Chapter 4

Linear Models and Matrix Algebra

For the one-commodity model (3.1), the solutions P^* and Q^* as expressed in (3.4) and (3.5), respectively, are relatively simple, even though a number of parameters are involved. As more and more commodities are incorporated into the model, such solution formulas quickly become cumbersome and unwieldy. That was why we had to resort to a little shorthand, even for the two-commodity case—in order that the solutions (3.14) and (3.15) can still be written in a relatively concise fashion. We did not attempt to tackle any three- or four-commodity models, even in the linear version, primarily because we did not yet have at our disposal a method suitable for handling a large system of simultaneous equations. Such a method is found in *matrix algebra*, the subject of this chapter and the next.

Matrix algebra can enable us to do many things. In the first place, it provides a compact way of writing an equation system, even an extremely large one. Second, it leads to a way of testing the existence of a solution by evaluation of a *determinant*—a concept closely related to that of a matrix. Third, it gives a method of finding that solution (if it exists). Since equation systems are encountered not only in static analysis but also in comparative-static and dynamic analyses and in optimization problems, you will find ample application of matrix algebra in almost every chapter that is to follow. This is why it is desirable to introduce matrix algebra early.

However, one slight catch is that matrix algebra is applicable only to *linear*-equation systems. How realistically linear equations can describe actual economic relationships depends, of course, on the nature of the relationships in question. In many cases, even if some sacrifice of realism is entailed by the assumption of linearity, an assumed linear relationship can produce a sufficiently close approximation to an actual nonlinear relationship to warrant its use.

In other cases, while preserving the nonlinearity in the model, we can effect a transformation of variables so as to obtain a linear relation to work with. For example, the nonlinear function

$$y = ax^b$$

can be readily transformed, by taking the logarithm on both sides, into the function

$$\log y = \log a + b \log x$$

which is linear in the two variables $(\log y)$ and $(\log x)$. (Logarithms will be discussed in more detail in Chap. 10.). More importantly, in many applications such as comparative-static analysis and optimization problems, discussed subsequently, although the original formulation of the economic model is nonlinear in nature, linear equation systems will emerge in the course of analysis. Thus the linearity restriction is not nearly as restrictive as it may first appear.

4.1 Matrices and Vectors

The two-commodity market model (3.12) can be written—after eliminating the quantity variables—as a system of two linear equations, as in (3.13′),

$$\begin{aligned} c_1 P_1 + c_2 P_2 &= -c_0 \\ \gamma_1 P_1 + \gamma_2 P_2 &= -\gamma_0 \end{aligned}$$

where the parameters c_0 and γ_0 appear to the right of the equals sign. In general, a system of m linear equations in n variables $(x_1, x_2, \ldots, x_n)$ can also be arranged into such a format:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m \end{aligned} \tag{4.1}$$

In (4.1), the variable x_1 appears only within the leftmost column, and in general the variable x_j appears only in the jth column on the left side of the equals sign. The double-subscripted parameter symbol a_{ij} represents the coefficient appearing in the ith equation and attached to the jth variable. For example, a_{21} is the coefficient in the second equation, attached to the variable x_1. The parameter d_i which is unattached to any variable, on the other hand, represents the constant term in the ith equation. For instance, d_1 is the constant term in the first equation. All subscripts are therefore keyed to the specific locations of the variables and parameters in (4.1).

Matrices as Arrays

There are essentially three types of ingredients in the equation system (4.1). The first is the set of coefficients a_{ij}; the second is the set of variables $x_1, \ldots, x_n$; and the last is the set of constant terms $d_1, \ldots, d_m$. If we arrange the three sets as three rectangular arrays and label them, respectively, as A, x, and d (without subscripts), then we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix} \tag{4.2}$$

As a simple example, given the linear-equation system

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned} \tag{4.3}$$

we can write

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} \tag{4.4}$$

Each of the three arrays in (4.2) or (4.4) constitutes a *matrix.*

A matrix is defined as a rectangular array of numbers, parameters, or variables. The members of the array, referred to as the *elements* of the matrix, are usually enclosed in brackets, as in (4.2), or sometimes in parentheses or with double vertical lines: || ||. Note that in matrix A (the *coefficient matrix* of the equation system), the elements are separated not by commas but by blank spaces only. As a shorthand device, the array in matrix A can be written more simply as

$$A = [a_{ij}] \qquad \begin{pmatrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{pmatrix}$$

Inasmuch as the location of each element in a matrix is unequivocally fixed by the subscript, every matrix is an ordered set.

Vectors as Special Matrices

The number of rows and the number of columns in a matrix together define the *dimension* of the matrix. Since matrix A in (4.2) contains m rows and n columns, it is said to be of dimension $m \times n$ (read "m by n"). It is important to remember that the row number always precedes the column number; this is in line with the way the two subscripts in a_{ij} are ordered. In the special case where $m = n$, the matrix is called a *square matrix;* thus the matrix A in (4.4) is a 3×3 square matrix.

Some matrices may contain only one column, such as x and d in (4.2) or (4.4). Such matrices are given the special name *column vectors.* In (4.2), the dimension of x is $n \times 1$, and that of d is $m \times 1$; in (4.4) both x and d are 3×1. If we arranged the variables x_j in a horizontal array, though, there would result a $1 \times n$ matrix, which is called a *row vector.* For notation purposes, a row vector is often distinguished from a column vector by the use of a primed symbol:

$$x' = [x_1 \quad x_2 \quad \cdots \quad x_n]$$

You may observe that a vector (whether row or column) is merely an ordered n-tuple, and as such it may sometimes be interpreted as a point in an n-dimensional space. In turn, the $m \times n$ matrix A can be interpreted as an ordered set of m row vectors or as an ordered set of n column vectors. These ideas will be followed up in Chap. 5.

An issue of more immediate interest is how the matrix notation can enable us, as promised, to express an equation system in a compact way. With the matrices defined in (4.4), we can express the equation system (4.3) simply as

$$Ax = d$$

In fact, if A, x, and d are given the meanings in (4.2), then even the general-equation system in (4.1) can be written as $Ax = d$. The compactness of this notation is thus unmistakable.

However, the equation $Ax = d$ prompts at least two questions. How do we multiply two matrices A and x? What is meant by the equality of Ax and d? Since matrices involve whole blocks of numbers, the familiar algebraic operations defined for single numbers are not directly applicable, and there is a need for a new set of operational rules.

EXERCISE 4.1

1. Rewrite the market model (3.1) in the format of (4.1), and show that, if the three variables are arranged in the order Q_d, Q_s, and P, the coefficient matrix will be

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$$

   How would you write the vector of constants?
2. Rewrite the market model (3.12) in the format of (4.1) with the variables arranged in the following order: Q_{d1}, Q_{s1}, Q_{d2}, Q_{s2}, P_1, P_2. Write out the coefficient matrix, the variable vector, and the constant vector.
3. Can the market model (3.6) be rewritten in the format of (4.1)? Why?
4. Rewrite the national-income model (3.23) in the format of (4.1), with Y as the first variable. Write out the coefficient matrix and the constant vector.
5. Rewrite the national-income model of Exercise 3.5-1 in the format of (4.1), with the variables in the order Y, T, and C. [*Hint:* Watch out for the multiplicative expression $b(Y - T)$ in the consumption function.]

4.2 Matrix Operations

As a preliminary, let us first define the word *equality*. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be *equal* if and only if they have the same dimension and have identical elements in the corresponding locations in the array. In other words, $A = B$ if and only if $a_{ij} = b_{ij}$ for all values of i and j. Thus, for example, we find

$$\begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

As another example, if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$, this will mean that $x = 7$ and $y = 4$.

Addition and Subtraction of Matrices

Two matrices can be added if and only if they have the same dimension. When this dimensional requirement is met, the matrices are said to be *conformable for addition*. In that case, the addition of $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined as the addition of each pair of corresponding elements.

Example 1

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 4+2 & 9+0 \\ 2+0 & 1+7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}$$

Example 2

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

In general, we may state the rule thus:

$$[a_{ij}] + [b_{ij}] = [c_{ij}] \qquad \text{where } c_{ij} = a_{ij} + b_{ij}$$

Note that the sum matrix $[c_{ij}]$ must have the same dimension as the component matrices $[a_{ij}]$ and $[b_{ij}]$.

The subtraction operation $A - B$ can be similarly defined if and only if A and B have the same dimension. The operation entails the result

$$[a_{ij}] - [b_{ij}] = [d_{ij}] \qquad \text{where } d_{ij} = a_{ij} - b_{ij}$$

Example 3

$$\begin{bmatrix} 19 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 19-6 & 3-8 \\ 2-1 & 0-3 \end{bmatrix} = \begin{bmatrix} 13 & -5 \\ 1 & -3 \end{bmatrix}$$

The subtraction operation $A - B$ may be considered alternatively as an addition operation involving a matrix A and another matrix $(-1)B$. This, however, raises the question of what is meant by the multiplication of a matrix by a single number (here, -1).

Scalar Multiplication

To multiply a matrix by a number—or in matrix-algebra terminology, by a *scalar*—is to multiply *every* element of that matrix by the given scalar.

Example 4

$$7\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 21 & -7 \\ 0 & 35 \end{bmatrix}$$

Example 5

$$\frac{1}{2}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{21} & \frac{1}{2}a_{22} \end{bmatrix}$$

From these examples, the rationale of the name scalar should become clear, for it "scales up (or down)" the matrix by a certain multiple. The scalar can, of course, be a negative number as well.

Example 6

$$-1\begin{bmatrix} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & -d_1 \\ -a_{21} & -a_{22} & -d_2 \end{bmatrix}$$

Note that if the matrix on the left represents the coefficients *and* the constant terms in the simultaneous equations

$$a_{11}x_1 + a_{12}x_2 = d_1$$
$$a_{21}x_1 + a_{22}x_2 = d_2$$

then multiplication by the scalar -1 will amount to multiplying both sides of both equations by -1, thereby changing the sign of every term in the system.

Multiplication of Matrices

Whereas a scalar can be used to multiply a matrix of any dimension, the multiplication of two matrices is contingent upon the satisfaction of a different dimensional requirement.

Suppose that, given two matrices A and B, we want to find the product AB. The conformability condition for multiplication is that the *column* dimension of A (the "lead" matrix in the expression AB) must be equal to the row dimension of B (the "lag" matrix). For instance, if

$$\underset{(1\times 2)}{A} = [a_{11} \quad a_{12}] \qquad \underset{(2\times 3)}{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \tag{4.5}$$

the product AB then *is* defined, since A has *two columns* and B has *two rows*—precisely the same number.[†] This can be checked at a glance by comparing the *second* number in the dimension indicator for A, which is (1×2), with the *first* number in the dimension indicator for B, (2×3). On the other hand, the reverse product BA is *not* defined in this case, because B (now the lead matrix) has *three* columns while A (the lag matrix) has only *one* row; hence the conformability condition is violated.

In general, if A is of dimension $m \times n$ and B is of dimension $p \times q$, the matrix product AB will be defined if and only if $n = p$. If defined, moreover, the product matrix AB will have the dimension $m \times q$ the same number of *rows* as the lead matrix A and the same number of *columns* as the lag matrix B. For the matrices given in (4.5), AB will be 1×3.

It remains to define the exact procedure of multiplication. For this purpose, let us take the matrices A and B in (4.5) for illustration. Since the product AB is defined and is expected to be of dimension 1×3, we may write in general (using the symbol C rather than c' for the row vector) that

$$AB = C = [c_{11} \quad c_{12} \quad c_{13}]$$

Each element in the product matrix C, denoted by c_{ij}, is defined as a sum of products, to be computed from the elements in the ith *row* of the lead matrix A, and those in the jth *column* of the lag matrix B. To find c_{11}, for instance, we should take the *first row* in A (since $i = 1$) and the *first column* in B (since $j = 1$)—as shown in the top panel of Fig. 4.1—and then pair the elements together sequentially, multiply out each pair, and take the sum of the resulting products, to get

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} \tag{4.6}$$

Similarly, for c_{12}, we take the *first row* in A (since $i = 1$) and the *second column* in B (since $j = 2$), and calculate the indicated sum of products—in accordance with the lower panel of Fig. 4.1—as follows:

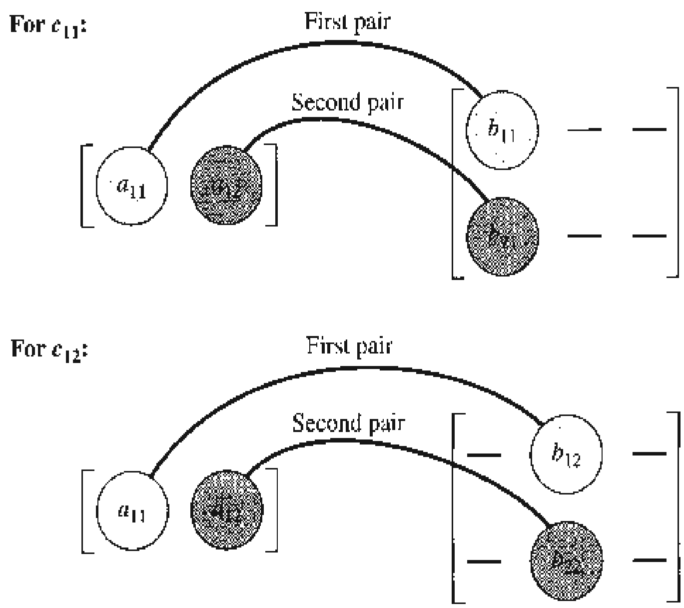
$$c_{12} = a_{11}b_{12} + a_{12}b_{22} \tag{4.6'}$$

By the same token, we should also have

$$c_{13} = a_{11}b_{13} + a_{12}b_{23} \tag{4.6''}$$

[†] The matrix A, being a row vector, would normally be denoted by a'. We use the symbol A here to stress the fact that the multiplication rule being explained applies to matrices in general, not only to the product of one vector and one matrix.

FIGURE 4.1



It is the particular pairing requirement in this process which necessitates the matching of the column dimension of the lead matrix and the row dimension of the lag matrix before multiplication can be performed.

The multiplication procedure illustrated in Fig. 4.1 can also be described by using the concept of the *inner product* of two vectors. Given two vectors u and v with n elements each, say, $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$, arranged *either* as two rows *or* as two columns *or* as one row and one column, their inner product, written as $u \cdot v$ (with a dot in the middle), is defined as

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

This is a sum of products of corresponding elements, and hence the inner product of two vectors is a scalar.

Example 7

If, after a shopping trip, we arrange the quantities purchased of n goods as a row vector $Q' = [Q_1 \quad Q_2 \quad \cdots \quad Q_n]$, and list the prices of those goods in a price vector $P' = [P_1 \quad P_2 \quad \cdots \quad P_n]$, then the inner product of these two vectors is

$$Q' \cdot P' = Q_1 P_1 + Q_2 P_2 + \cdots + Q_n P_n = \text{total purchase cost}$$

Using this concept, we can describe the element c_{ij} in the product matrix $C = AB$ simply as the inner product of the ith row of the lead matrix A and the jth column of the lag matrix B. By examining Fig. 4.1, we can easily verify the validity of this description.

The rule of multiplication just outlined applies with equal validity when the dimensions of A and B are other than those illustrated in Fig. 4.1; the only prerequisite is that the conformability condition be met.

Example 8

Given

$$\underset{(3\times 2)}{A} = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad \underset{(2\times 1)}{B} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

find AB. The product AB is indeed defined because A has two columns and B has two rows. Their product matrix should be 3×1, a column vector:

$$AB = \begin{bmatrix} 1(5)+3(9) \\ 2(5)+8(9) \\ 4(5)+0(9) \end{bmatrix} = \begin{bmatrix} 32 \\ 82 \\ 20 \end{bmatrix}$$

Example 9

Given

$$\underset{(3\times3)}{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \underset{(3\times3)}{B} = \begin{bmatrix} 0 & -\frac{1}{5} & \frac{3}{10} \\ -1 & \frac{1}{5} & \frac{7}{10} \\ 0 & \frac{2}{5} & -\frac{1}{10} \end{bmatrix}$$

find AB. The same rule of multiplication now yields a very special product matrix:

$$AB = \begin{bmatrix} 0+1+0 & -\frac{3}{5}-\frac{1}{5}+\frac{4}{5} & \frac{9}{10}-\frac{7}{10}-\frac{2}{10} \\ 0+0+0 & -\frac{1}{5}+0+\frac{6}{5} & \frac{3}{10}+0-\frac{3}{10} \\ 0+0+0 & -\frac{4}{5}+0+\frac{4}{5} & \frac{12}{10}+0-\frac{2}{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This last matrix—a square matrix with 1s in its *principal diagonal* (the diagonal running from northwest to southeast) and 0s everywhere else—exemplifies the important type of matrix known as the *identity matrix*. This will be further discussed in Section 4.5.

Example 10

Let us now take the matrix A and the vector x as defined in (4.4) and find Ax. The product matrix is a 3×1 column vector:

$$Ax = \underset{(3\times3)}{\begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}} \underset{(3\times1)}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{(3\times1)}{\begin{bmatrix} 6x_1 + 3x_2 + x_3 \\ x_1 + 4x_2 - 2x_3 \\ 4x_1 - x_2 + 5x_3 \end{bmatrix}}$$

Note: The product on the right is a *column* vector, its corpulent appearance notwithstanding! When we write $Ax = d$, therefore, we have

$$\begin{bmatrix} 6x_1 + 3x_2 + x_3 \\ x_1 + 4x_2 - 2x_3 \\ 4x_1 - x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

which, according to the definition of matrix equality, is equivalent to the statement of the entire equation system in (4.3).

Note that, to use the matrix notation $Ax = d$, it is necessary, because of the conformability condition, to arrange the variables x_j into a *column* vector, even though these variables are listed in a horizontal order in the original equation system.

Example 11

The simple national-income model in two endogenous variables Y and C,

$$Y = C + I_0 + G_0$$
$$C = a + bY$$

can be rearranged into the standard format of (4.1) as follows:

$$Y - C = I_0 + G_0$$
$$-bY + C = a$$

Hence the coefficient matrix A, the vector of variables x, and the vector of constants d are

$$\underset{(2\times2)}{A} = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \qquad \underset{(2\times1)}{x} = \begin{bmatrix} Y \\ C \end{bmatrix} \qquad \underset{(2\times1)}{d} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

Let us verify that this given system can be expressed by the equation $Ax = d$.

By the rule of matrix multiplication, we have

$$Ax = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} 1(Y) + (-1)(C) \\ -b(Y) + 1(C) \end{bmatrix} = \begin{bmatrix} Y - C \\ -bY + C \end{bmatrix}$$

Thus the matrix equation $Ax = d$ would give us

$$\begin{bmatrix} Y - C \\ -bY + C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

Since matrix equality means the equality between corresponding elements, it is clear that the equation $Ax = d$ does precisely represent the original equation system, as expressed in the (4.1) format.

The Question of Division

While matrices, like numbers, can undergo the operations of addition, subtraction, and multiplication—subject to the conformability conditions—it is not possible to divide one matrix by another. That is, we cannot write A/B.

For two numbers a and b, the quotient a/b (with $b \neq 0$) can be written alternatively as ab^{-1} or $b^{-1}a$, where b^{-1} represents the *inverse* or *reciprocal* of b. Since $ab^{-1} = b^{-1}a$, the quotient expression a/b can be used to represent both ab^{-1} and $b^{-1}a$. The case of matrices is different. Applying the concept of inverses to matrices, we may in certain cases (discussed in Sec. 4.6) define a matrix B^{-1} that is the inverse of matrix B. But from the discussion of the conformability condition it follows that, if AB^{-1} is defined, there can be no assurance that $B^{-1}A$ is also defined. Even if AB^{-1} and $B^{-1}A$ are indeed both defined, they still may not represent the same product. Hence the expression A/B cannot be used without ambiguity, and it must be avoided. Instead, you must specify whether you are referring to AB^{-1} or $B^{-1}A$—provided that the inverse B^{-1} does exist and that the matrix product in question is defined. Inverse matrices will be further discussed in Sec. 4.6.

The $\sum$ Notation

The use of subscripted symbols not only helps in designating the locations of parameters and variables but also lends itself to a flexible shorthand for denoting sums of terms, such as those which arose during the process of matrix multiplication.

The summation shorthand makes use of the Greek letter Σ (sigma, for "sum"). To express the sum of x_1, x_2, and x_3, for instance, we may write

$$x_1 + x_2 + x_3 = \sum_{j=1}^{3} x_j$$

which is read as "the sum of x_j as j ranges from 1 to 3." The symbol j, called the *summation index,* takes only integer values. The expression x_j represents the *summand* (that which is to be summed), and it is in effect a function of j. Aside from the letter j, summation indices are also commonly denoted by i or k, such as

$$\sum_{i=3}^{7} x_i = x_3 + x_4 + x_5 + x_6 + x_7$$

$$\sum_{k=0}^{n} x_k = x_0 + x_1 + \cdots + x_n$$

The application of $\sum$ notation can be readily extended to cases in which the x term is prefixed with a coefficient or in which each term in the sum is raised to some integer power. For instance, we may write:

$$\sum_{j=1}^{3} ax_j = ax_1 + ax_2 + ax_3 = a(x_1 + x_2 + x_3) = a\sum_{j=1}^{3} x_j$$

$$\sum_{j=1}^{3} a_j x_j = a_1x_1 + a_2x_2 + a_3x_3$$

$$\begin{aligned}\sum_{i=0}^{n} a_i x^i &= a_0x^0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n \\ &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\end{aligned}$$

The last example, in particular, shows that the expression $\sum_{i=0}^{n} a_i x^i$ can in fact be used as a shorthand form of the general polynomial function of (2.4).

It may be mentioned in passing that, whenever the context of the discussion leaves no ambiguity as to the range of summation, the symbol $\sum$ can be used alone, without an index attached (such as $\sum x_i$), or with only the index letter underneath (such as $\sum_i x_i$).

Let us apply the $\sum$ shorthand to matrix multiplication. In (4.6), (4.6′), and (4.6″), each element of the product matrix $C = AB$ is defined as a sum of terms, which may now be rewritten as follows:

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} = \sum_{k=1}^{2} a_{1k}b_{k1}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} = \sum_{k=1}^{2} a_{1k}b_{k2}$$

$$c_{13} = a_{11}b_{13} + a_{12}b_{23} = \sum_{k=1}^{2} a_{1k}b_{k3}$$

In each case, the first subscript of c_{1j} is reflected in the first subscript of a_{1k}, and the second subscript of c_{1j} is reflected in the second subscript of b_{kj} in the $\sum$ expression. The index k, on the other hand, is a "dummy" subscript; it serves to indicate which particular pair of elements is being multiplied, but it does not show up in the symbol c_{1j}.

Extending this to the multiplication of an $m \times n$ matrix $A = [a_{ik}]$ and an $n \times p$ matrix $B = [b_{kj}]$, we may now write the elements of the $m \times p$ product matrix $AB = C = [c_{ij}]$ as

$$c_{11} = \sum_{k=1}^{n} a_{1k}b_{k1} \qquad c_{12} = \sum_{k=1}^{n} a_{1k}b_{k2} \qquad \cdots$$

or more generally,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \qquad \begin{pmatrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, p \end{pmatrix}$$

This last equation represents yet another way of stating the rule of multiplication for the matrices defined above.

EXERCISE 4.2

1. Given $A = \begin{bmatrix} 7 & -1 \\ 6 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 4 \\ 3 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 8 & 3 \\ 6 & 1 \end{bmatrix}$, find:

   (*a*) $A + B$ (*b*) $C - A$ (*c*) $3A$ (*d*) $4B + 2C$

2. Given $A = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$:

   (*a*) Is AB defined? Calculate AB. Can you calculate BA? Why?

   (*b*) Is BC defined? Calculate BC. Is CB defined? If so, calculate CB. Is it true that $BC = CB$?

3. On the basis of the matrices given in Example 9, is the product BA defined? If so, calculate the product. In this case do we have $AB = BA$?

4. Find the product matrices in the following (in each case, append beneath every matrix a dimension indicator):

   (*a*) $\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 1 \\ 3 & 5 \end{bmatrix}$ (*c*) $\begin{bmatrix} 3 & 5 & 0 \\ 4 & 2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

   (*b*) $\begin{bmatrix} 6 & 5 & -1 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 0 & 1 \end{bmatrix}$ (*d*) $[a \quad b \quad c] \begin{bmatrix} 7 & 0 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$

5. In Example 7, if we arrange the quantities and prices as column vectors instead of row vectors, is $Q \cdot P$ defined? Can we express the total purchase cost as $Q \cdot P$? As $Q' \cdot P$? As $Q \cdot P'$?

6. Expand the following summation expressions:

   (*a*) $\sum_{i=2}^{5} x_i$ (*d*) $\sum_{i=1}^{n} a_i x^{i-1}$

   (*b*) $\sum_{i=5}^{8} a_i x_i$ (*e*) $\sum_{i=0}^{3} (x + i)^2$

   (*c*) $\sum_{i=1}^{4} bx_i$

7. Rewrite the following in $\sum$ notation:
   (*a*) $x_1(x_1 - 1) + 2x_2(x_2 - 1) + 3x_3(x_3 - 1)$
   (*b*) $a_2(x_3 + 2) + a_3(x_4 + 3) + a_4(x_5 + 4)$
   (*c*) $\frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^n} \quad (x \neq 0)$
   (*d*) $1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^n} \quad (x \neq 0)$
8. Show that the following are true:
   (*a*) $\left(\sum_{i=0}^{n} x_i\right) + x_{n+1} = \sum_{i=0}^{n+1} x_i$
   (*b*) $\sum_{j=1}^{n} ab_j y_j = a \sum_{j=1}^{n} b_j y_j$
   (*c*) $\sum_{j=1}^{n} (x_j + y_j) = \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} y_j$

4.3 Notes on Vector Operations

In Secs. 4.1 and 4.2, vectors are considered as a special type of matrix. As such, they qualify for the application of all the algebraic operations discussed. Owing to their dimensional peculiarities, however, some additional comments on vector operations are useful.

Multiplication of Vectors

An $m \times 1$ column vector u, and a $1 \times n$ row vector v', yield a product matrix uv' of dimension $m \times n$.

Example 1 Given $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v' = [1 \quad 4 \quad 5]$, we can get

$$uv' = \begin{bmatrix} 3(1) & 3(4) & 3(5) \\ 2(1) & 2(4) & 2(5) \end{bmatrix} = \begin{bmatrix} 3 & 12 & 15 \\ 2 & 8 & 10 \end{bmatrix}$$

Since each row in u consists of one element only, as does each column in v', each element of uv' turns out to be a single product instead of a sum of products. The product uv' is a 2×3 matrix, even though what we started out with are a pair of vectors.

On the other hand, given a $1 \times n$ row vector u' and an $n \times 1$ column vector v, the product $u'v$ will be of dimension 1×1.

Example 2 Given $u' = [3 \quad 4]$ and $v = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$, we have

$$u'v = [3(9) + 4(7)] = [55]$$

As written, $u'v$ is a matrix, despite the fact that only a single element is present. However, 1×1 matrices behave exactly like scalars with respect to addition and multiplication: $[4] + [8] = [12]$, just as $4 + 8 = 12$; and $[3]\,[7] = [21]$, just as $3(7) = 21$. Moreover, 1×1

matrices possess no major properties that scalars do not have. In fact, there is a one-to-one correspondence between the set of all scalars and the set of all 1×1 matrices whose elements are scalars. For this reason, we may redefine $u'v$ to be the *scalar* corresponding to the 1×1 product matrix. For Example 2, we can accordingly write $u'v = 55$. Such a product is called a *scalar product*.[†] Remember, however, that while a 1×1 matrix can be treated as a scalar, a scalar cannot be replaced by a 1×1 matrix at will if further calculation is to be carried out, because complications regarding conformability conditions may arise.

Example 3 Given a row vector $u' = [3 \quad 6 \quad 9]$, find $u'u$. Since u is merely the column vector with the elements of u' arranged vertically, we have

$$u'u = [3 \quad 6 \quad 9]\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = (3)^2 + (6)^2 + (9)^2$$

where we have omitted the brackets from the 1×1 product matrix on the right. Note that the product $u'u$ gives the sum of squares of the elements of u.

In general, if $u' = [u_1 \quad u_2 \quad \cdots \quad u_n]$, then $u'u$ will be the sum of squares (a scalar) of the elements u_j:

$$u'u = u_1^2 + u_2^2 + \cdots + u_n^2 = \sum_{j=1}^{n} u_j^2$$

Had we calculated the inner product $u \cdot u$ (or $u' \cdot u'$), we would have, of course, obtained exactly the same result.

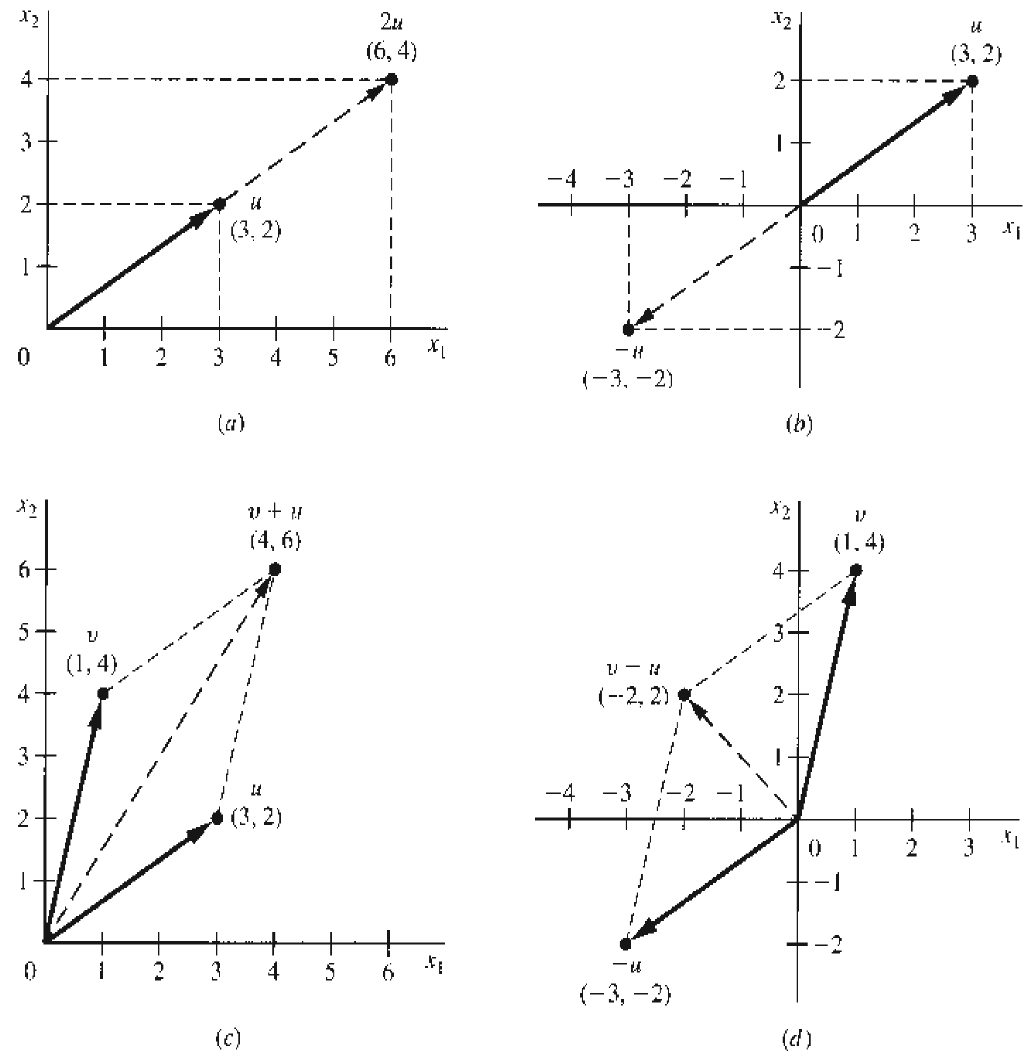
To conclude, it is important to distinguish between the meanings of uv' (a matrix larger than 1×1) and $u'v$ (a 1×1 matrix, or a scalar). Observe, in particular, that a scalar product must have a *row* vector as the lead matrix and a *column* vector as the lag matrix; otherwise the product cannot be 1×1.

Geometric Interpretation of Vector Operations

It was mentioned earlier that a column or row vector with n elements (referred to hereafter as an *n-vector*) can be viewed as an n-tuple, and hence as a point in an n-dimensional space (referred to hereafter as an n-space). Let us elaborate on this idea. In Fig. 4.2*a*, a point (3, 2) is plotted in a 2-space and is labeled u. This is the geometric counterpart of the vector $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ or the vector $u' = [3 \quad 2]$, both of which indicate in this context one and the same ordered pair. If an arrow (a directed-line segment) is drawn from the point of origin (0, 0) to the point u, it will specify the unique straight route by which to reach the destination point u from the point of origin. Since a unique arrow exists for each point, we can regard the vector u as graphically represented *either* by the point (3, 2), *or* by the corresponding arrow. Such an arrow, which emanates from the origin (0, 0) like the hand of a clock, with a definite length and a definite direction, is called a *radius vector*.

[†] The concept of scalar product is thus akin to the concept of inner product of two vectors with the same number of elements in each, which also yields a scalar. Recall, however, that the inner product is exempted from the conformability condition for multiplication, so that we may write it as $u \cdot v$. In the case of scalar product (denoted without a dot between the two vector symbols), on the other hand, we can express it only as a row vector multiplied by a column vector, with the row vector in the lead.

FIGURE 4.2



Following this new interpretation of a vector, it becomes possible to give geometric meanings to (*a*) the scalar multiplication of a vector, (*b*) the addition and subtraction of vectors, and more generally, (*c*) the so-called linear combination of vectors.

First, if we plot the vector $\begin{bmatrix} 6 \\ 4 \end{bmatrix} = 2u$ in Fig. 4.2*a*, the resulting arrow will overlap the old one but will be twice as long. In fact, the multiplication of vector u by any scalar k will produce an overlapping arrow, but the arrowhead will be relocated, unless $k = 1$. If the scalar multiplier is $k > 1$, the arrow will be extended out (scaled up); if $0 < k < 1$, the arrow will be shortened (scaled down); if $k = 0$, the arrow will shrink into the point of origin—which represents a *null vector*, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. A negative scalar multiplier will even reverse the direction of the arrow. If the vector u is multiplied by -1, for instance, we get $-u = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, and this plots in Fig. 4.2*b* as an arrow of the same length as u but diametrically opposite in direction.

Next, consider the addition of two vectors, $v = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. The sum $v + u = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ can be directly plotted as the broken arrow in Fig. 4.2*c*. If we construct a parallelogram with the two vectors u and v (solid arrows) as two of its sides, however, the diagonal of the parallelogram will turn out exactly to be the arrow representing the vector sum $v + u$. In general, a vector sum can be obtained geometrically from a parallelogram. Moreover, this method can also give us the *vector difference* $v - u$, since the latter is equivalent to the *sum* of v and $(-1)u$. In Fig. 4.2*d*, we first reproduce the vector v and the negative vector $-u$ from diagrams *c* and *b*, respectively, and then construct a parallelogram. The resulting diagonal represents the vector difference $v - u$.

It takes only a simple extension of these results to interpret geometrically a linear combination (i.e., a linear sum or difference) of vectors. Consider the simple case of

$$3v + 2u = 3\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

The scalar multiplication aspect of this operation involves the relocation of the respective arrowheads of the two vectors v and u, and the addition aspect calls for the construction of a parallelogram. Beyond these two basic graphical operations, there is nothing new in a linear combination of vectors. This is true even if there are more terms in the linear combination, as in

$$\sum_{i=1}^{n} k_i v_i = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$$

where k_i are a set of scalars but the subscripted symbols v_i now denote a set of vectors. To form this sum, the first two terms may be added first, and then the resulting sum is added to the third, and so forth, till all terms are included.

Linear Dependence

A set of vectors $v_1, \ldots, v_n$ is said to be *linearly dependent* if (and only if) any one of them can be expressed as a linear combination of the remaining vectors; otherwise they are *linearly independent*.

Example 4

The three vectors $v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ are linearly dependent because v_3 is a linear combination of v_1 and v_2:

$$3v_1 - 2v_2 = \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = v_3$$

Note that this last equation is alternatively expressible as

$$3v_1 - 2v_2 - v_3 = 0$$

where $0 \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ represents a null vector (also called the *zero vector*).

Example 5

The two row vectors $v_1' = [5 \quad 12]$ and $v_2' = [10 \quad 24]$ are linearly dependent because

$$2v_1' = 2[5 \quad 12] = [10 \quad 24] = v_2'$$

The fact that one vector is a multiple of another vector illustrates the simplest case of linear combination. Note again that this last equation may be written equivalently as

$$2v_1' - v_2' = 0'$$

where $0'$ represents the null row vector $[0 \quad 0]$.

With the introduction of null vectors, linear dependence may be redefined as follows. A set of m-vectors $v_1, \dots, v_n$ is *linearly dependent* if and only if there exists a set of scalars $k_1, \dots, k_n$ (not all zero) such that

$$\sum_{i=1}^{n} k_i v_i = \underset{(m\times 1)}{0}$$

If this equation can be satisfied *only when* $k_i = 0$ for all i, on the other hand, these vectors are linearly independent.

The concept of linear dependence admits of an easy geometric interpretation also. Two vectors u and $2u$—one being a multiple of the other—are obviously dependent. Geometrically, in Fig. 4.2*a*, their arrows lie on a single straight line. The same is true of the two dependent vectors u and $-u$ in Fig. 4.2*b*. In contrast, the two vectors u and v of Fig. 4.2*c* are linearly *independent*, because it is impossible to express one as a multiple of the other. Geometrically, their arrows do not lie on a single straight line.

When more than two vectors in the 2-space are considered, there emerges this significant conclusion: once we have found two linearly *independent* vectors in the 2-space (say, u and v), all the other vectors in that space will be expressible as a linear combination of these (u and v). In Fig. 4.2*c* and *d*, it has already been illustrated how the two simple linear combinations $v + u$ and $v - u$ can be found. Furthermore, by extending, shortening, and reversing the given vectors u and v and then combining these into various parallelograms, we can generate an infinite number of new vectors, which will exhaust the set of all 2-vectors. Because of this, any set of three or more 2-vectors (three or more vectors in a 2-space) must be linearly dependent. Two of them can be independent, but then the third must be a linear combination of the first two.

Vector Space

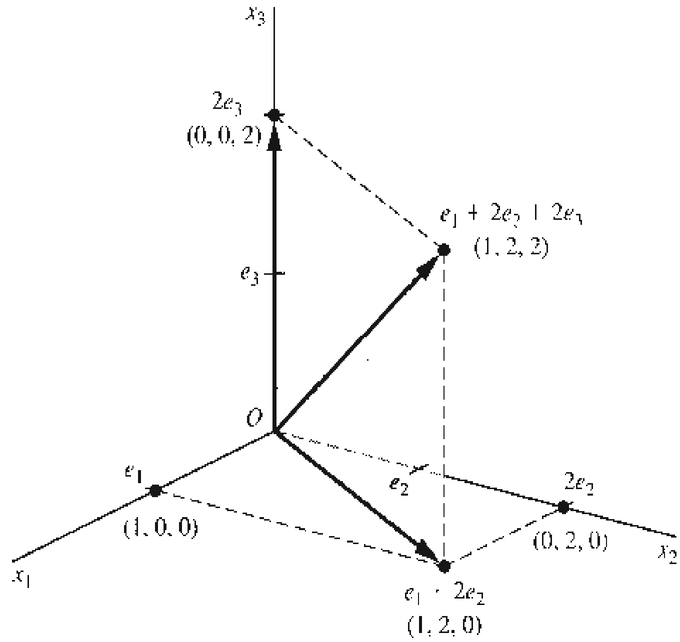
The totality of the 2-vectors generated by the various linear combinations of two independent vectors u and v constitutes the two-dimensional *vector space*. Since we are dealing only with vectors with real-valued elements, this vector space is none other than R^2, the 2-space we have been referring to all along. The 2-space cannot be generated by a single 2-vector, because linear combinations of the latter can only give rise to the set of vectors lying on a single straight line. Nor does the generation of the 2-space require more than two linearly independent 2-vectors—at any rate, it would be impossible to find more than two.

The two linearly independent vectors u and v are said to *span* the 2-space. They are also said to constitute a *basis* for the 2-space. Note that we said *a* basis, not *the* basis, because any pair of 2-vectors can serve in that capacity as long as they are linearly independent. In particular, consider the two vectors $[1 \quad 0]$ and $[0 \quad 1]$, which are called *unit vectors*. The first one plots as an arrow lying along the horizontal axis, and the second, an arrow lying along the vertical axis. Because they are linearly independent, they can serve as a basis for the 2-space, and we do in fact ordinarily think of the 2-space as spanned by its two axes, which are nothing but the extended versions of the two unit vectors.

By analogy, the three-dimensional vector space is the totality of 3-vectors, and it must be spanned by exactly three linearly independent 3-vectors. As an illustration, consider the set of three unit vectors

$$e_1 \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad e_2 \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad e_3 \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.7}$$

FIGURE 4.3



where each e_i is a vector with 1 as its ith element and with zeros elsewhere.† These three vectors are obviously linearly independent; in fact, their arrows lie on the three axes of the 3-space in Fig. 4.3. Thus they span the 3-space, which implies that the entire 3-space (R^3, in our framework) can be generated from these unit vectors. For example, the vector $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ can be considered as the linear combination $e_1 + 2e_2 + 2e_3$. Geometrically, we can first add the vectors e_1 and $2e_2$ in Fig. 4.3 by the parallelogram method, in order to get the vector represented by the point (1, 2, 0) in the x_1x_2 plane, and then add the latter vector to $2e_3$—via the parallelogram constructed in the shaded vertical plane—to obtain the desired final result, at the point (1, 2, 2).

The further extension to n-space should be obvious. The n-space can be defined as the totality of n-vectors. Though nongraphable, we can still think of the n-space as being spanned by a total of n (n-element) unit vectors that are all linearly independent. Each n-vector, being an ordered n-tuple, represents a *point* in the n-space, or an arrow extending from the point of origin (i.e., the n-element null vector) to the said point. And any given set of n linearly independent n-vectors is, in fact, capable of generating the entire n-space. Since, in our discussion, each element of the n-vector is restricted to be a real number, this n-space is in fact R^n.

The n-space we have referred to is sometimes more specifically called the *Euclidean n-space* (named after Euclid). To explain this latter concept, we must first comment briefly on the concept of *distance* between two vector points. For any pair of vector points u and v in a given space, the distance from u to v is some real-valued function

$$d = d(u, v)$$

with the following properties: (1) when u and v coincide, the distance is zero; (2) when the two points are distinct, the distance from u to v and the distance from v to u are represented

† The symbol e may be associated with the German word *eins*, for "one."

by an identical positive real number; and (3) the distance between u and v is never longer than the distance from u to w (a point distinct from u and v) plus the distance from w to v. Expressed symbolically,

$$\begin{aligned} d(u, v) &= 0 && (\text{for } u = v) \\ d(u, v) &= d(v, u) > 0 && (\text{for } u \neq v) \\ d(u, v) &\leq d(u, w) + d(w, v) && (\text{for } w \neq u, v) \end{aligned}$$

The last property is known as the *triangular inequality*, because the three points u, v, and w together will usually define a triangle.

When a vector space has a distance function defined that fulfills the previous three properties, it is called a *metric space*. However, note that the distance $d(u, v)$ has been discussed only in general terms. Depending on the specific form assigned to the d function, there may result a variety of metric spaces. The so-called Euclidean space is one specific type of metric space, with a distance function defined as follows. Let point u be the n-tuple $(a_1, a_2, \ldots, a_n)$ and point v be the n-tuple $(b_1, b_2, \ldots, b_n)$; then the Euclidean distance function is

$$d(u, v) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

where the square root is taken to be positive. As can be easily verified, this specific distance function satisfies all three properties previously enumerated. Applied to the two-dimensional space in Fig. 4.2a, the distance between the two points (6, 4) and (3, 2) is found to be

$$\sqrt{(6-3)^2 + (4-2)^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

This result is seen to be consistent with *Pythagoras's theorem*, which states that the length of the hypotenuse of a right-angled triangle is equal to the (positive) square root of the sum of the squares of the lengths of the other two sides. For if we take (6, 4) and (3, 2) to be u and v, and plot a new point w at (6, 2), we shall indeed have a right-angled triangle with the lengths of its horizontal and vertical sides equal to 3 and 2, respectively, and the length of the hypotenuse (the distance between u and v) equal to $\sqrt{3^2 + 2^2} = \sqrt{13}$.

The Euclidean distance function can also be expressed in terms of the square root of a scalar product of two vectors. Since u and v denote the two n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, we can write a column vector $u - v$, with elements $a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n$. What goes under the square-root sign in the Euclidean distance function is, of course, simply the sum of squares of these n elements, which, in view of Example 3 of this section, can be written as the scalar product $(u - v)'(u - v)$. Hence we have

$$d(u, v) = \sqrt{(u - v)'(u - v)}$$

EXERCISE 4.3

1. Given $u' = [5 \quad 1 \quad 3]$, $v' = [3 \quad 1 \quad -1]$, $w' = [7 \quad 5 \quad 8]$, and $x' = [x_1 \quad x_2 \quad x_3]$, write out the column vectors, u, v, w, and x, and find

   (a) uv' (c) xx' (e) $u'v$ (g) $u'u$
   (b) uw' (d) $v'u$ (f) $w'x$ (h) $x'x$

2. Given $w = \begin{bmatrix} 3 \\ 2 \\ 16 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, and $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$:

   (*a*) Which of the following are defined: $w'x$, $x'y'$, xy', $y'y$, zz', yw', $x \cdot y$?

   (*b*) Find all the products that are defined.

3. Having sold n items of merchandise at quantities $Q_1, \ldots, Q_n$ and prices $P_1, \ldots, P_n$, how would you express the total revenue in (*a*) $\sum$ notation and (*b*) vector notation?

4. Given two nonzero vectors w_1 and w_2, the angle θ ($0° \le \theta \le 180°$) they form is related to the scalar product $w_1'w_2$ ($= w_2'w_1$) as follows:

$$\theta \text{ is a(n)} \begin{Bmatrix} \text{acute} \\ \text{right} \\ \text{obtuse} \end{Bmatrix} \text{angle if and only if } w_1'w_2 \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$$

   Verify this by computing the scalar product for each of the following pair of vectors (see Figs. 4.2 and 4.3):

   (*a*) $w_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

   (*b*) $w_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $w_2 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

   (*c*) $w_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $w_2 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

   (*d*) $w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

   (*e*) $w_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

5. Given $u = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, find the following graphically:

   (*a*) $2v$
   (*b*) $u - v$
   (*c*) $u - v$
   (*d*) $v - u$
   (*e*) $2u + 3v$
   (*f*) $4u - 2v$

6. Since the 3-space is spanned by the three unit vectors defined in (4.7), any other 3-vector should be expressible as a linear combination of e_1, e_2, and e_3. Show that the following 3-vectors can be so expressed:

   (*a*) $\begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 25 \\ -2 \\ 1 \end{bmatrix}$ (*c*) $\begin{bmatrix} -1 \\ 6 \\ 9 \end{bmatrix}$ (*d*) $\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$

7. In the three-dimensional Euclidean space, what is the distance between the following points?

   (*a*) (3, 2, 8) and (0, −1, 5) (*b*) (9, 0, 4) and (2, 0, −4)

8. The triangular inequality is written with the *weak* inequality sign $\le$, rather than the strict inequality sign $<$. Under what circumstances would the "=" part of the inequality apply?

9. Express the length of a radius vector v in the Euclidean n-space (i.e., the distance from the origin to point v) by using each of the following:

   (*a*) scalars (*b*) a scalar product (*c*) an inner product

4.4 Commutative, Associative, and Distributive Laws

In ordinary scalar algebra, the additive and multiplicative operations obey the commutative, associative, and distributive laws as follows:

Commutative law of addition:	$a + b = b + a$
Commutative law of multiplication:	$ab = ba$
Associative law of addition:	$(a + b) + c = a + (b + c)$
Associative law of multiplication:	$(ab)c = a(bc)$
Distributive law:	$a(b + c) = ab + ac$

These have been referred to during the discussion of the similarly named laws applicable to the union and intersection of sets. Most, but not all, of these laws also apply to matrix operations—the significant exception being the commutative law of multiplication.

Matrix Addition

Matrix addition is commutative as well as associative. This follows from the fact that matrix addition calls only for the addition of the corresponding elements of two matrices, and that the order in which each pair of corresponding elements is added is immaterial. In this context, incidentally, the subtraction operation $A - B$ can simply be regarded as the addition operation $A + (-B)$, and thus no separate discussion is necessary.

The commutative and associative laws can be stated as follows:

Commutative law $\quad A + B = B + A$

PROOF $\quad A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A$

Example 1 Given $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$, we find that

$$A + B = B + A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

Associative law $\quad (A + B) + C = A + (B + C)$

PROOF
$$\begin{aligned}(A + B) + C &= [a_{ij} + b_{ij}] + [c_{ij}] = [a_{ij} + b_{ij} + c_{ij}] \\ &= [a_{ij}] + [b_{ij} + c_{ij}] = A + (B + C)\end{aligned}$$

Example 2 Given $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, we find that

$$(v_1 + v_2) - v_3 = \begin{bmatrix} 12 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

which is equal to

$$v_1 + (v_2 - v_3) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

Applied to the linear combination of vectors $k_1v_1 + \cdots + k_nv_n$, the associative law permits us to select any pair of terms for addition (or subtraction) first, instead of having to follow the sequence in which the n terms are listed.

Matrix Multiplication

Matrix multiplication is *not* commutative, that is,

$$AB \neq BA$$

As explained previously, even when AB is defined, BA may not be; but even if both products are defined, the general rule is still $AB \neq BA$.

Example 3 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$; then

$$AB = \begin{bmatrix} 1(0)+2(6) & 1(-1)+2(7) \\ 3(0)+4(6) & 3(-1)+4(7) \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$$

but

$$BA = \begin{bmatrix} 0(1)-1(3) & 0(2)-1(4) \\ 6(1)+7(3) & 6(2)+7(4) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$$

Example 4 Let u' be 1×3 (a row vector); then the corresponding column vector u must be 3×1. The product $u'u$ will be 1×1, but the product uu' will be 3×3. Thus, obviously, $u'u \neq uu'$.

In view of the general rule $AB \neq BA$, the terms *premultiply* and *postmultiply* are often used to specify the order of multiplication. In the product AB, the matrix B is said to be *pre*multiplied by A, and A to be *post*multiplied by B.

There do exist interesting exceptions to the rule $AB \neq BA$, however. One such case is when A is a square matrix and B is an identity matrix. Another is when A is the inverse of B, that is, when $A = B^{-1}$. Both of these will be taken up again later. It should also be remarked here that the scalar multiplication of a matrix *does* obey the commutative law; thus, if k is a scalar, then

$$kA = Ak$$

Although it is not in general commutative, matrix multiplication *is* associative.

Associative law $\qquad (AB)C = A(BC) = ABC$

In forming the product ABC, the conformability condition must naturally be satisfied by each *adjacent* pair of matrices. If A is $m \times n$ and if C is $p \times q$, then conformability requires that B be $n \times p$:

$$\underset{(m\times n)}{A} \quad \underset{(n\times p)}{B} \quad \underset{(p\times q)}{C}$$

Note the dual appearance of n and p in the dimension indicators. If the conformability condition is met, the associative law states that any *adjacent* pair of matrices may be multiplied out first, provided that the product is duly inserted in the exact place of the original pair.

Example 5 If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$, then

$$x'Ax = x'(Ax) = [x_1 \quad x_2]\begin{bmatrix} a_{11}x_1 \\ a_{22}x_2 \end{bmatrix} = a_{11}x_1^2 + a_{22}x_2^2$$

Exactly the same result comes from

$$(x'A)x = [a_{11}x_1 \quad a_{22}x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + a_{22}x_2^2$$

In Example 5, the square matrix A has nonzero elements a_{11} and a_{22} in the principal diagonal, and zeros everywhere else. Such a matrix is called a *diagonal matrix*. When a diagonal matrix A appears in the product $x'Ax$, the resulting product gives a "weighted" sum of squares, the weights for the x_1^2 and the x_2^2 terms being supplied by the elements in the diagonal of A. This result is in contrast to the scalar product $x'x$, which yields a simple (unweighted) sum of squares.

Example 6

Let the economic ideal be defined as the national-income level Y^0 coupled with the inflation rate p^0. And suppose that we view any positive deviation of the actual income Y from Y^0 to be equally undesirable as a negative deviation of the same magnitude, and similarly for deviations of the actual inflation rate p from p^0. Then we may write a *social-loss function* such as

$$\Lambda = \alpha(Y - Y^0)^2 + \beta(p - p^0)^2$$

where α and β are the weights assigned to the two sources of social loss. If deviations of Y are considered to be the more serious type of loss, then α should exceed β. Note that the squaring of the deviations produces two effects. First, upon squaring, a positive deviation will receive the same loss value as a negative deviation of the same numerical magnitude. Second, squaring causes the larger deviations to show up much more significantly in the social-loss measure than minor deviations. Such a social-loss function can be expressed, if desired, by the matrix product

$$[Y - Y^0 \quad p - p^0]\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} Y - Y^0 \\ p - p^0 \end{bmatrix}$$

Matrix multiplication is also distributive.

Distributive law $A(B + C) = AB + AC$ [premultiplication by A]

$(B + C)A = BA + CA$ [postmultiplication by A]

In each case, the conformability conditions for addition as well as for multiplication must, of course, be observed.

EXERCISE 4.4

1. Given $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 7 \\ 8 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 4 \\ 1 & 9 \end{bmatrix}$, verify that
   (*a*) $(A + B) + C = A + (B + C)$
   (*b*) $(A + B) - C = A + (B - C)$
2. The subtraction of a matrix B may be considered as the addition of the matrix $(-1)B$. Does the commutative law of addition permit us to state that $A - B = B - A$? If not, how would you correct the statement?
3. Test the associative law of multiplication with the following matrices:

$$A = \begin{bmatrix} 5 & 3 \\ 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -8 & 0 & 7 \\ 1 & 3 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 7 & 1 \end{bmatrix}$$

4. Prove that for any two scalars g and k
   (*a*) $k(A + B) = kA + kB$
   (*b*) $(g + k)A = gA + kA$
   (*Note:* To *prove* a result, you cannot use specific examples.)
5. For (*a*) through (*d*) find $C = AB$.
   (*a*) $A = \begin{bmatrix} 12 & 14 \\ 20 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 9 \\ 0 & 2 \end{bmatrix}$
   (*b*) $A = \begin{bmatrix} 4 & 7 \\ 9 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 8 & 5 \\ 2 & 6 & 7 \end{bmatrix}$
   (*c*) $A = \begin{bmatrix} 7 & 11 \\ 2 & 9 \\ 10 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 12 & 4 & 5 \\ 3 & 6 & 1 \end{bmatrix}$
   (*d*) $A = \begin{bmatrix} 6 & 2 & 5 \\ 7 & 9 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 10 & 1 \\ 11 & 3 \\ 2 & 9 \end{bmatrix}$
   (*e*) Find (i) $C = AB$, and (ii) $D = BA$, if
   $$A = \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix} \qquad B = [3 \quad 6 \quad -2]$$
6. Prove that $(A + B)(C + D) = AC + AD + BC + BD$.
7. If the matrix A in Example 5 had all its four elements nonzero, would $x'Ax$ still give a weighted sum of squares? Would the associative law still apply?
8. Name some situations or contexts where the notion of a weighted or unweighted sum of squares may be relevant.

4.5 Identity Matrices and Null Matrices

Identity Matrices

We have referred earlier to the term *identity matrix*. Such a matrix is defined as a *square* (repeat: square) matrix with 1s in its principal diagonal and 0s everywhere else. It is denoted by the symbol I, or I_n, in which the subscript n serves to indicate its row (as well as column) dimension. Thus,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But both of these can also be denoted by I.

The importance of this special type of matrix lies in the fact that it plays a role similar to that of the number 1 in scalar algebra. For any number a, we have $1(a) = a(1) = a$. Similarly, for any matrix A, we have

$$IA = AI = A \tag{4.8}$$

Example 1

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix}$, then

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A$$

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A$$

Because A is 2×3, premultiplication and postmultiplication of A by I would call for identity matrices of different dimensions, namely, I_2 and I_3, respectively. But in case A is $n \times n$, then the same identity matrix I_n can be used, so that (4.8) becomes $I_n A = A I_n$, thus illustrating an exception to the rule that matrix multiplication is not commutative.

The special nature of identity matrices makes it possible, during the multiplication process, to *insert* or *delete* an identity matrix without affecting the matrix product. This follows directly from (4.8). Recalling the associative law, we have, for instance,

$$\underset{(m\times n)}{A}\ \underset{(n\times n)}{I}\ \underset{(n\times p)}{B} = (AI)B = \underset{(m\times n)}{A}\ \underset{(n\times p)}{B}$$

which shows that the presence or absence of I does not affect the product. Observe that dimension conformability is preserved whether or not I appears in the product.

An interesting case of (4.8) occurs when $A = I_n$, for then we have

$$AI_n = (I_n)^2 = I_n$$

which states that an identity matrix squared is equal to itself. A generalization of this result is that

$$(I_n)^k = I_n \qquad (k = 1, 2, \ldots)$$

An identity matrix remains unchanged when it is multiplied by itself any number of times. Any matrix with such a property (namely, $AA = A$) is referred to as an *idempotent matrix*.

Null Matrices

Just as an identity matrix I plays the role of the number 1, a *null matrix*—or *zero matrix*—denoted by 0, plays the role of the number 0. A null matrix is simply a matrix whose elements are all zero. Unlike I, the zero matrix is not restricted to being square. Thus it is possible to write

$$\underset{(2\times 2)}{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \underset{(2\times 3)}{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so forth. A square null matrix is idempotent, but a nonsquare one is not. (Why?)

As the counterpart of the number 0, null matrices obey the following rules of operation (subject to conformability) with regard to addition and multiplication:

$$\underset{(m\times n)}{A} + \underset{(m\times n)}{0} = \underset{(m\times n)}{0} + \underset{(m\times n)}{A} = \underset{(m\times n)}{A}$$

$$\underset{(m\times n)}{A}\ \underset{(n\times p)}{0} = \underset{(m\times p)}{0} \qquad \text{and} \qquad \underset{(q\times m)}{0}\ \underset{(m\times n)}{A} = \underset{(q\times n)}{0}$$

Note that, in multiplication, the null matrix to the left of the equals sign and the one to the right may be of different dimensions.

Example 2

$$A + 0 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$$

Example 3

$$\underset{(2\times3)}{A}\ \underset{(3\times1)}{0} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underset{(2\times1)}{0}$$

To the left, the null matrix is a 3×1 null vector; to the right, it is a 2×1 null vector.

Idiosyncrasies of Matrix Algebra

Despite the apparent similarities between matrix algebra and scalar algebra, the case of matrices does display certain idiosyncrasies that serve to warn us not to "borrow" from scalar algebra too unquestioningly. We have already seen that, in general, $AB \neq BA$ in matrix algebra. Let us look at two more such idiosyncrasies of matrix algebra.

For one thing, in the case of scalars, the equation $ab = 0$ always implies that either a or b is zero, but this is not so in matrix multiplication. Thus, we have

$$AB = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

although neither A nor B is itself a zero matrix.

As another illustration, for scalars, the equation $cd = ce$ (with $c \neq 0$) implies that $d = e$. The same does not hold for matrices. Thus, given

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

we find that

$$CD = CE = \begin{bmatrix} 5 & 8 \\ 15 & 24 \end{bmatrix}$$

even though $D \neq E$.

These strange results actually pertain only to the special class of matrices known as *singular matrices*, of which the matrices A, B, and C are examples. (Roughly, these matrices contain a row which is a multiple of another row.) Nevertheless, such examples do reveal the pitfalls of unwarranted extension of algebraic theorems to matrix operations.

EXERCISE 4.5

Given $A = \begin{bmatrix} -1 & 5 & 7 \\ 0 & -2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 9 \\ 6 \\ 0 \end{bmatrix}$, and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:

1. Calculate: (*a*) AI (*b*) IA (*c*) Ix (*d*) $x'I$

   Indicate the dimension of the identity matrix used in each case.

2. Calculate: (*a*) Ab (*b*) AIb (*c*) $x'IA$ (*d*) $x'A$

   Does the insertion of I in (*b*) affect the result in (*a*)? Does the deletion of I in (*d*) affect the result in (*c*)?
3. What is the dimension of the null matrix resulting from each of the following?

   (*a*) Premultiply A by a 5×2 null matrix.

   (*b*) Postmultiply A by a 3×6 null matrix.

   (*c*) Premultiply b by a 2×3 null matrix.

   (*d*) Postmultiply x by a 1×5 null matrix.
4. Show that the diagonal matrix

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

   can be idempotent only if each diagonal element is either 1 or 0. How many different numerical idempotent diagonal matrices of dimension $n \times n$ can be constructed altogether from such a matrix?

4.6 Transposes and Inverses

When the rows and columns of a matrix A are interchanged—so that its first row becomes the first column, and vice versa—we obtain the *transpose* of A, which is denoted by A' or A^T. The prime symbol is by no means new to us; it was used earlier to distinguish a row vector from a column vector. In the newly introduced terminology, a row vector x' constitutes the transpose of the column vector x. The superscript T in the alternative symbol is obviously shorthand for the word transpose.

Example 1

Given $\underset{(2\times 3)}{A} = \begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$ and $\underset{(2\times 2)}{B} = \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$, we can interchange the rows and columns and write

$$\underset{(3\times 2)}{A'} = \begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix} \qquad \text{and} \qquad \underset{(2\times 2)}{B'} = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$$

By definition, if a matrix A is $m \times n$, then its transpose A' must be $n \times m$. An $n \times n$ square matrix, however, possesses a transpose with the same dimension.

Example 2

If $C = \begin{bmatrix} 9 & -1 \\ 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$, then

$$C' = \begin{bmatrix} 9 & 2 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad D' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$$

Here, the dimension of each transpose is identical with that of the original matrix.

In D', we also note the remarkable result that D' inherits not only the dimension of D but also the original array of elements! The fact that $D' = D$ is the result of the symmetry of the elements with reference to the principal diagonal. Considering the principal diagonal in D as a mirror, the elements located to its northeast are exact images of the elements to its southwest; hence the first row reads identically with the first column, and so forth. The matrix D exemplifies the special class of square matrices known as *symmetric matrices*. Another example of such a matrix is the identity matrix I, which, as a symmetric matrix, has the transpose $I' = I$.

Properties of Transposes

The following properties characterize transposes:

$$(A')' = A \tag{4.9}$$

$$(A + B)' = A' + B' \tag{4.10}$$

$$(AB)' = B'A' \tag{4.11}$$

The first says that the transpose of the transpose is the original matrix—a rather self-evident conclusion.

The second property may be verbally stated thus: The transpose of a sum is the sum of the transposes.

Example 3 If $A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$, then

$$(A + B)' = \begin{bmatrix} 6 & 1 \\ 16 & 1 \end{bmatrix}' = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}$$

and

$$A' + B' = \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}$$

The third property is that the transpose of a product is the product of the transposes *in reverse order.* To appreciate the necessity for the reversed order, let us examine the dimension conformability of the two products on the two sides of (4.11). If we let A be $m \times n$ and B be $n \times p$, then AB will be $m \times p$, and $(AB)'$ will be $p \times m$. For equality to hold, it is necessary that the right-hand expression $B'A'$ be of the identical dimension. Since B' is $p \times n$ and A' is $n \times m$, the product $B'A'$ is indeed $p \times m$, as required. The dimension of $B'A'$ thus works out. Note that, on the other hand, the product $A'B'$ is not even defined unless $m = p$.

Example 4 Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$, we have

$$(AB)' = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}' = \begin{bmatrix} 12 & 24 \\ 13 & 25 \end{bmatrix}$$

and

$$B'A' = \begin{bmatrix} 0 & 6 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 24 \\ 13 & 25 \end{bmatrix}$$

This verifies the property.

Inverses and Their Properties

For a given matrix A, the transpose A' is always derivable. On the other hand, its *inverse* matrix—another type of "derived" matrix—may or may not exist. The inverse of matrix A, denoted by A^{-1}, is defined only if A is a square matrix, in which case the inverse is the matrix that satisfies the condition

$$AA^{-1} = A^{-1}A = I \tag{4.12}$$

That is, whether A is pre- or postmultiplied by A^{-1}, the product will be the same identity matrix. This is another exception to the rule that matrix multiplication is not commutative.

The following points are worth noting:

1. Not every square matrix has an inverse—squareness is a *necessary* condition, but *not* a *sufficient* condition, for the existence of an inverse. If a square matrix A has an inverse, A is said to be *nonsingular;* if A possesses no inverse, it is called a *singular* matrix.
2. If A^{-1} does exist, then the matrix A can be regarded as the inverse of A^{-1}, just as A^{-1} is the inverse of A. In short, A and A^{-1} are inverses of each other.
3. If A is $n \times n$, then A^{-1} must also be $n \times n$; otherwise it cannot be conformable for *both* pre- and postmultiplication. The identity matrix produced by the multiplication will also be $n \times n$.
4. If an inverse exists, then it is unique. To prove its uniqueness, let us suppose that B has been found to be an inverse for A, so that

   $$AB = BA = I$$

   Now assume that there is another matrix C such that $AC = CA = I$. By premultiplying both sides of $AB = I$ by C, we find that

   $$CAB = CI(= C) \qquad \text{[by (4.8)]}$$

   Since $CA = I$ by assumption, the preceding equation is reducible to

   $$IB = C \qquad \text{or} \qquad B = C$$

   That is, B and C must be one and the same inverse matrix. For this reason, we can speak of *the* (as against *an*) inverse of A.
5. The two parts of condition (4.12)—namely, $AA^{-1} = I$ and $A^{-1}A = I$—actually imply each other, so that satisfying either equation is sufficient to establish the inverse relationship between A and A^{-1}. To prove this, we should show that if $AA^{-1} = I$, and if there is a matrix B such that $BA = I$, then $B = A^{-1}$ (so that $BA = I$ must in effect be the equation $A^{-1}A = I$). Let us postmultiply both sides of the given equation $BA = I$ by A^{-1}; then

   $$\begin{aligned} (BA)A^{-1} &= IA^{-1} \\ B(AA^{-1}) &= IA^{-1} \qquad \text{[associative law]} \\ BI &= IA^{-1} \qquad [AA^{-1} = I \text{ by assumption}] \end{aligned}$$

   Therefore, as required,

   $$B = A^{-1} \qquad \text{[by (4.8)]}$$

Analogously, it can be demonstrated that, if $A^{-1}A = I$, then the only matrix C which yields $CA^{-1} = I$ is $C = A$.

Example 5

Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$; then, since the scalar multiplier ($\frac{1}{6}$) in B can be moved to the rear (commutative law), we can write

$$AB = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This establishes B as the inverse of A, and vice versa. The reverse multiplication, as expected, also yields the same identity matrix:

$$BA = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The following three properties of inverse matrices are of interest. If A and B are nonsingular matrices with dimension $n \times n$, then

$$(A^{-1})^{-1} = A \tag{4.13}$$

$$(AB)^{-1} = B^{-1}A^{-1} \tag{4.14}$$

$$(A')^{-1} = (A^{-1})' \tag{4.15}$$

The first says that the inverse of an inverse is the original matrix. The second states that the inverse of a product is the product of the inverses *in reverse order*. And the last one means that the inverse of the transpose is the transpose of the inverse. Note that in these statements the existence of the inverses and the satisfaction of the conformability condition are presupposed.

The validity of (4.13) is fairly obvious, but let us prove (4.14) and (4.15). Given the product AB, let us find its inverse—call it C. From (4.12) we know that $CAB = I$; thus, postmultiplication of both sides by $B^{-1}A^{-1}$ will yield

$$CABB^{-1}A^{-1} = IB^{-1}A^{-1}\,(= B^{-1}A^{-1}) \tag{4.16}$$

But the left side is reducible to

$$\begin{aligned} CA(BB^{-1})A^{-1} &= CAIA^{-1} && \text{[by (4.12)]} \\ &= CAA^{-1} = CI = C && \text{[by (4.12) and (4.8)]} \end{aligned}$$

Substitution of this into (4.16) then tells us that $C = B^{-1}A^{-1}$ or, in other words, that the inverse of AB is equal to $B^{-1}A^{-1}$, as alleged. In this proof, the equation $AA^{-1} = A^{-1}A = I$ was utilized twice. Note that the application of this equation is permissible if and only if a matrix and its inverse are strictly adjacent to each other in a product. We may write $AA^{-1}B = IB = B$, but *never* $ABA^{-1} = B$.

The proof of (4.15) is as follows. Given A', let us find its inverse—call it D. By definition, we then have $DA' = I$. But we know that

$$(AA^{-1})' = I' = I$$

produces the same identity matrix. Thus we may write

$$DA' = (AA^{-1})'$$
$$= (A^{-1})'A' \qquad \text{[by (4.11)]}$$

Postmultiplying both sides by $(A')^{-1}$, we obtain

$$DA'(A')^{-1} = (A^{-1})'A'(A')^{-1}$$

or

$$D = (A^{-1})' \qquad \text{[by (4.12)]}$$

Thus, the inverse of A' is equal to $(A^{-1})'$, as alleged.

In the proofs just presented, mathematical operations were performed on whole blocks of numbers. If those blocks of numbers had not been treated as mathematical entities (matrices), the same operations would have been much more lengthy and involved. The beauty of matrix algebra lies precisely in its simplification of such operations.

Inverse Matrix and Solution of Linear-Equation System

The application of the concept of inverse matrix to the solution of a simultaneous-equation system is immediate and direct. Referring to the equation system in (4.3), we pointed out earlier that it can be written in matrix notation as

$$\underset{(3\times3)}{A}\ \underset{(3\times1)}{x} = \underset{(3\times1)}{d} \tag{4.17}$$

where A, x, and d are as defined in (4.4). Now if the inverse matrix A^{-1} exists, the premultiplication of both sides of the equation (4.17) by A^{-1} will yield

$$A^{-1}Ax = A^{-1}d$$

or

$$\underset{(3\times1)}{x} = \underset{(3\times3)}{A^{-1}}\ \underset{(3\times1)}{d} \tag{4.18}$$

The left side of (4.18) is a column vector of variables, whereas the right-hand product is a column vector of certain known numbers. Thus, by definition of the equality of matrices or vectors, (4.18) shows the set of values of the variables that satisfy the equation system, i.e., the solution values. Furthermore, since A^{-1} is unique if it exists, $A^{-1}d$ must be a unique vector of solution values. We shall therefore write the x vector in (4.18) as x^*, to indicate its status as a (unique) solution.

Methods of testing the existence of the inverse and of its calculation will be discussed in Chap. 5. It may be stated here, however, that the inverse of the matrix A in (4.4) is

$$A^{-1} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}$$

Thus (4.18) will turn out to be

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}\begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

which gives the solution: $x_1^* = 2$, $x_2^* = 3$, and $x_3^* = 1$.

The upshot is that, as one way of finding the solution of a linear-equation system $Ax = d$, where the coefficient matrix A is nonsingular, is to first find the inverse A^{-1}, and then postmultiply A^{-1} by the constant vector d. The product $A^{-1}d$ will then give the solution values of the variables.

Example 6

As shown in Example 11 of Sec. 4.2, the simple national-income model

$$Y = C + I_0 + G_0$$
$$C = a - bY$$

can be written in matrix notation as $Ax = d$, where

$$A = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \qquad x = \begin{bmatrix} Y \\ C \end{bmatrix} \qquad \text{and} \qquad d = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

The inverse of matrix A is (see explanation in Sec. 5.6)

$$A^{-1} = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$$

Thus the solution of the model is $x^* = A^{-1}d$, or

$$\begin{bmatrix} Y^* \\ C^* \end{bmatrix} = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}\begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix} = \frac{1}{1-b}\begin{bmatrix} I_0 + G_0 + a \\ b(I_0 + G_0) + a \end{bmatrix}$$

EXERCISE 4.6

1. Given $A = \begin{bmatrix} 0 & 4 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -8 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & 9 \\ 6 & 1 & 1 \end{bmatrix}$, find A', B', and C'.
2. Use the matrices given in Prob. 1 to verify that
   (*a*) $(A + B)' = A' + B'$ (*b*) $(AC)' = C'A'$
3. Generalize the result (4.11) to the case of a product of three matrices by proving that, for any conformable matrices A, B, and C, the equation $(ABC)' = C'B'A'$ holds.
4. Given the following four matrices, test whether any one of them is the inverse of another:

$$D = \begin{bmatrix} 1 & 12 \\ 0 & 3 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 1 \\ 6 & 8 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & -4 \\ 0 & \frac{1}{3} \end{bmatrix} \qquad G = \begin{bmatrix} 4 & -\frac{1}{2} \\ -3 & \frac{1}{2} \end{bmatrix}$$

5. Generalize the result (4.14) by proving that, for any conformable nonsingular matrices A, B, and C, the equation $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ holds.
6. Let $A = I - X(X'X)^{-1}X'$.
   (*a*) Must A be square? Must $(X'X)$ be square? Must X be square?
   (*b*) Show that matrix A is idempotent. [*Note:* If X' and X are not square, it is inappropriate to apply (4.14).]

4.7 Finite Markov Chains

A common application of matrix algebra is found in what is known as Markov processes or Markov chains. *Markov processes* are used to measure or estimate movements over time. This involves the use of a Markov transition matrix, where each value in the transition