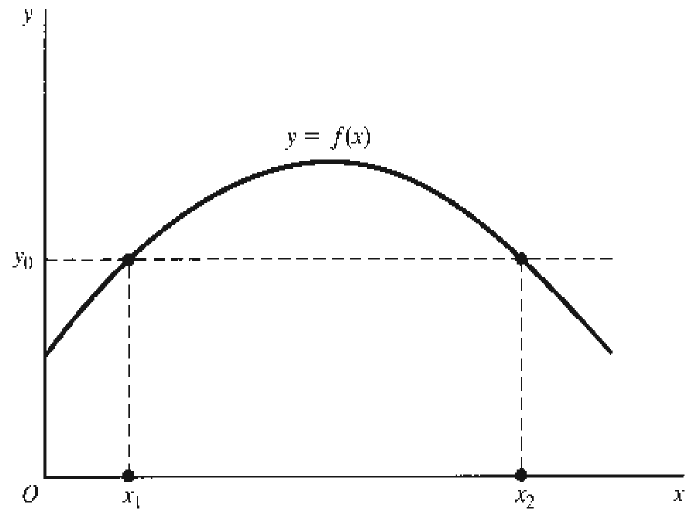
FIGURE 2.6



may thus be interpreted to mean a rule by which the set x is "mapped" ("transformed") into the set y. Thus we may write

$$f: x \to y$$

where the arrow indicates mapping, and the letter f symbolically specifies a rule of mapping. Since f represents a *particular* rule of mapping, a different functional notation must be employed to denote another function that may appear in the same model. The customary symbols (besides f) used for this purpose are g, F, G, the Greek letters ϕ (phi) and ψ (psi), and their capitals, Φ and Ψ. For instance, two variables y and z may both be functions of x, but if one function is written as $y = f(x)$, the other should be written as $z = g(x)$, or $z = \phi(x)$. It is also permissible, however, to write $y = y(x)$ and $z = z(x)$, thereby dispensing with the symbols f and g altogether.

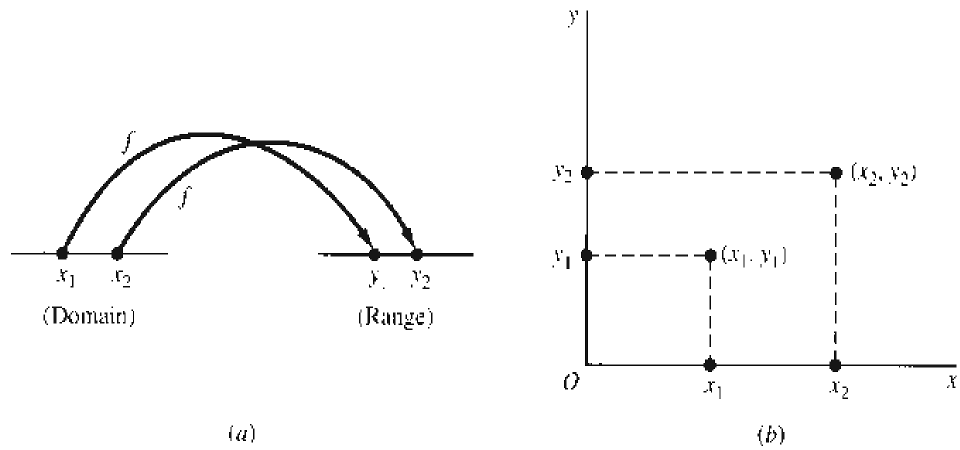
In the function $y = f(x)$, x is referred to as the *argument* of the function, and y is called the *value* of the function. We shall also alternatively refer to x as the *independent variable* and y as the *dependent variable*. The set of all permissible values that x can take in a given context is known as the *domain* of the function, which may be a subset of the set of all real numbers. The y value into which an x value is mapped is called the *image* of that x value. The set of all images is called the *range* of the function, which is the set of all values that the y variable can take. Thus the domain pertains to the independent variable x, and the range has to do with the dependent variable y.

As illustrated in Fig. 2.7*a*, we may regard the function f as a rule for mapping each point on some line segment (the domain) into some point on another line segment (the range). By placing the domain on the x axis and the range on the y axis, as in Fig. 2.7*b*, however, we immediately obtain the familiar two-dimensional graph, in which the association between x values and y values is specified by a set of ordered pairs such as (x_1, y_1) and (x_2, y_2).

In economic models, behavioral equations usually enter as functions. Since most variables in economic models are by their nature restricted to being nonnegative real numbers,[†] their domains are also so restricted. This is why most geometric representations in

[†] We say "nonnegative" rather than "positive" when zero values are permissible.

FIGURE 2.7



economics are drawn only in the first quadrant. In general, we shall not bother to specify the domain of every function in every economic model. When no specification is given, it is to be understood that the domain (and the range) will only include numbers for which a function makes economic sense.

Example 5 The total cost C of a firm per day is a function of its daily output Q: $C = 150 + 7Q$. The firm has a capacity limit of 100 units of output per day. What are the domain and the range of the cost function? Inasmuch as Q can vary only between 0 and 100, the domain is the set of values $0 \leq Q \leq 100$; or more formally,

$$\text{Domain} = \{Q \mid 0 \leq Q \leq 100\}$$

As for the range, since the function plots as a straight line, with the minimum C value at 150 (when $Q = 0$) and the maximum C value at 850 (when $Q = 100$), we have

$$\text{Range} = \{C \mid 150 \leq C \leq 850\}$$

Beware, however, that the extreme values of the range may not always occur where the extreme values of the domain are attained.

EXERCISE 2.4

1. Given $S_1 = \{3, 6, 9\}$, $S_2 = \{a, b\}$, and $S_3 = \{m, n\}$, find the Cartesian products:
   (*a*) $S_1 \times S_2$ (*b*) $S_2 \times S_3$ (*c*) $S_3 \times S_1$
2. From the information in Prob. 1, find the Cartesian product $S_1 \times S_2 \times S_3$.
3. In general, is it true that $S_1 \times S_2 = S_2 \times S_1$? Under what conditions will these two Cartesian products be equal?
4. Does any of the following, drawn in a rectangular coordinate plane, represent a function?
   (*a*) A circle (*c*) A rectangle
   (*b*) A triangle (*d*) A downward-sloping straight line
5. If the domain of the function $y = 5 + 3x$ is the set $\{x \mid 1 \leq x \leq 9\}$, find the range of the function and express it as a set.

6. For the function $y = -x^2$, if the domain is the set of all nonnegative real numbers, what will its range be?
7. In the theory of the firm, economists consider the total cost C to be a function of the output level Q: $C = f(Q)$.
   (*a*) According to the definition of a function, should each cost figure be associated with a unique level of output?
   (*b*) Should each level of output determine a unique cost figure?
8. If an output level Q_1 can be produced at a cost of C_1, then it must also be possible (by being less efficient) to produce Q_1 at a cost of $C_1 + \$1$, or $C_1 + \$2$, and so on. Thus it would seem that output Q does not uniquely determine total cost C. If so, to write $C = f(Q)$ would violate the definition of a function. How, in spite of the this reasoning, would you justify the use of the function $C = f(Q)$?

2.5 Types of Function

The expression $y = f(x)$ is a general statement to the effect that a mapping is possible, but the actual rule of mapping is not thereby made explicit. Now let us consider several specific types of function, each representing a different rule of mapping.

Constant Functions

A function whose range consists of only one element is called a *constant function*. As an example, we cite the function

$$y = f(x) = 7$$

which is alternatively expressible as $y = 7$ or $f(x) = 7$, whose value stays the same regardless of the value of x. In the coordinate plane, such a function will appear as a horizontal straight line. In national-income models, when investment I is exogenously determined, we may have an investment function of the form $I = \$100$ million, or $I = I_0$, which exemplifies the constant function.

Polynomial Functions

The constant function is actually a "degenerate" case of what are known as *polynomial functions*. The word *polynomial* means "multiterm," and a polynomial function of a single variable x has the general form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad \textbf{(2.4)}$$

in which each term contains a coefficient as well as a nonnegative-integer power of the variable x. (As will be explained later in this section, we can write $x^1 = x$ and $x^0 = 1$ in general; thus the first two terms may be taken to be $a_0 x^0$ and $a_1 x^1$, respectively.) Note that, instead of the symbols $a, b, c, \ldots$, we have employed the subscripted symbols a_0, $a_1, \ldots, a_n$ for the coefficients. This is motivated by two considerations: (1) we can economize on symbols, since only the letter a is "used up" in this way; and (2) the subscript helps to pinpoint the location of a particular coefficient in the entire equation. For instance, in (2.4), a_2 is the coefficient of x^2, and so forth.

Depending on the value of the integer n (which specifies the highest power of x), we have several subclasses of polynomial function:

Case of $n = 0$:	$y = a_0$	[*constant* function]
Case of $n = 1$:	$y = a_0 + a_1 x$	[*linear* function]
Case of $n = 2$:	$y = a_0 + a_1 x + a_2 x^2$	[*quadratic* function]
Case of $n = 3$:	$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$	[*cubic* function]

and so forth. The superscript indicators of the powers of x are called *exponents*. The highest power involved, i.e., the value of n, is often called the *degree* of the polynomial function; a quadratic function, for instance, is a second-degree polynomial, and a cubic function is a third-degree polynomial.† The order in which the several terms appear to the right of the equals sign is inconsequential; they may be arranged in descending order of power instead. Also, even though we have put the symbol y on the left, it is also acceptable to write $f(x)$ in its place.

When plotted in the coordinate plane, a linear function will appear as a straight line, as illustrated in Fig. 2.8*a*. When $x = 0$, the linear function yields $y = a_0$; thus the ordered pair $(0, a_0)$ is on the line. This gives us the so-called y intercept (or *vertical intercept*), because it is at this point that the vertical axis intersects the line. The other coefficient, a_1, measures the *slope* (the steepness of incline) of our line. This means that a unit increase in x will result in an increment in y in the amount of a_1. What Fig. 2.8*a* illustrates is the case of $a_1 > 0$, involving a positive slope and thus an upward-sloping line; if $a_1 < 0$, the line will be downward-sloping.

A quadratic function, on the other hand, plots as a *parabola*—roughly, a curve with a single built-in bump or wiggle. The particular illustration in Fig. 2.8*b* implies a negative a_2; in the case of $a_2 > 0$, the curve will "open" the other way, displaying a valley rather than a hill. The graph of a cubic function will, in general, manifest two wiggles, as illustrated in Fig. 2.8*c*. These functions will be used quite frequently in the economic models subsequently discussed.

Rational Functions

A function such as

$$y = \frac{x - 1}{x^2 + 2x + 4}$$

in which y is expressed as a ratio of two polynomials in the variable x, is known as a *rational function*. According to this definition, any polynomial function must itself be a rational function, because it can always be expressed as a ratio to 1, and 1 is a constant function.

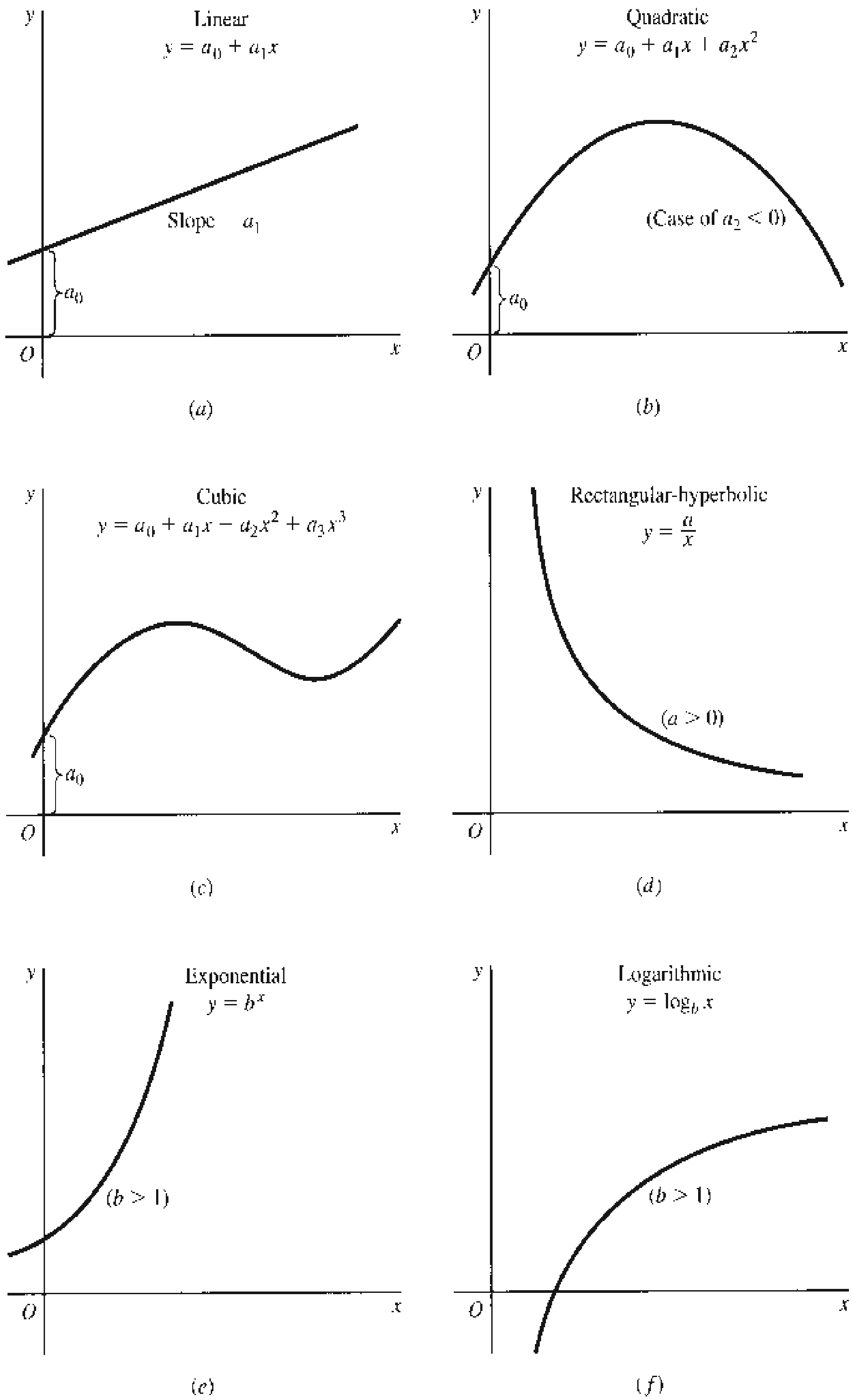
A special rational function that has interesting applications in economics is the function

$$y = \frac{a}{x} \qquad \text{or} \qquad xy = a$$

which plots as a *rectangular hyperbola*, as in Fig. 2.8*d*. Since the product of the two variables is always a fixed constant in this case, this function may be used to represent that special demand curve—with price P and quantity Q on the two axes—for which the total

† In the several equations just cited, the last coefficient (a_n) is always assumed to be nonzero; otherwise the function would degenerate into a lower-degree polynomial.

FIGURE 2.8



expenditure PQ is constant at all levels of price. (Such a demand curve is the one with a unitary elasticity at each point on the curve.) Another application is to the average fixed cost (AFC) curve. With AFC on one axis and output Q on the other, the AFC curve must be rectangular-hyperbolic because AFC $\times$ Q(= total fixed cost) is a fixed constant.

The rectangular hyperbola drawn from $xy = a$ never meets the axes, even if extended indefinitely upward and to the right. Rather, the curve approaches the axes *asymptotically:* as y becomes very large, the curve will come ever closer to the y axis but never actually reach it, and similarly for the x axis. The axes constitute the *asymptotes* of this function.

Nonalgebraic Functions

Any function expressed in terms of polynomials and/or roots (such as square root) of polynomials is an *algebraic function.* Accordingly, the functions discussed thus far are all algebraic.

However, *exponential functions* such as $y = b^x$, in which the independent variable appears in the exponent, are *nonalgebraic.* The closely related *logarithmic functions,* such as $y = \log_b x$, are also nonalgebraic. These two types of function have a special role to play in certain types of economic applications, and it is pedagogically desirable to postpone their discussion to Chap. 10. Here, we simply preview their general graphic shapes in Fig. 2.8*e* and *f.* Other types of nonalgebraic function are the *trigonometric* (or *circular*) *functions,* which we shall discuss in Chap. 16 in connection with dynamic analysis. We should add here that nonalgebraic functions are also known by the more esoteric name of *transcendental functions.*

A Digression on Exponents

In discussing polynomial functions, we introduced the term *exponents* as indicators of the power to which a variable (or number) is to be raised. The expression 6^2 means that 6 is to be raised to the second power; that is, 6 is to be multiplied by itself, or $6^2 \equiv 6 \times 6 = 36$. In general, we define, for a positive integer n,

$$x^n \equiv \underbrace{x \times x \times \cdots \times x}_{n \text{ terms}}$$

and as a special case, we note that $x^1 = x$. From the general definition, it follows that for positive integers m and n, exponents obey the following rules:

Rule I $\quad x^m \times x^n = x^{m+n} \qquad$ (for example, $x^3 \times x^4 = x^7$)

PROOF

$$x^m \times x^n = (\underbrace{x \times x \times \cdots \times x}_{m \text{ terms}})(\underbrace{x \times x \times \cdots \times x}_{n \text{ terms}})$$
$$= \underbrace{x \times x \times \cdots \times x}_{m-n \text{ terms}} = x^{m+n}$$

Note that in this proof, we did not assign any specific value to the number x, or to the exponents m and n. Thus the result obtained is *generally* true. It is for this reason that the demonstration given constitutes a proof, as against a mere verification. The same can be said about the proof of Rule II which follows.

Rule II $\quad \dfrac{x^m}{x^n} = x^{m-n} \qquad (x \neq 0) \qquad \left(\text{for example, } \dfrac{x^4}{x^3} = x\right)$

PROOF

$$\frac{x^m}{x^n} = \frac{\overbrace{x \times x \times \cdots \times x}^{m \text{ terms}}}{\underbrace{x \times x \times \cdots \times x}_{n \text{ terms}}} = \underbrace{x \times x \times \cdots \times x}_{m-n \text{ terms}} = x^{m-n}$$

because the n terms in the denominator cancel out n of the m terms in the numerator. Note that the case of $x = 0$ is ruled out in the statement of this rule. This is because when $x = 0$, the expression x^m/x^n would involve division by zero, which is undefined.

What if $m < n$, say, $m = 2$ and $n = 5$? In that case we get, according to Rule II, $x^{m-n} = x^{-3}$, a *negative power* of x. What does this mean? The answer is actually supplied by Rule II itself: When $m = 2$ and $n = 5$, we have

$$\frac{x^2}{x^5} = \frac{x \times x}{x \times x \times x \times x \times x} = \frac{1}{x \times x \times x} = \frac{1}{x^3}$$

Thus $x^{-3} = 1/x^3$, and this may be generalized into another rule:

Rule III $$x^{-n} = \frac{1}{x^n} \qquad (x \neq 0)$$

To raise a (nonzero) number to a power of *negative n* is to take the *reciprocal* of its nth power.

Another special case in the application of Rule II is when $m = n$, which yields the expression $x^{m-n} = x^{m-m} = x^0$. To interpret the meaning of raising a number x to the zeroth power, we can write out the term x^{m-m} in accordance with Rule II, with the result that $x^m/x^m = 1$. Thus we may conclude that any (nonzero) number raised to the zeroth power is equal to 1. (The expression 0^0 is undefined.) This may be expressed as another rule:

Rule IV $$x^0 = 1 \qquad (x \neq 0)$$

As long as we are concerned only with polynomial functions, only (nonnegative) integer powers are required. In exponential functions, however, the exponent is a variable that can take noninteger values as well. In order to interpret a number such as $x^{1/2}$, let us consider the fact that, by Rule I, we have

$$x^{1/2} \times x^{1/2} = x^1 = x$$

Since $x^{1/2}$ multiplied by itself is x, $x^{1/2}$ must be the square root of x. Similarly, $x^{1/3}$ can be shown to be the cube root of x. In general, therefore, we can state the following rule:

Rule V $$x^{1/n} = \sqrt[n]{x}$$

Two other rules obeyed by exponents are

Rule VI $$(x^m)^n = x^{mn}$$

Rule VII $$x^m \times y^m = (xy)^m$$

EXERCISE 2.5

1. Graph the functions
   (*a*) $y = 16 + 2x$ (*b*) $y = 8 - 2x$ (*c*) $y = 2x + 12$
   (In each case, consider the domain as consisting of nonnegative real numbers only.)
2. What is the major difference between (*a*) and (*b*) in Prob. 1? How is this difference reflected in the graphs? What is the major difference between (*a*) and (*c*)? How do their graphs reflect it?

3. Graph the functions

   (*a*) $y = -x^2 + 5x - 2$ (*b*) $y = x^2 + 5x - 2$

   with the set of values $-5 \le x \le 5$ constituting the domain. It is well known that the sign of the coefficient of the x^2 term determines whether the graph of a quadratic function will have a "hill" or a "valley." On the basis of the present problem, which sign is associated with the hill? Supply an intuitive explanation for this.
4. Graph the function $y = 36/x$, assuming that x and y can take positive values only. Next, suppose that both variables can take negative values as well; how must the graph be modified to reflect this change in assumption?
5. Condense the following expressions:

   (*a*) $x^4 \times x^{15}$ (*b*) $x^a \times x^b \times x^c$ (*c*) $x^3 \times y^3 \times z^3$
6. Find: (*a*) x^3/x^{-3} (*b*) $(x^{1/2} \times x^{1/3})/x^{2/3}$
7. Show that $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$. Specify the rules applied in each step.
8. Prove Rule VI and Rule VII.

2.6 Functions of Two or More Independent Variables

Thus far, we have considered only functions of a single independent variable, $y = f(x)$. But the concept of a function can be readily extended to the case of two or more independent variables. Given a function

$$z = g(x, y)$$

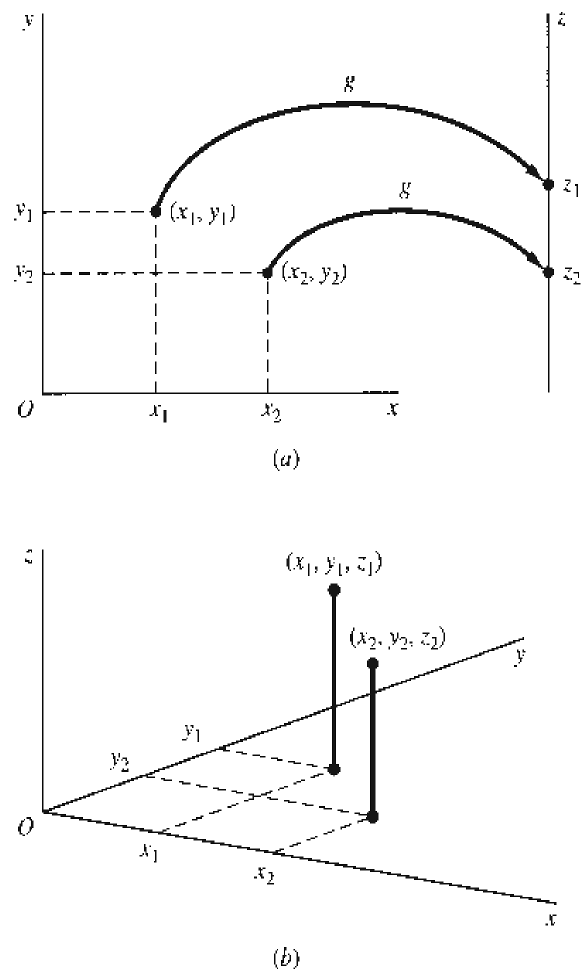
a given pair of x and y values will uniquely determine a value of the dependent variable z. Such a function is exemplified by

$$z = ax + by \qquad \text{or} \qquad z = a_0 - a_1x + a_2x^2 + b_1y + b_2y^2$$

Just as the function $y = f(x)$ maps a point in the domain into a point in the range, the function g will do precisely the same. However, the domain is in this case no longer a set of numbers but a set of ordered pairs (x, y), because we can determine z only when *both x and y* are specified. The function g is thus a mapping from a point in a two-dimensional space into a point on a line segment (i.e., a point in a one-dimensional space), such as from the point (x_1, y_1) into the point z_1 or from (x_2, y_2) into z_2 in Fig. 2.9*a*.

If a vertical z axis is erected perpendicular to the xy plane, as is done in diagram *b*, however, there will result a three-dimensional space in which the function g can be given a graphical representation as follows. The domain of the function will be some subset of the points in the xy plane, and the value of the function (value of z) for a given point in the domain—say, (x_1, y_1)—can be indicated by the height of a vertical line planted on that point. The association between the three variables is thus summarized by the ordered triple (x_1, y_1, z_1), which is a specific point in the three-dimensional space. The locus of such ordered triples, which will take the form of a *surface*, then constitutes the graph of the function g. Whereas the function $y = f(x)$ is a set of ordered *pairs*, the function $z = g(x, y)$ will be a set of ordered *triples*. We shall have many occasions to use functions of this type

FIGURE 2.9



in economic models. One ready application is in the area of production functions. Suppose that output is determined by the amounts of capital (K) and labor (L) employed; then we can write a production function in the general form $Q = Q(K, L)$.

The possibility of further extension to the cases of three or more independent variables is now self-evident. With the function $y = h(u, v, w)$, for example, we can map a point in the three-dimensional space, (u_1, v_1, w_1), into a point in a one-dimensional space (y_1). Such a function might be used to indicate that a consumer's utility is a function of his or her consumption of three different commodities, and the mapping is from a three-dimensional commodity space into a one-dimensional utility space. But this time it will be physically impossible to graph the function, because for that task a four-dimensional diagram is needed to picture the ordered quadruples, but the world in which we live is only three-dimensional. Nonetheless, in view of the intuitive appeal of geometric analogy, we can continue to refer to an ordered quadruple (u_1, v_1, w_1, y_1) as a "point" in the four-dimensional space. The locus of such points will give the (nongraphable) "graph" of the function $y = h(u, v, w)$, which is called a *hypersurface.* These terms, viz., point and hypersurface, are also carried over to the general case of the n-dimensional space.

Functions of more than one variable can be classified into various types, too. For instance, a function of the form

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

is a *linear* function, whose characteristic is that every variable is raised to the first power only. A *quadratic* function, on the other hand, involves first and second powers of one or more independent variables, but the sum of exponents of the variables appearing in any single term must not exceed 2.

Note that instead of denoting the independent variables by x, u, v, w, etc., we have switched to the symbols $x_1, x_2, \ldots, x_n$. The latter notation, like the system of subscripted coefficients, has the merit of economy of alphabet, as well as of an easier accounting of the number of variables involved in a function.

2.7 Levels of Generality

In discussing the various types of function, we have without explicit notice introduced examples of functions that pertain to varying levels of generality. In certain instances, we have written functions in the form

$$y = 7 \qquad y = 6x + 4 \qquad y = x^2 - 3x + 1 \qquad \text{(etc.)}$$

Not only are these expressed in terms of numerical coefficients, but they also indicate specifically whether each function is constant, linear, or quadratic. In terms of graphs, each such function will give rise to a well-defined unique curve. In view of the numerical nature of these functions, the solutions of the model based on them will emerge as numerical values also. The drawback is that, if we wish to know how our analytical conclusion will change when a different set of numerical coefficients comes into effect, we must go through the reasoning process afresh each time. Thus, the results obtained from specific functions have very little generality.

On a more general level of discussion and analysis, there are functions in the form

$$y = a \qquad y = a + bx \qquad y = a + bx + cx^2 \qquad \text{(etc.)}$$

Since parameters are used, each function represents not a single curve but a whole family of curves. The function $y = a$, for instance, encompasses not only the specific cases $y = 0$, $y = 1$, and $y = 2$ but also $y = \frac{1}{3}$, $y = -5, \ldots,$ ad infinitum. With parametric functions, the outcome of mathematical operations will also be in terms of parameters. These results are more general in the sense that, by assigning various values to the parameters appearing in the solution of the model, a whole family of specific answers may be obtained without having to repeat the reasoning process anew.

In order to attain an even higher level of generality, we may resort to the general function statement $y = f(x)$, or $z = g(x, y)$. When expressed in this form, the function is not restricted to being either linear, quadratic, exponential, or trigonometric—all of which are subsumed under the notation. The analytical result based on such a general formulation will therefore have the most general applicability. As will be found below, however, in order to obtain economically meaningful results, it is often necessary to impose certain qualitative restrictions on the general functions built into a model, such as the restriction that a demand function have a negatively sloped graph or that a consumption function have a graph with a positive slope of less than 1.

To sum up the present chapter, the structure of a mathematical economic model is now clear. In general, it will consist of a system of equations, which may be definitional, behavioral, or in the nature of equilibrium conditions.[†] The behavioral equations are usually in the form of functions, which may be linear or nonlinear, numerical or parametric, and with one independent variable or many. It is through these that the analytical assumptions adopted in the model are given mathematical expression.

In attacking an analytical problem, therefore, the first step is to select the appropriate variables—exogenous as well as endogenous—for inclusion in the model. Next, we must translate into equations the set of chosen analytical assumptions regarding the human, institutional, technological, legal, and other behavioral aspects of the environment affecting the working of the variables. Only then can we attempt to derive a set of conclusions through relevant mathematical operations and manipulations and to give them appropriate economic interpretations.

[†] Inequalities may also enter as an important ingredient of a model, but we shall not worry about them for the time being.