3. *Pound* $(i = 3)$:

$$\text{Total available pounds} = (\$ \to £) + (€ \to £) + (¥ \to £) + (\text{KD} \to £)$$
$$= .625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53}$$

$$\text{Total distributed pounds} = (£ \to \$) + (£ \to €) + (£ \to ¥) + (£ \to \text{KD})$$
$$= x_{31} + x_{32} + x_{34} + x_{35}$$

Thus, the constraint is

$$x_{31} + x_{32} + x_{34} + x_{35} - .625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53} = 0$$

4. *Yen* $(i = 4)$:

$$\text{Total available yen} = (\$ \to ¥) + (€ \to ¥) + (£ \to ¥) + (\text{KD} \to ¥)$$
$$= 105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}$$

$$\text{Total distributed yen} = (¥ \to \$) + (¥ \to €) + (¥ \to £) + (¥ \to \text{KD})$$
$$= x_{41} + x_{42} + x_{43} + x_{45}$$

Thus, the constraint is

$$x_{41} + x_{42} + x_{43} + x_{45} - \left(105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}\right) = 0$$

5. *KD* $(i = 5)$:

$$\text{Total available KDs} = (\text{KD} \to \$) + (\text{KD} \to €) + (\text{KD} \to £) + (\text{KD} \to ¥)$$
$$= .342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}$$

$$\text{Total distributed KDs} = (\$ \to \text{KD}) + (€ \to \text{KD}) + (£ \to \text{KD}) + (¥ \to \text{KD})$$
$$= x_{51} + x_{52} + x_{53} + x_{54}$$

Thus, the constraint is

$$x_{51} + x_{52} + x_{53} + x_{54} - (.342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}) = 0$$

The only remaining constraints are the transaction limits, which are 5 million dollars, 3 million euros, 3.5 million pounds, 100 million yen, and 2.8 million KDs. These can be translated as

$$x_{1j} \le 5, j = 2, 3, 4, 5$$
$$x_{2j} \le 3, j = 1, 3, 4, 5$$
$$x_{3j} \le 3.5, j = 1, 2, 4, 5$$
$$x_{4j} \le 100, j = 1, 2, 3, 5$$
$$x_{5j} \le 2.8, j = 1, 2, 3, 4$$

The complete model is now given as

$$\text{Maximize } z = y$$

subject to

$$\begin{aligned}
&y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right) = 5\\
&x_{21} + x_{23} + x_{24} + x_{25} - \left(.769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}\right) = 0\\
&x_{31} + x_{32} + x_{34} + x_{35} - \left(.625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53}\right) = 0\\
&x_{41} + x_{42} + x_{43} + x_{45} - \left(105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}\right) = 0\\
&x_{51} + x_{52} + x_{53} + x_{54} - (.342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}) = 0
\end{aligned}$$

$$\begin{aligned}
x_{1j} &\le 5, j = 2, 3, 4, 5\\
x_{2j} &\le 3, j = 1, 3, 4, 5\\
x_{3j} &\le 3.5, j = 1, 2, 4, 5\\
x_{4j} &\le 100, j = 1, 2, 3, 5\\
x_{5j} &\le 2.8, j = 1, 2, 3, 4\\
x_{ij} &\ge 0, \text{for all } i \text{ and } j
\end{aligned}$$

Solution:

The optimum solution (using file amplEx2.3-2.txt or solverEx2.3-2.xls) is: 2.

Solution	Interpretation
$y = 5.09032$	Final holdings = \$5,090,320. Net dollar gain = \$90,320, which represents a 1.8064% rate of return
$x_{12} = 1.46206$	Buy \$1,462,060 worth of euros
$x_{15} = 5$	Buy \$5,000,000 worth of KD
$x_{25} = 3$	Buy €3,000,000 worth of KD
$x_{31} = 3.5$	Buy £3,500,000 worth of dollars
$x_{32} = 0.931495$	Buy £931,495 worth of euros
$x_{41} = 100$	Buy ¥100,000,000 worth of dollars
$x_{42} = 100$	Buy ¥100,000,000 worth of euros
$x_{43} = 100$	Buy ¥100,000,000 worth of pounds
$x_{53} = 2.085$	Buy KD2,085,000 worth of pounds
$x_{54} = .96$	Buy KD960,000 worth of yen

Remarks. At first it may appear that the solution is nonsensical because it calls for using $x_{12} + x_{15} = 1.46206 + 5 = 6.46206$, or \$6,462,060 to buy euros and KDs when the initial dollar amount is only \$5,000,000. Where do the extra dollars come from? What happens in practice is that the given solution is submitted to the currency dealer as *one* order, meaning we do not wait until we accumulate enough currency of a certain type before making a buy. In the end, the net result of all these transactions is a net cost of \$5,000,000 to the investor. This can be seen by summing up all the dollar transactions in the solution:

$$I = y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right)$$

$$= 5.09032 + 1.46206 + 5 - \left(\tfrac{3.5}{.625} + \tfrac{100}{105}\right) = 5$$

Notice that x_{21}, x_{31}, x_{41} and x_{51} are in euro, pound, yen, and KD, respectively, and hence must be converted to dollars.

PROBLEM SET 2.3B

1. Modify the arbitrage model to account for a commission that amounts to .1% of any currency buy. Assume that the commission does not affect the circulating funds and that it is collected after the entire order is executed. How does the solution compare with that of the original model?

*2. Suppose that the company is willing to convert the initial \$5 million to any other currency that will provide the highest rate of return. Modify the original model to determine which currency is the best.

3. Suppose the initial amount I = \$7 million and that the company wants to convert it optimally to a combination of euros, pounds, and yen. The final mix may not include more than €2 million, £3 million, and ¥200 million. Modify the original model to determine the optimal buying mix of the three currencies.

4. Suppose that the company wishes to buy \$6 million. The transaction limits for different currencies are the same as in the original problem. Devise a buying schedule for this transaction, given that mix may not include more than €3 million, £2 million, and KD2 million.

5. Suppose that the company has \$2 million, €5 million, £4 million. Devise a buy-sell order that will improve the overall holdings converted to yen.

2.3.3 Investment

Today's investors are presented with multitudes of investment opportunities. Examples of investment problems are capital budgeting for projects, bond investment strategy, stock portfolio selection, and establishment of bank loan policy. In many of these situations, linear programming can be used to select the optimal mix of opportunities that will maximize return while meeting the investment conditions set by the investor.

Example 2.3-3 (Loan Policy Model)

Thriftem Bank is in the process of devising a loan policy that involves a maximum of \$12 million. The following table provides the pertinent data about available types of loans.

Type of loan	Interest rate	Bad-debt ratio
Personal	.140	.10
Car	.130	.07
Home	.120	.03
Farm	.125	.05
Commercial	.100	.02

Bad debts are unrecoverable and produce no interest revenue.

Competition with other financial institutions requires that the bank allocate at least 40% of the funds to farm and commercial loans. To assist the housing industry in the region, home loans must equal at least 50% of the personal, car, and home loans. The bank also has a stated policy of not allowing the overall ratio of bad debts on all loans to exceed 4%.

Mathematical Model: The situation seeks to determine the amount of loan in each category, thus leading to the following definitions of the variables:

$$x_1 = \text{personal loans (in millions of dollars)}$$
$$x_2 = \text{car loans}$$
$$x_3 = \text{home loans}$$
$$x_4 = \text{farm loans}$$
$$x_5 = \text{commercial loans}$$

The objective of the Thriftem Bank is to maximize its net return, the difference between interest revenue and lost bad debts. The interest revenue is accrued only on loans in good standing. Thus, because 10% of personal loans are lost to bad debt, the bank will receive interest on only 90% of the loan—that is, it will receive 14% interest on $.9x_1$ of the original loan x_1. The same reasoning applies to the remaining four types of loans. Thus,

$$\begin{aligned}\text{Total interest} &= .14(.9x_1) + .13(.93x_2) + .12(.97x_3) + .125(.95x_4) + .1(.98x_5)\\ &= .126x_1 + .1209x_2 + .1164x_3 + .11875x_4 + .098x_5\end{aligned}$$

We also have

$$\text{Bad debt} = .1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5$$

The objective function is thus expressed as

$$\begin{aligned}\text{Maximize } z &= \text{Total interest—Bad debt}\\ &= (.126x_1 + .1209x_2 + .1164x_3 + .11875x_4 + .098x_5)\\ &\quad - (.1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5)\\ &= .026x_1 + .0509x_2 + .0864x_3 + .06875x_4 + .078x_5\end{aligned}$$

The problem has five constraints:

1. *Total funds should not exceed \$12 (million):*

$$x_1 + x_2 + x_3 + x_4 + x_5 \le 12$$

2. *Farm and commercial loans equal at least 40% of all loans:*

$$x_4 + x_5 \ge .4(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$.4x_1 + .4x_2 + .4x_3 - .6x_4 - .6x_5 \le 0$$

3. *Home loans should equal at least 50% of personal, car, and home loans:*

$$x_3 \geq .5(x_1 + x_2 + x_3)$$

or

$$.5x_1 + .5x_2 - .5x_3 \leq 0$$

4. *Bad debts should not exceed 4% of all loans:*

$$.1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5 \leq .04(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$.06x_1 + .03x_2 - .01x_3 + .01x_4 - .02x_5 \leq 0$$

5. *Nonnegativity:*

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

A subtle assumption in the preceding formulation is that all loans are issued at approximately the same time. This assumption allows us to ignore differences in the time value of the funds allocated to the different loans.

Solution:

The optimal solution is

$$z = .99648, x_1 = 0, x_2 = 0, x_3 = 7.2, x_4 = 0, x_5 = 4.8$$

Remarks.

1. You may be wondering why we did not define the right-hand side of the second constraint as $.4 \times 12$ instead of $.4(x_1 + x_2 + x_3 + x_4 + x_5)$. After all, it seems logical that the bank would want to loan out all \$12 (million). The answer is that the second usage does not "rob" the model of this possibility. If the optimum solution needs all \$12 (million), the given constraint will allow it. But there are two important reasons why you should not use $.4 \times 12$: (1) If other constraints in the model are such that all \$12 (million) *cannot* be used (for example, the bank may set caps on the different loans), then the choice $.4 \times 12$ could lead to an infeasible or incorrect solution. (2) If you want to experiment with the effect of changing available funds (say from \$12 to \$13 million) on the optimum solution, there is a real chance that you may forget to change $.4 \times 12$ to $.4 \times 13$, in which case the solution you get will not be correct. A similar reasoning applies to the left-hand side of the fourth constraint.
2. The optimal solution calls for allocating all \$12 million: \$7.2 million to home loans and \$4.8 million to commercial loans. The remaining categories receive none. The return on the investment is computed as

$$\text{Rate of return} = \frac{z}{12} = \frac{.99648}{12} = .08034$$

This shows that the combined annual rate of return is 8.034%, which is less than the best *net* interest rate (=.0864 for home loans), and one wonders why the optimum does not take advantage of this opportunity. The answer is that the restriction stipulating that farm and commercial loans account for at least 40% of all loans (constraint 2) forces the solution to allocate \$4.8 million to commercial loans at the lower *net* rate of .078, hence lowering the overall interest rate to $\frac{.0864 \times 7.2 + .078 \times 4.8}{12} = .08034$. In fact, if we remove constraint 2, the optimum will allocate all the funds to home loans at the higher 8.64% rate.

PROBLEM SET 2.3C

1. Fox Enterprises is considering six projects for possible construction over the next four years. The expected (present value) returns and cash outlays for the projects are given below. Fox can undertake any of the projects partially or completely. A partial undertaking of a project will prorate both the return and cash outlays proportionately.

	Cash outlay ($1000)				
Project	*Year 1*	*Year 2*	*Year 3*	*Year 4*	Return ($1000)
1	10.5	14.4	2.2	2.4	32.40
2	8.3	12.6	9.5	3.1	35.80
3	10.2	14.2	5.6	4.2	17.75
4	7.2	10.5	7.5	5.0	14.80
5	12.3	10.1	8.3	6.3	18.20
6	9.2	7.8	6.9	5.1	12.35
Available funds ($1000)	60.0	70.0	35.0	20.0	

(a) Formulate the problem as a linear program, and determine the optimal project mix that maximizes the total return. Ignore the time value of money.

(b) Suppose that if a portion of project 2 is undertaken then at least an equal portion of project 6 must undertaken. Modify the formulation of the model and find the new optimal solution.

(c) In the original model, suppose that any funds left at the end of a year are used in the next year. Find the new optimal solution, and determine how much each year "borrows" from the preceding year. For simplicity, ignore the time value of money.

(d) Suppose in the original model that the yearly funds available for any year can be exceeded, if necessary, by borrowing from other financial activities within the company. Ignoring the time value of money, reformulate the LP model, and find the optimum solution. Would the new solution require borrowing in any year? If so, what is the rate of return on borrowed money?

*2. Investor Doe has $10,000 to invest in four projects. The following table gives the cash flow for the four investments.

	Cash flow ($1000) at the start of				
Project	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Year 5*
1	−1.00	0.50	0.30	1.80	1.20
2	−1.00	0.60	0.20	1.50	1.30
3	0.00	−1.00	0.80	1.90	0.80
4	−1.00	0.40	0.60	1.80	0.95

The information in the table can be interpreted as follows: For project 1, $1.00 invested at the start of year 1 will yield $.50 at the start of year 2, $.30 at the start of year 3, $1.80 at the start of year 4, and $1.20 at the start of year 5. The remaining entries can be interpreted similarly. The entry 0.00 indicates that no transaction is taking place. Doe has the additional option of investing in a bank account that earns 6.5% annually. All funds accumulated at the end of one year can be reinvested in the following year. Formulate the problem as a linear program to determine the optimal allocation of funds to investment opportunities.