	Operation 2 Totals	460	2	460	400	20
E8	Operation 3 Totals	400	0	420	1E+30	20

FIGURE 3.15

Excel Solver sensitivity analysis report for the TOYCO model

Figure 3.15 provides the Solver TOYCO model (file solverTOYCO.xls) and its sensitivity analysis report. After you click Solve in the **Solver Parameters** dialogue box, the new dialogue box **Solver Results** will give you the opportunity to request further details about the solution, including the important sensitivity analysis report. The report will be stored in a separate Excel sheet, as shown by the choices on the bottom of the screen. You can then click **Sensitivity Report 1** to view the results. The report is similar to TORA's with three exceptions: (1) The reduced cost carries an opposite sign. (2) The name *shadow price* replaces the name *dual price*. (3) The optimality ranges are for the changes d_j and D_i rather than for the total objective coefficients and constraints on the right-hand side. The differences are minor and the interpretation of the results remains the same.

In AMPL, the sensitivity analysis report is readily available. File amplTOYCO.txt provides the code necessary to determine the sensitivity analysis output. It requires the following additional statements:

```
option solver cplex;
option cplex_options 'sensitivity';
solve;
#-----------------------------sensitivity analysis
display oper.down,oper.current,oper.up,oper.dual>a.out;
display x.down,x.current,x.up,x.rc>a.out;
```

The CPLEX `option` statements are needed to be able to obtain the standard sensitivity analysis report. In the TOYCO model, the indexed variables and constraints use the root names `x` and `oper`, respectively. Using these names, the suggestive suffixes `.down`, `.current`, and `.up` in the `display` statements automatically generate the formatted sensitivity analysis report in Figure 3.16. The suffixes `.dual` and `.rc` provide the dual price and the reduced cost.

An alternative to AMPL's standard sensitivity analysis report is to actually solve the LP model for a range of values for the objective coefficients and the right-hand side of the constraints. AMPL automates this process through the use of `commands` (see Section A.7). Suppose in the TOYCO model, file amplTOYCO.txt, that we want to investigate the effect of making changes in `b[1]`, the total available time for operation 1. We can do so by moving `solve` and `display` from amplTOYCO.txt to a new file, which we arbitrarily name analysis.txt:

```
repeat while b[1]<=500
{
solve;
display z, x;
let b[1]:=b[1]+1;
};
```

Next, enter the following lines at the ampl prompt:

```
ampl: model amplTOYCO.txt;
ampl: commands analysis.txt;
```

```
: oper.down oper.current oper.up oper.dual    :=
1      230        430        440      1
2      440        460        860      2
3      400        420      1e+20      0
;

:     x.down      x.current     x.up   x.rc    :=
1   -1e+20             3           7    -4
2        0             2          10     0
3        2.33333       5       1e+20     0
;
```

FIGURE 3.16
AMPL sensitivity analysis report for the TOYCO model

The first line will provide the model and its data and the second line will provide the optimum solutions starting with `b[1]` at 430 (the initial value given in amplTOYCO.txt) and continuing in increments of 1 until `b[1]` reaches 500. An examination of the output will then allow us to study the sensitivity of the optimum solution to changes in `b[1]`. Similar procedures can be followed with other coefficients including the case of making simultaneous changes.

PROBLEM SET 3.6E[7]

1. Consider Problem 1, Set 2.3c (Chapter 2). Use the dual price to decide if it is worthwhile to increase the funding for year 4.
2. Consider Problem 2, Set 2.3c (Chapter 2).
   - **(a)** Use the dual prices to determine the overall return on investment.
   - **(b)** If you wish to spend $1000 on pleasure at the end of year 1, how would this affect the accumulated amount at the start of year 5?
3. Consider Problem 3, Set 2.3c (Chapter 2).
   - **(a)** Give an economic interpretation of the dual prices of the model.
   - **(b)** Show how the dual price associated with the upper bound on borrowed money at the beginning of the third quarter can be derived from the dual prices associated with the balance equations representing the in-out cash flow at the five designated dates of the year.
4. Consider Problem 4, Set 2.3c (Chapter 2). Use the dual prices to determine the rate of return associated with each year.

*5. Consider Problem 5, Set 2.3c (Chapter 2). Use the dual price to determine if it is worthwhile for the executive to invest more money in the plans.

6. Consider Problem 6, Set 2.3c (Chapter 2). Use the dual price to decide if it is advisable for the gambler to bet additional money.
7. Consider Problem 1, Set 2.3d (Chapter 2). Relate the dual prices to the unit production costs of the model.
8. Consider Problem 2, Set 2.3d (Chapter 2). Suppose that any additional capacity of machines 1 and 2 can be acquired only by using overtime. What is the maximum cost per hour the company should be willing to incur for either machine?

*9. Consider Problem 3, Set 2.3d (Chapter 2).
   - **(a)** Suppose that the manufacturer can purchase additional units of raw material *A* at $12 per unit. Would it be advisable to do so?
   - **(b)** Would you recommend that the manufacturer purchase additional units of raw material *B* at $5 per unit?

10. Consider Problem 10, Set 2.3e (Chapter 2).
    - **(a)** Which of the specification constraints impacts the optimum solution adversely?
    - **(b)** What is the most the company should pay per ton of each ore?

---

[7]Before answering the problems in this set, you are expected to generate the sensitivity analysis report using AMPL, Solver, or TORA.

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CHAPTER 4

Duality and Post-Optimal Analysis

Chapter Guide. Chapter 3 dealt with the sensitivity of the optimal solution by determining the ranges for the model parameters that will keep the optimum basic solution unchanged. A natural sequel to sensitivity analysis is *post-optimal analysis*, where the goal is to determine the new optimum that results from making targeted changes in the model parameters. Although post-optimal analysis can be carried out using the simplex tableau computations in Section 3.6, this chapter is based entirely on the dual problem.

At a minimum, you will need to study the dual problem and its economic interpretation (Sections 4.1, 4.2, and 4.3). The mathematical definition of the dual problem in Section 4.1 is purely abstract. Yet, when you study Section 4.3, you will see that the dual problem leads to intriguing economic interpretations of the LP model, including *dual prices* and *reduced costs*. It also provides the foundation for the development of the new *dual simplex algorithm*, a prerequisite for post-optimal analysis. The dual simplex algorithm is also needed for integer programming in Chapter 9.

The *generalized simplex algorithm* in Section 4.4.2 is intended to show that the simplex method is not rigid, in the sense that you can modify the rules to handle problems that start both infeasible and nonoptimal. However, this material may be skipped without loss of continuity.

You may use TORA's interactive mode to reinforce your understanding of the computational details of the dual simplex method.

This chapter includes 14 solved examples, 56 end-of-section problems, and 2 cases. The cases are in Appendix E on the CD.

4.1 DEFINITION OF THE DUAL PROBLEM

The **dual** problem is an LP defined directly and systematically from the **primal** (or original) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other.

In most LP treatments, the dual is defined for various forms of the primal depending on the sense of optimization (maximization or minimization), types of constraints

($\le$, $\ge$, or $=$), and orientation of the variables (nonnegative or unrestricted). This type of treatment is somewhat confusing, and for this reason we offer a *single* definition that automatically subsumes *all* forms of the primal.

Our definition of the dual problem requires expressing the primal problem in the *equation form* presented in Section 3.1 (all the constraints are equations with nonnegative right-hand side and all the variables are nonnegative). This requirement is consistent with the format of the simplex starting tableau. Hence, any results obtained from the primal optimal solution will apply directly to the associated dual problem.

To show how the dual problem is constructed, define the primal in *equation form* as follows:

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = 1, 2, \ldots, m$$

$$x_j \ge 0,\ j = 1, 2, \ldots, n$$

The variables x_j, $j = 1, 2, \ldots, n$, include the surplus, slack, and artificial variables, if any.

Table 4.1 shows how the dual problem is constructed from the primal. Effectively, we have

1. A dual variable is defined for each primal (constraint) equation.
2. A dual constraint is defined for each primal variable.
3. The constraint (column) coefficients of a primal variable define the left-hand-side coefficients of the dual constraint and its objective coefficient define the right-hand side.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

TABLE 4.1 Construction of the Dual from the Primal

	Primal variables						
	x_1	x_2	$\ldots$	x_j	$\ldots$	x_n	
Dual variables	c_1	c_2	$\ldots$	c_j	$\ldots$	c_n	Right-hand side
y_1	a_{11}	a_{12}	$\ldots$	a_{1j}	$\ldots$	a_{1n}	b_1
y_2	a_{21}	a_{22}	$\ldots$	a_{2j}	$\ldots$	a_{2n}	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\ldots$	a_{mj}	$\ldots$	a_{mn}	b_m
				↑ jth dual constraint			↑ Dual objective coefficients

TABLE 4.2 Rules for Constructing the Dual Problem

Primal problem objective[a]	Dual problem		
	Objective	*Constraints type*	*Variables sign*
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

[a] All primal constraints are equations with nonnegative right-hand side and all the variables are nonnegative.

The rules for determining the sense of optimization (maximization or minimization), the type of the constraint ($\leq$, $\geq$, or $=$), and the sign of the dual variables are summarized in Table 4.2. Note that the sense of optimization in the dual is always opposite to that of the primal. An easy way to remember the constraint type in the dual (i.e., $\leq$ or $\geq$) is that if the dual objective is *minimization* (i.e., pointing *down*), then the constraints are all of the type $\geq$ (i.e., pointing *up*). The opposite is true when the dual objective is maximization.

The following examples demonstrate the use of the rules in Table 4.2 and also show that our definition incorporates all forms of the primal automatically.

Example 4.1-1

Primal	Primal in equation form	Dual variables
Maximize $z = 5x_1 + 12x_2 + 4x_3$	Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 + x_3 \leq 10$	$x_1 + 2x_2 + x_3 + x_4 = 10$	y_1
$2x_1 - x_2 + 3x_3 = 8$	$2x_1 - x_2 + 3x_3 + 0x_4 = 8$	y_2
$x_1, x_2, x_3 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Minimize } w = 10y_1 + 8y_2$$

subject to

$$\begin{aligned} y_1 + 2y_2 &\geq 5 \\ 2y_1 - y_2 &\geq 12 \\ y_1 + 3y_2 &\geq 4 \\ \left.\begin{aligned} y_1 + 0y_2 &\geq 0 \\ y_1, y_2 &\text{ unrestricted} \end{aligned}\right\} &\Rightarrow (y_1 \geq 0,\ y_2 \text{ unrestricted}) \end{aligned}$$

Example 4.1-2

Primal	Primal in equation form	Dual variables
Minimize $z = 15x_1 + 12x_2$	Minimize $z = 15x_1 + 12x_2 + 0x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 \geq 3$	$x_1 + 2x_2 - x_3 + 0x_4 = 3$	y_1
$2x_1 - 4x_2 \leq 5$	$2x_1 - 4x_2 + 0x_3 + x_4 = 5$	y_2
$x_1, x_2 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Maximize } w = 3y_1 + 5y_2$$

subject to

$$\begin{aligned} y_1 + 2y_2 &\le 15 \\ 2y_1 - 4y_2 &\le 12 \\ \left.\begin{aligned} -y_1 \quad &\le 0 \\ y_2 &\le 0 \\ y_1, y_2 \text{ unrestricted} \end{aligned}\right\} &\Rightarrow (y_1 \ge 0, y_2 \le 0) \end{aligned}$$

Example 4.1-3

Primal	Primal in equation form	Dual variables
	Substitute $x_1 = x_1^+ - x_1^-$	
Maximize $z = 5x_1 + 6x_2$	Maximize $z = 5x_1^+ - 5x_1^- + 6x_2$	
subject to	subject to	
$x_1 + 2x_2 = 5$	$x_1^- - x_1^+ + 2x_2 = 5$	y_1
$-x_1 + 5x_2 \ge 3$	$-x_1^- + x_1^+ + 5x_2 - x_3 = 3$	y_2
$4x_1 + 7x_2 \le 8$	$4x_1^- - 4x_1^+ + 7x_2 + x_4 = 8$	y_3
x_1 unrestricted, $x_2 \ge 0$	$x_1^-, x_1^+, x_2, x_3, x_4 \ge 0$	

Dual Problem

$$\text{Minimize } z = 5y_1 + 3y_2 + 8y_3$$

subject to

$$\left.\begin{aligned} y_1 - y_2 + 4y_3 &\ge 5 \\ -y_1 + y_2 - 4y_3 &\ge -5 \end{aligned}\right\} \Rightarrow (y_1 - y_2 + 4y_3 = 5)$$

$$2y_1 + 5y_2 + 7y_3 \ge 6$$

$$\left.\begin{aligned} -y_2 \quad &\ge 0 \\ y_3 &\ge 0 \\ y_1, y_2, y_3 \text{ unrestricted} \end{aligned}\right\} \Rightarrow (y_1 \text{ unrestricted}, y_2 \le 0, y_3 \ge 0)$$

The first and second constraints are replaced by an equation. The general rule in this case is that an unrestricted primal variable always corresponds to an equality dual constraint. Conversely, a primal equation produces an unrestricted dual variable, as the first primal constraint demonstrates.

Summary of the Rules for Constructing the Dual. The general conclusion from the preceding examples is that the variables and constraints in the primal and dual problems are defined by the rules in Table 4.3. It is a good exercise to verify that these explicit rules are subsumed by the general rules in Table 4.2.

TABLE 4.3 Rules for Constructing the Dual Problem

Maximization problem		Minimization problem
Constraints		*Variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	Unrestricted
Variables		*Constraints*
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

Note that the table does not use the designation primal and dual. What matters here is the sense of optimization. If the primal is maximization, then the dual is minimization, and vice versa.

PROBLEM SET 4.1A

1. In Example 4.1-1, derive the associated dual problem if the sense of optimization in the primal problem is changed to minimization.

*2. In Example 4.1-2, derive the associated dual problem given that the primal problem is augmented with a third constraint, $3x_1 + x_2 = 4$.

3. In Example 4.1-3, show that even if the sense of optimization in the primal is changed to minimization, an unrestricted primal variable always corresponds to an equality dual constraint.

4. Write the dual for each of the following primal problems:

   (a) Maximize $z = -5x_1 + 2x_2$
   subject to

$$-x_1 + x_2 \leq -2$$
$$2x_1 + 3x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

   (b) Minimize $z = 6x_1 + 3x_2$
   subject to

$$6x_1 - 3x_2 + x_3 \geq 2$$
$$3x_1 + 4x_2 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

   ***(c)** Maximize $z = x_1 + x_2$
   subject to

$$2x_1 + x_2 = 5$$
$$3x_1 - x_2 = 6$$
$$x_1, x_2 \text{ unrestricted}$$

*5. Consider Example 4.1-1. The application of the simplex method to the primal requires the use of an artificial variable in the second constraint of the standard primal to secure a starting basic solution. Show that the presence of an artificial primal in equation form variable does not affect the definition of the dual because it leads to a redundant dual constraint.

6. True or False?
   (a) The dual of the dual problem yields the original primal.
   (b) If the primal constraint is originally in equation form, the corresponding dual variable is necessarily unrestricted.
   (c) If the primal constraint is of the type $\leq$, the corresponding dual variable will be nonnegative (nonpositive) if the primal objective is maximization (minimization).
   (d) If the primal constraint is of the type $\geq$, the corresponding dual variable will be nonnegative (nonpositive) if the primal objective is minimization (maximization).
   (e) An unrestricted primal variable will result in an equality dual constraint.

4.2 PRIMAL-DUAL RELATIONSHIPS

Changes made in the original LP model will change the elements of the current optimal tableau, which in turn may affect the optimality and/or the feasibility of the current solution. This section introduces a number of primal-dual relationships that can be used to recompute the elements of the optimal simplex tableau. These relationships will form the basis for the economic interpretation of the LP model as well as for post-optimality analysis.

This section starts with a brief review of matrices, a convenient tool for carrying out the simplex tableau computations.

4.2.1 Review of Simple Matrix Operations

The simplex tableau computations use only three elementary matrix operations: (row vector) $\times$ (matrix), (matrix) $\times$ (column vector), and (scalar) $\times$ (matrix). These operations are summarized here for convenience. First, we introduce some matrix definitions:[1]

1. A *matrix*, **A**, of size $(m \times n)$ is a rectangular array of elements with m rows and n columns.
2. A *row vector*, **V**, of size m is a $(1 \times m)$ matrix.
3. A *column vector*, **P**, of size n is an $(n \times 1)$ matrix.

These definitions can be represented mathematically as

$$\mathbf{V} = (v_1, v_2, \ldots, v_m),\ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \vdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \vdots & a_{mn} \end{pmatrix},\ \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{pmatrix}$$

[1] Appendix D on the CD provides a more complete review of matrices.

1. (Row vector $\times$ matrix, VA). The operation is defined only if the size of the row vector $\mathbf{V}$ equals the number of rows of $\mathbf{A}$. In this case,

$$\mathbf{VA} = \left(\sum_{i=1}^{m} v_i a_{i1}, \sum_{i=1}^{m} v_i a_{i2}, \ldots, \sum_{i=1}^{m} v_i a_{in}\right)$$

For example,

$$(11, 22, 33)\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = (1 \times 11 + 3 \times 22 + 5 \times 33, 2 \times 11 + 4 \times 22 + 6 \times 33)$$

$$= (242, 308)$$

2. (Matrix $\times$ column vector, AP). The operation is defined only if the number of columns of $\mathbf{A}$ equals the size of column vector $\mathbf{P}$. In this case,

$$\mathbf{AP} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} p_j \\ \sum_{j=1}^{n} a_{2j} p_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} p_j \end{pmatrix}$$

As an illustration, we have

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}\begin{pmatrix} 11 \\ 22 \\ 33 \end{pmatrix} = \begin{pmatrix} 1 \times 11 + 3 \times 22 + 5 \times 33 \\ 2 \times 11 + 4 \times 22 + 6 \times 33 \end{pmatrix} = \begin{pmatrix} 242 \\ 308 \end{pmatrix}$$

3. (Scalar $\times$ matrix, αA). Given the scalar (or constant) quantity α, the multiplication operation $\alpha\mathbf{A}$ will result in a matrix of the same size as $\mathbf{A}$ whose (i, j)th element equals αa_{ij}. For example, given $\alpha = 10$,

$$(10)\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{pmatrix}$$

In general, $\alpha\mathbf{A} = \mathbf{A}\alpha$. The same operation is extended equally to the multiplication of vectors by scalars. For example, $\alpha\mathbf{V} = \mathbf{V}\alpha$ and $\alpha\mathbf{P} = \mathbf{P}\alpha$.

PROBLEM SET 4.2A

1. Consider the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \mathbf{P}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{V}_1 = (11, 22), \mathbf{V}_2 = (-1, -2, -3)$$

In each of the following cases, indicate whether the given matrix operation is legitimate, and, if so, calculate the result.

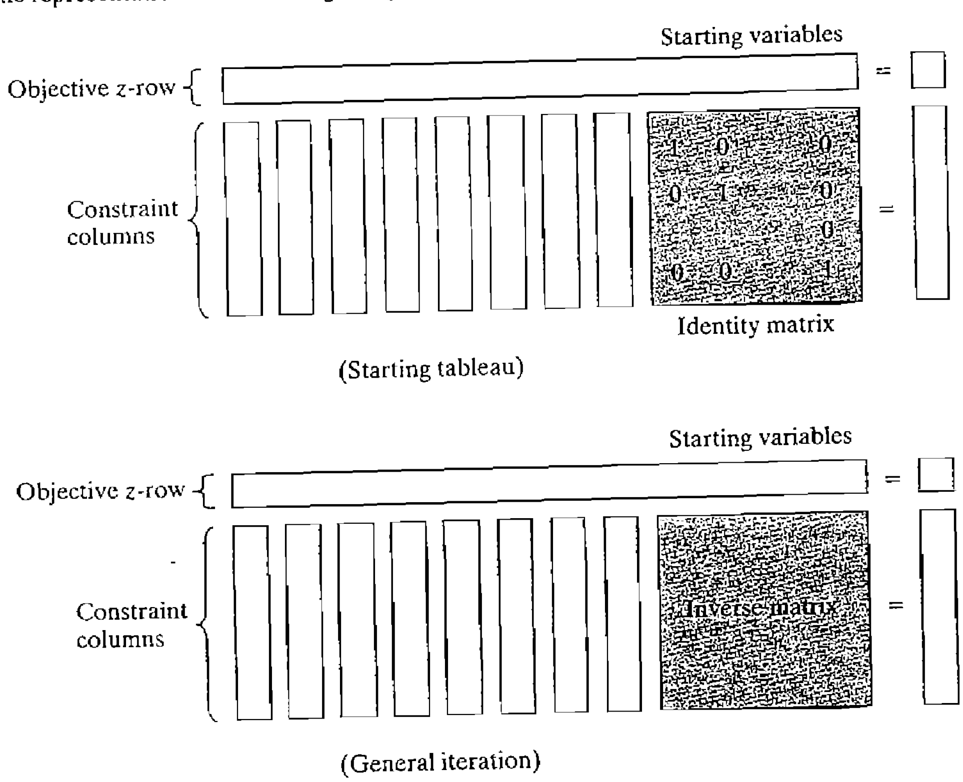
*(a) $\mathbf{AV}_1$
(b) $\mathbf{AP}_1$
(c) $\mathbf{AP}_2$
(d) $\mathbf{V}_1\mathbf{A}$
*(e) $\mathbf{V}_2\mathbf{A}$
(f) $\mathbf{P}_1\mathbf{P}_2$
(g) $\mathbf{V}_1\mathbf{P}_1$

4.2.2 Simplex Tableau Layout

In Chapter 3, we followed a specific format for setting up the simplex tableau. This format is the basis for the development in this chapter.

Figure 4.1 gives a schematic representation of the *starting* and *general* simplex tableaus. In the starting tableau, the constraint coefficients under the starting variables form an **identity matrix** (all main-diagonal elements equal 1 and all off-diagonal elements equal zero). With this arrangement, subsequent iterations of the simplex tableau generated by the Gauss-Jordan row operations (see Chapter 3) will modify the elements of the identity matrix to produce what is known as the **inverse matrix**. As we will see in the remainder of this chapter, the inverse matrix is key to computing all the elements of the associated simplex tableau.

FIGURE 4.1
Schematic representation of the starting and general simplex tableaus



PROBLEM SET 4.2B

1. Consider the optimal tableau of Example 3.3-1.
   *(a) Identify the optimal inverse matrix.
   (b) Show that the right-hand side equals the inverse multiplied by the original right-hand side vector of the original constraints.
2. Repeat Problem 1 for the last tableau of Example 3.4-1.

4.2.3 Optimal Dual Solution

The primal and dual solutions are so closely related that the optimal solution of either problem directly yields (with little additional computation) the optimal solution to the other. Thus, in an LP model in which the number of variables is considerably smaller than the number of constraints, computational savings may be realized by solving the dual, from which the primal solution is determined automatically. This result follows because the amount of simplex computation depends largely (though not totally) on the number of constraints (see Problem 2, Set 4.2c).

This section provides two methods for determining the dual values. Note that the dual of the dual is itself the primal, which means that the dual solution can also be used to yield the optimal primal solution automatically.

Method 1.

$$\begin{pmatrix}\text{Optimal value of} \\ \textit{dual} \text{ variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal } z\text{-coefficient of } \textit{starting} \text{ variable } x_i \\ + \\ \textit{Original} \text{ objective coefficient of } x_i\end{pmatrix}$$

Method 2.

$$\begin{pmatrix}\text{Optimal values} \\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of} \\ \text{original objective coefficients} \\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal} \\ \text{inverse}\end{pmatrix}$$

The elements of the row vector must appear in the same order in which the basic variables are listed in the *Basic* column of the simplex tableau.

Example 4.2-1

Consider the following LP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 10 \\ 2x_1 - x_2 + 3x_3 &= 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

To prepare the problem for solution by the simplex method, we add a slack x_4 in the first constraint and an artificial R in the second. The resulting primal and the associated dual problems are thus defined as follows:

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR$ subject to $x_1 + 2x_2 + x_3 + x_4 = 10$ $2x_1 - x_2 + 3x_3 + R = 8$ $x_1, x_2, x_3, x_4, R \geq 0$	Minimize $w = 10y_1 + 8y_2$ subject to $y_1 + 2y_2 \geq 5$ $2y_1 - y_2 \geq 12$ $y_1 + 3y_2 \geq 4$ $y_1 \geq 0$ $y_2 \geq -M$ ($\Rightarrow y_2$ unrestricted)

Table 4.4 provides the optimal primal tableau.

We now show how the optimal dual values are determined using the two methods described at the start of this section.

Method 1. In Table 4.4, the starting primal variables x_4 and R uniquely correspond to the dual variables y_1 and y_2, respectively. Thus, we determine the optimum dual solution as follows:

Starting primal basic variables	x_4	R
z-equation coefficients	$\frac{29}{5}$	$-\frac{2}{5} + M$
Original objective coefficient	0	$-M$
Dual variables	y_1	y_2
Optimal dual values	$\frac{29}{5} + 0 = \frac{29}{5}$	$-\frac{2}{5} + M + (-M) = -\frac{2}{5}$

Method 2. The optimal inverse matrix, highlighted under the starting variables x_4 and R, is given in Table 4.4 as

$$\text{Optimal inverse} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

First, we note that the optimal primal variables are listed in the tableau in *row order* as x_2 and then x_1. This means that the elements of the original objective coefficients for the two variables must appear in the same order—namely,

$$\begin{aligned}(\text{Original objective coefficients}) &= (\text{Coefficient of } x_2, \text{coefficient of } x_1) \\ &= (12, 5)\end{aligned}$$

TABLE 4.4 Optimal Tableau of the Primal of Example 4.2-1

Basic	x_1	x_2	x_3	x_4	R	Solution
z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5} + M$	$54\frac{4}{5}$
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Thus, the optimal dual values are computed as

$$(y_1, y_2) = \begin{pmatrix}\text{Original objective} \\ \text{coefficients of } x_2, x_1\end{pmatrix} \times (\text{Optimal inverse})$$

$$= (12, 5)\begin{pmatrix}\frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$$

$$= \left(\tfrac{29}{5}, -\tfrac{2}{5}\right)$$

Primal-dual objective values. Having shown how the optimal dual values are determined, next we present the relationship between the primal and dual objective values. For any pair of *feasible* primal and dual solutions,

$$\begin{pmatrix}\text{Objective value in the} \\ \textit{maximization} \text{ problem}\end{pmatrix} \leq \begin{pmatrix}\text{Objective value in the} \\ \textit{minimization} \text{ problem}\end{pmatrix}$$

At the optimum, the relationship holds as a strict equation. The relationship does not specify which problem is primal and which is dual. Only the sense of optimization (maximization or minimization) is important in this case.

The optimum cannot occur with z strictly less than w (i.e., $z < w$) because, no matter how close z and w are, there is always room for improvement, which contradicts optimality as Figure 4.2 demonstrates.

Example 4.2-2

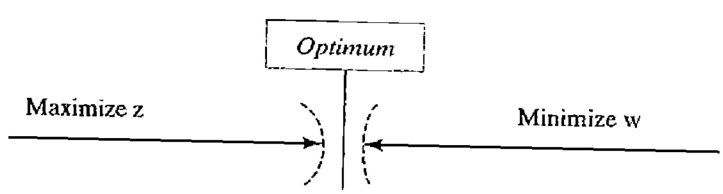
In Example 4.2-1, $\left(x_1 = 0, x_2 = 0, x_3 = \frac{8}{3}\right)$ and $(y_1 = 6, y_2 = 0)$ are feasible primal and dual solutions. The associated values of the objective functions are

$$z = 5x_1 + 12x_2 + 4x_3 = 5(0) + 12(0) + 4\left(\tfrac{8}{3}\right) = 10\tfrac{2}{3}$$
$$w = 10y_1 + 8y_2 = 10(6) + 8(0) = 60$$

Thus, $z \left(= 10\frac{2}{3}\right)$ for the maximization problem (primal) is less than w $(= 60)$ for the minimization problem (dual). The optimum value of $z \left(= 54\frac{4}{5}\right)$ falls within the range $\left(10\frac{2}{3}, 60\right)$.

FIGURE 4.2

Relationship between maximum z and minimum w



PROBLEM SET 4.2C

1. Find the optimal value of the objective function for the following problem by inspecting only its dual. (Do not solve the dual by the simplex method.)

$$\text{Minimize } z = 10x_1 + 4x_2 + 5x_3$$

subject to

$$5x_1 - 7x_2 + 3x_3 \geq 50$$
$$x_1, x_2, x_3 \geq 0$$

2. Solve the dual of the following problem, then find its optimal solution from the solution of the dual. Does the solution of the dual offer computational advantages over solving the primal directly?

$$\text{Minimize } z = 5x_1 + 6x_2 + 3x_3$$

subject to

$$\begin{aligned}
5x_1 + 5x_2 + 3x_3 &\geq 50 \\
x_1 + x_2 - x_3 &\geq 20 \\
7x_1 + 6x_2 - 9x_3 &\geq 30 \\
5x_1 + 5x_2 + 5x_3 &\geq 35 \\
2x_1 + 4x_2 - 15x_3 &\geq 10 \\
12x_1 + 10x_2 &\geq 90 \\
x_2 - 10x_3 &\geq 20 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

*3. Consider the following LP:

$$\text{Maximize } z = 5x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned}
x_1 + 5x_2 + 2x_3 &= 30 \\
x_1 - 5x_2 - 6x_3 &\leq 40 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

Given that the artificial variable x_4 and the slack variable x_5 form the starting basic variables and that M was set equal to 100 when solving the problem, the *optimal* tableau is given as

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	0	23	7	105	0	150
x_1	1	5	2	1	0	30
x_5	0	−10	−8	−1	1	10

Write the associated dual problem and determine its optimal solution in two ways.

4. Consider the following LP:

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

The starting solution consists of artificial x_4 and x_5 for the first and second constraints and slack x_6 for the third constraint. Using $M = 100$ for the artificial variables, the optimal tableau is given as

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	0	−98.6	−100	−.2	3.4
x_1	1	0	0	.4	0	−.2	.4
x_2	0	1	0	.2	0	.6	1.8
x_3	0	0	1	1	−1	1	1.0

Write the associated dual problem and determine its optimal solution in two ways.

5. Consider the following LP:

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

subject to

$$x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 + x_4 = 8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Using x_3 and x_4 as starting variables, the optimal tableau is given as

Basic	x_1	x_2	x_3	x_4	Solution
z	2	0	0	3	16
x_3	.75	0	1	−.25	2
x_2	.25	1	0	.25	2

Write the associated dual problem and determine its optimal solution in two ways.

***6.** Consider the following LP:

$$\text{Maximize } z = x_1 + 5x_2 + 3x_3$$

subject to

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 - x_2 = 4$$
$$x_1, x_2, x_3 \geq 0$$

The starting solution consists of x_3 in the first constraint and an artificial x_4 in the second constraint with $M = 100$. The optimal tableau is given as

Basic	x_1	x_2	x_3	x_4	Solution
z	0	2	0	99	5
x_3	1	2.5	1	−.5	1
x_1	0	−.5	0	.5	2

4.2

Write the associated dual problem and determine its optimal solution in two ways.

7. Consider the following set of inequalities:

$$\begin{aligned} 2x_1 + 3x_2 &\leq 12 \\ -3x_1 + 2x_2 &\leq -4 \\ 3x_1 - 5x_2 &\leq 2 \\ x_1 &\text{ unrestricted} \\ x_2 &\geq 0 \end{aligned}$$

A feasible solution can be found by augmenting the trivial objective function Maximize $z = x_1 + x_2$ and then solving the problem. Another way is to solve the dual; from which a solution for the set of inequalities can be found. Apply the two methods.

8. Estimate a range for the optimal objective value for the following LPs:

***(a)** Minimize $z = 5x_1 + 2x_2$
subject to

$$\begin{aligned} x_1 - x_2 &\geq 3 \\ 2x_1 + 3x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) Maximize $z = x_1 + 5x_2 + 3x_3$
subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 &= 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(c) Maximize $z = 2x_1 + x_2$
subject to

$$\begin{aligned} x_1 - x_2 &\leq 10 \\ 2x_1 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(d) Maximize $z = 3x_1 + 2x_2$
subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 3 \\ 3x_1 + 4x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$