for any value of x, the given function is strictly increasing, and an inverse function exists. To solve the given equation for x may not be such an easy task, but the derivative of the inverse function can nevertheless be found quickly by use of the inverse-function rule:

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{5x^4 + 1}$$

The inverse-function rule is, strictly speaking, applicable only when the function involved is a one-to-one mapping. In fact, however, we do have some leeway. For instance, when dealing with a U-shaped curve (not strictly monotonic), we may consider the downward- and the upward-sloping segments of the curve as representing two *separate* functions, each with a restricted domain, and each being strictly monotonic in the restricted domain. To each of these, the inverse-function rule can then again be applied.

EXERCISE 7.3

1. Given $y = u^3 + 2u$, where $u = 5 - x^2$, find dy/dx by the chain rule.
2. Given $w = ay^2$ and $y = bx^2 + cx$, find dw/dx by the chain rule.
3. Use the chain rule to find dy/dx for the following:
   (*a*) $y = (3x^2 - 13)^3$ (*b*) $y = (7x^3 - 5)^9$ (*c*) $y = (ax + b)^5$
4. Given $y = (16x + 3)^{-2}$, use the chain rule to find dy/dx. Then rewrite the function as $y = 1/(16x + 3)^2$ and find dy/dx by the quotient rule. Are the answers identical?
5. Given $y = 7x + 21$, find its inverse function. Then find dy/dx and dx/dy, and verify the inverse-function rule. Also verify that the graphs of the two functions bear a mirror-image relationship to each other.
6. Are the following functions strictly monotonic?
   (*a*) $y = -x^6 + 5 \quad (x > 0)$
   (*b*) $y = 4x^5 + x^3 + 3x$
   For each strictly monotonic function, find dx/dy by the inverse-function rule.

7.4 Partial Differentiation

Hitherto, we have considered only the derivatives of functions of a single independent variable. In comparative-static analysis, however, we are likely to encounter the situation in which several parameters appear in a model, so that the equilibrium value of each endogenous variable may be a function of more than one parameter. Therefore, as a final preparation for the application of the concept of derivative to comparative statics, we must learn how to find the derivative of a function of more than one variable.

Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, \ldots, x_n) \tag{7.12}$$

where the variables x_i $(i = 1, 2, \ldots, n)$ are all *independent* of one another, so that each can vary by itself without affecting the others. If the variable x_1 undergoes a change Δx_1 while

$x_2, \ldots, x_n$ all remain fixed, there will be a corresponding change in y, namely, Δy. The difference quotient in this case can be expressed as

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_1} \tag{7.13}$$

If we take the limit of $\Delta y/\Delta x_1$ as $\Delta x_1 \to 0$, that limit will constitute a derivative. We call it the *partial derivative* of y with respect to x_1, to indicate that all the other independent variables in the function are held constant when taking this particular derivative. Similar partial derivatives can be defined for infinitesimal changes in the other independent variables. The process of taking partial derivatives is called *partial differentiation.*

Partial derivatives are assigned distinctive symbols. In lieu of the letter d (as in dy/dx), we employ the symbol ∂, which is a variant of the Greek δ (lowercase delta). Thus we shall now write $\partial y/\partial x_i$, which is read: "the partial derivative of y with respect to x_i." The partial-derivative symbol sometimes is also written as $\frac{\partial}{\partial x_i} y$; in that case, its $\partial/\partial x_i$ part can be regarded as an operator symbol instructing us to take the partial derivative of (some function) with respect to the variable x_i. Since the function involved here is denoted in (7.12) by f, it is also permissible to write $\partial f/\partial x_i$.

Is there also a partial-derivative counterpart for the symbol $f'(x)$ that we used before? The answer is yes. Instead of f', however, we now use f_1, f_2, etc., where the subscript indicates which independent variable (alone) is being allowed to vary. If the function in (7.12) happens to be written in terms of unsubscripted variables, such as $y = f(u, v, w)$, then the partial derivatives may be denoted by f_u, f_v, and f_w rather than f_1, f_2, and f_3.

In line with these notations, and on the basis of (7.12) and (7.13), we can now define

$$f_1 \equiv \frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

as the first in the set of n partial derivatives of the function f.

Techniques of Partial Differentiation

Partial differentiation differs from the previously discussed differentiation primarily in that we must hold $(n-1)$ independent variables *constant* while allowing *one* variable to vary. Inasmuch as we have learned how to handle *constants* in differentiation, the actual differentiation should pose little problem.

Example 1

Given $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$, find the partial derivatives. When finding $\partial y/\partial x_1$, (or f_1), we must bear in mind that x_2 is to be treated as a constant during differentiation. As such, x_2 will drop out in the process if it is an *additive* constant (such as the term $4x_2^2$) but will be retained if it is a *multiplicative* constant (such as in the term x_1x_2). Thus we have

$$\frac{\partial y}{\partial x_1} \equiv f_1 = 6x_1 + x_2$$

Similarly, by treating x_1 as a constant, we find that

$$\frac{\partial y}{\partial x_2} \equiv f_2 = x_1 + 8x_2$$

Note that, like the primitive function f, both partial derivatives are themselves functions of the variables x_1 and x_2. That is, we may write them as two derived functions

$$f_1 = f_1(x_1, x_2) \qquad \text{and} \qquad f_2 = f_2(x_1, x_2)$$

For the point $(x_1, x_2) = (1, 3)$ in the domain of the function f, for example, the partial derivatives will take the following specific values:

$$f_1(1, 3) = 6(1) + 3 = 9 \qquad \text{and} \qquad f_2(1, 3) = 1 + 8(3) = 25$$

Example 2 Given $y = f(u, v) = (u + 4)(3u + 2v)$, the partial derivatives can be found by use of the product rule. By holding v constant, we have

$$f_u = (u + 4)(3) + 1(3u + 2v) = 2(3u + v + 6)$$

Similarly, by holding u constant, we find that

$$f_v = (u + 4)(2) + 0(3u + 2v) = 2(u + 4)$$

When $u = 2$ and $v = 1$, these derivatives will take the following values:

$$f_u(2, 1) = 2(13) = 26 \qquad \text{and} \qquad f_v(2, 1) = 2(6) = 12$$

Example 3 Given $y = (3u - 2v)/(u^2 + 3v)$, the partial derivatives can be found by use of the quotient rule:

$$\frac{\partial y}{\partial u} = \frac{3(u^2 + 3v) - 2u(3u - 2v)}{(u^2 + 3v)^2} = \frac{-3u^2 + 4uv + 9v}{(u^2 + 3v)^2}$$

$$\frac{\partial y}{\partial v} = \frac{-2(u^2 + 3v) - 3(3u - 2v)}{(u^2 + 3v)^2} = \frac{-u(2u + 9)}{(u^2 + 3v)^2}$$

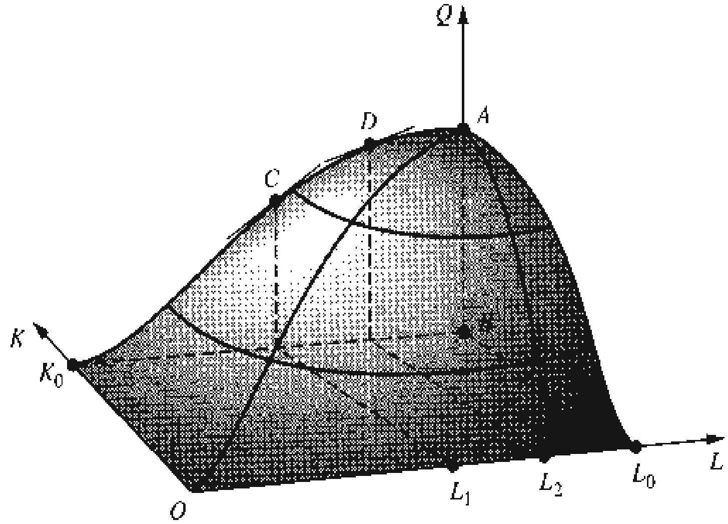
Geometric Interpretation of Partial Derivatives

As a special type of derivative, a partial derivative is a measure of the instantaneous rates of change of some variable, and in that capacity it again has a geometric counterpart in the slope of a particular curve.

Let us consider a production function $Q = Q(K, L)$, where Q, K, and L denote output, capital input, and labor input, respectively. This function is a particular two-variable version of (7.12), with $n = 2$. We can therefore define two partial derivatives $\partial Q/\partial K$ (or Q_K) and $\partial Q/\partial L$ (or Q_L). The partial derivative Q_K relates to the rates of change of output with respect to infinitesimal changes in capital, while labor input is held constant. Thus Q_K symbolizes the marginal-physical-product-of-capital (MPP_K) function. Similarly, the partial derivative Q_L is the mathematical representation of the MPP_L function.

Geometrically, the production function $Q = Q(K, L)$ can be depicted by a *production surface* in a 3-space, such as is shown in Fig. 7.4. The variable Q is plotted vertically, so that for any point (K, L) in the base plane (KL plane), the height of the surface will indicate the output Q. The domain of the function should consist of the entire nonnegative quadrant of the base plane, but for our purposes it is sufficient to consider a subset of it, the

FIGURE 7.4



rectangle OK_0BL_0. As a consequence, only a small portion of the production surface is shown in the figure.

Let us now hold capital fixed at the level K_0 and consider only variations in the input L. By setting $K = K_0$, all points in our (curtailed) domain become irrelevant except those on the line segment K_0B. By the same token, only the curve K_0CDA (a cross section of the production surface) is germane to the present discussion. This curve represents a total-physical-product-of-labor (TPP_L) curve for a fixed amount of capital $K = K_0$; thus we may read from its slope the rate of change of Q with respect to changes in L while K is held constant. It is clear, therefore, that the slope of a curve such as K_0CDA represents the geometric counterpart of the partial derivative Q_L. Once again, we note that the slope of a total (TPP_L) curve is its corresponding marginal ($\text{MPP}_L \equiv Q_L$) curve.

As mentioned earlier, a partial derivative is a function of all the independent variables of the primitive function. That Q_L is a function of L is immediately obvious from the K_0CDA curve itself. When $L = L_1$, the value of Q_L is equal to the slope of the curve at point C; but when $L = L_2$, the relevant slope is the one at point D. Why is Q_L also a function of K? The answer is that K can be fixed at various levels, and for each fixed level of K, there results a different TPP_L curve (a different cross section of the production surface), with inevitable repercussions on the derivative Q_L. Hence Q_L is also a function of K.

An analogous interpretation can be given to the partial derivative Q_K. If the labor input is held constant instead of K (say, at the level of L_0), the line segment L_0B will be the relevant subset of the domain, and the curve L_0A will indicate the relevant subset of the production surface. The partial derivative Q_K can then be interpreted as the slope of the curve L_0A—bearing in mind that the K axis extends from southeast to northwest in Fig. 7.4. It should be noted that Q_K is again a function of both the variables L and K.

Gradient Vector

All the partial derivatives of a function $y = f(x_1, x_2, \ldots, x_n)$ can be collected under a single mathematical entity called the *gradient vector*, or simply the *gradient*, of function f:

$$\text{grad } f(x_1, x_2, \ldots, x_n) = (f_1, f_2, \ldots, f_n)$$

where $f_i \equiv \partial y/\partial x_i$. Note that we are using parentheses rather than brackets here in writing the vector. Alternatively, the gradient can be denoted by $\nabla f(x_1, x_2, \ldots, x_n)$, where ∇ (read: "del") is the inverted version of the Greek letter Δ.

Since the function f has n arguments, there are altogether n partial derivatives; hence, grad f is an n-vector. When these derivatives are evaluated at a specific point $(x_{10}, x_{20}, \ldots, x_{n0})$ in the domain, we get grad $f(x_{10}, x_{20}, \ldots, x_{n0})$, a vector of specific derivative *values*.

Example 4 The gradient vector of the production function $Q = Q(K, L)$ is

$$\nabla Q = \nabla Q(K, L) = (Q_K, Q_L)$$

EXERCISE 7.4

1. Find $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for each of the following functions:
   (*a*) $y = 2x_1^3 - 11x_1^2x_2 + 3x_2^2$
   (*b*) $y = 7x_1 + 6x_1x_2^2 - 9x_2^3$
   (*c*) $y = (2x_1 + 3)(x_2 - 2)$
   (*d*) $y = (5x_1 + 3)/(x_2 - 2)$
2. Find f_x and f_y from the following:
   (*a*) $f(x, y) = x^2 + 5xy - y^3$
   (*b*) $f(x, y) = (x^2 - 3y)(x - 2)$
   (*c*) $f(x, y) = \dfrac{2x - 3y}{x + y}$
   (*d*) $f(x, y) = \dfrac{x^2 - 1}{xy}$
3. From the answers to Prob. 2, find $f_x(1, 2)$—the value of the partial derivative f_x when $x = 1$ and $y = 2$—for each function.
4. Given the production function $Q = 96K^{0.3}L^{0.7}$, find the MPP_K and MPP_L functions. Is MPP_K a function of K alone, or of both K and L? What about MPP_L?
5. If the utility function of an individual takes the form

   $$U = U(x_1, x_2) = (x_1 + 2)^2(x_2 + 3)^3$$

   where U is total utility, and x_1 and x_2 are the quantities of two commodities consumed:
   (*a*) Find the marginal-utility function of each of the two commodities.
   (*b*) Find the value of the marginal utility of the first commodity when 3 units of each commodity are consumed.
6. The total money supply M has two components: bank deposits D and cash holdings C, which we assume to bear a constant ratio $C/D = c$, $0 < c < 1$. The high-powered money H is defined as the sum of cash holdings held by the public and the reserves held by the banks. Bank reserves are a fraction of bank deposits, determined by the reserve ratio r, $0 < r < 1$.
   (*a*) Express the money supply M as a function of high-powered money H.
   (*b*) Would an increase in the reserve ratio r raise or lower the money supply?
   (*c*) How would an increase in the cash-deposit ratio c affect the money supply?
7. Write the gradients of the following functions:
   (*a*) $f(x, y, z) = x^2 + y^3 + z^4$
   (*b*) $f(x, y, z) = xyz$

7.5 Applications to Comparative-Static Analysis

Equipped with the knowledge of the various rules of differentiation, we can at last tackle the problem posed in comparative-static analysis: namely, how the equilibrium value of an endogenous variable will change when there is a change in any of the exogenous variables or parameters.

Market Model

First let us consider again the simple one-commodity market model of (3.1). That model can be written in the form of two equations:

$$Q = a - bP \qquad (a, b > 0) \qquad \text{[demand]}$$
$$Q = -c + dP \qquad (c, d > 0) \qquad \text{[supply]}$$

with solutions

$$P^* = \frac{a + c}{b + d} \tag{7.14}$$

$$Q^* = \frac{ad - bc}{b + d} \tag{7.15}$$

These solutions will be referred to as being in the *reduced form:* The two endogenous variables have been reduced to explicit expressions of the four mutually independent parameters a, b, c, and d.

To find how an infinitesimal change in one of the parameters will affect the value of P^*, one has only to differentiate (7.14) partially with respect to each of the parameters. If the *sign* of a partial derivative, say, $\partial P^*/\partial a$, can be determined from the given information about the parameters, we shall know the direction in which P^* will move when the parameter a changes; this constitutes a qualitative conclusion. If the magnitude of $\partial P^*/\partial a$ can be ascertained, it will constitute a quantitative conclusion.

Similarly, we can draw qualitative or quantitative conclusions from the partial derivatives of Q^* with respect to each parameter, such as $\partial Q^*/\partial a$. To avoid misunderstanding, however, a clear distinction should be made between the two derivatives $\partial Q^*/\partial a$ and $\partial Q/\partial a$. The latter derivative is a concept appropriate to the demand function taken alone, and without regard to the supply function. The derivative $\partial Q^*/\partial a$ pertains, on the other hand, to the equilibrium quantity in (7.15) which, being in the nature of a solution of the model, takes into account the interaction of demand and supply together. To emphasize this distinction, we shall refer to the partial derivatives of P^* and Q^* with respect to the parameters as *comparative-static derivatives.* The possibility of confusion between $\partial Q^*/\partial a$ and $\partial Q/\partial a$ is precisely the reason why we have chosen to use the asterisk notation, as in Q^* to denote the equilibrium value.

Concentrating on P^* for the time being, we can get the following four partial derivatives from (7.14):

$$\frac{\partial P^*}{\partial a} = \frac{1}{b + d} \qquad \left[\text{parameter } a \text{ has the coefficient } \frac{1}{b + d}\right]$$

$$\frac{\partial P^*}{\partial b} = \frac{0(b + d) - 1(a + c)}{(b + d)^2} = \frac{-(a + c)}{(b + d)^2} \qquad \text{[quotient rule]}$$

$$\frac{\partial P^*}{\partial c} = \frac{1}{b+d} \left(= \frac{\partial P^*}{\partial a}\right)$$

$$\frac{\partial P^*}{\partial d} = \frac{0(b+d) - 1(a+c)}{(b+d)^2} = \frac{-(a+c)}{(b+d)^2} \left(= \frac{\partial P^*}{\partial b}\right)$$

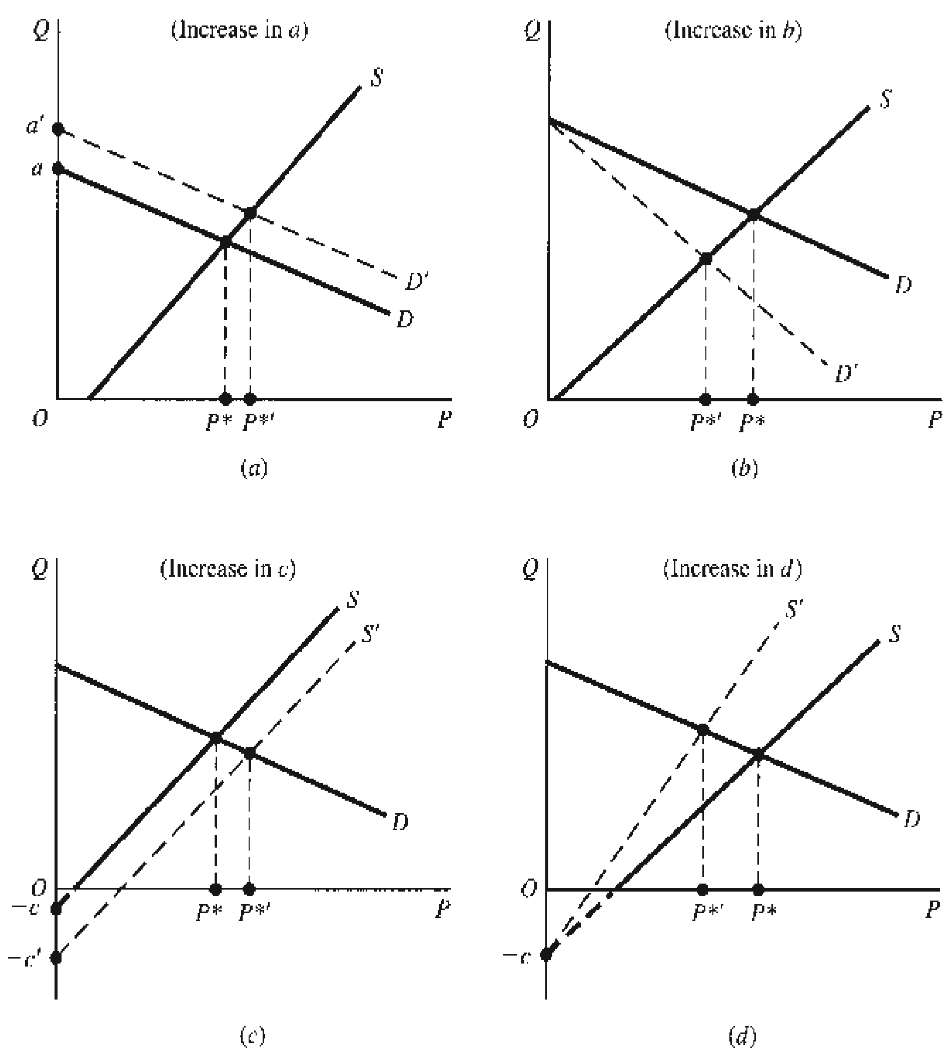
Since all the parameters are restricted to being positive in the present model, we can conclude that

$$\frac{\partial P^*}{\partial a} = \frac{\partial P^*}{\partial c} > 0 \qquad \text{and} \qquad \frac{\partial P^*}{\partial b} = \frac{\partial P^*}{\partial d} < 0 \tag{7.16}$$

For a fuller appreciation of the results in (7.16), let us look at Fig. 7.5, where each diagram shows a change in *one* of the parameters. As before, we are plotting Q (rather than P) on the vertical axis.

Figure 7.5a pictures an increase in the parameter a (to a'). This means a higher vertical intercept for the demand curve, and inasmuch as the parameter b (the slope parameter) is unchanged, the increase in a results in a parallel upward shift of the demand curve from D

FIGURE 7.5



to D'. The intersection of D' and the supply curve S determines an equilibrium price $P^{*\prime}$, which is greater than the old equilibrium price P^*. This corroborates the result that $\partial P^*/\partial a > 0$, although for the sake of exposition we have shown in Fig. 7.5*a* a much larger change in the parameter a than what the concept of derivative implies.

The situation in Fig. 7.5*c* has a similar interpretation; but since the increase takes place in the parameter c, the result is a parallel shift of the supply curve instead. Note that this shift is downward because the supply curve has a vertical intercept of $-c$; thus an increase in c would mean a change in the intercept, say, from -2 to -4. The graphical comparative-static result, that $P^{*\prime}$ exceeds P^*, again conforms to what the positive sign of the derivative $\partial P^*/\partial c$ would lead us to expect.

Figures 7.5*b* and 7.5*d* illustrate the effects of changes in the slope parameters b and d of the two functions in the model. An increase in b means that the slope of the demand curve will assume a larger numerical (absolute) value; i.e., it will become steeper. In accordance with the result $\partial P^*/\partial b < 0$, we find a decrease in P^* in this diagram. The increase in d that makes the supply curve steeper also results in a decrease in the equilibrium price. This is, of course, again in line with the negative sign of the comparative-static derivative $\partial P^*/\partial d$.

Thus far, all the results in (7.16) seem to have been obtainable graphically. If so, why should we bother to use differentiation at all? The answer is that the differentiation approach has at least two major advantages. First, the graphical technique is subject to a dimensional restriction, but differentiation is not. Even when the number of endogenous variables and parameters is such that the equilibrium state cannot be shown graphically, we can nevertheless apply the differentiation techniques to the problem. Second, the differentiation method can yield results that are on a higher level of generality. The results in (7.16) will remain valid, regardless of the specific values that the parameters a, b, c, and d take, as long as they satisfy the sign restrictions. So the comparative-static conclusions of this model are, in effect, applicable to an infinite number of combinations of (linear) demand and supply functions. In contrast, the graphical approach deals only with some specific members of the family of demand and supply curves, and the analytical result derived therefrom is applicable, strictly speaking, only to the specific functions depicted.

This discussion serves to illustrate the application of partial differentiation to comparative-static analysis of the simple market model, but only half of the task has actually been accomplished, for we can also find the comparative-static derivatives pertaining to Q^*. This we shall leave to you as an exercise.

National-Income Model

In place of the simple national-income model discussed in Chap. 3, let us now work with a slightly enlarged model with three endogenous variables, Y (national income), C (consumption), and T (taxes):

$$\begin{aligned} Y &= C + I_0 + G_0 \\ C &= \alpha + \beta(Y - T) \qquad (\alpha > 0; \quad 0 < \beta < 1) \\ T &= \gamma + \delta Y \qquad (\gamma > 0; \quad 0 < \delta < 1) \end{aligned} \tag{7.17}$$

The first equation in this system gives the equilibrium condition for national income, while the second and third equations show, respectively, how C and T are determined in the model.

The restrictions on the values of the parameters α, β, γ, and δ can be explained thus: α is positive because consumption is positive even if disposable income $(Y - T)$ is zero; β is a positive fraction because it represents the marginal propensity to consume; γ is positive because even if Y is zero the government will still have a positive tax revenue (from tax bases other than income); and finally, δ is a positive fraction because it represents an income tax rate, and as such it cannot exceed 100 percent. The exogenous variables I_0 (investment) and G_0 (government expenditure) are, of course, nonnegative. All the parameters and exogenous variables are assumed to be independent of one another, so that any one of them can be assigned a new value without affecting the others.

This model can be solved for Y^* by substituting the third equation of (7.17) into the second and then substituting the resulting equation into the first. The equilibrium income (in reduced form) is

$$Y^* = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta} \tag{7.18}$$

Similar equilibrium values can also be found for the endogenous variables C and T, but we shall concentrate on the equilibrium income.

From (7.18), there can be obtained six comparative-static derivatives. Among these, the following three have special policy significance:

$$\frac{\partial Y^*}{\partial G_0} = \frac{1}{1 - \beta + \beta\delta} > 0 \tag{7.19}$$

$$\frac{\partial Y^*}{\partial \gamma} = \frac{-\beta}{1 - \beta + \beta\delta} < 0 \tag{7.20}$$

$$\frac{\partial Y^*}{\partial \delta} = \frac{-\beta(\alpha - \beta\gamma + I_0 + G_0)}{(1 - \beta + \beta\delta)^2} = \frac{-\beta Y^*}{1 - \beta + \beta\delta} < 0 \qquad \text{[by (7.18)]} \tag{7.21}$$

The partial derivative in (7.19) gives us the *government-expenditure multiplier.* It has a positive sign here because β is less than 1, and $\beta\delta$ is greater than zero. If numerical values are given for the parameters β and δ, we can also find the numerical value of this multiplier from (7.19). The derivative in (7.20) may be called the *nonincome-tax multiplier,* because it shows how a change in γ, the government revenue from nonincome-tax sources, will affect the equilibrium income. This multiplier is negative in the present model because the denominator in (7.20) is positive and the numerator is negative. Lastly, the partial derivative in (7.21)—which is not in the nature of a multiplier, since it does not relate a dollar change to another dollar change as the derivatives in (7.19) and (7.20) do—tells us the extent to which an increase in the income tax rate δ will lower the equilibrium income.

Again, note the difference between the two derivatives $\partial Y^*/\partial G_0$ and $\partial Y/\partial G_0$. The former is derived from (7.18), the expression for the equilibrium income. The latter, obtainable from the first equation in (7.17), is $\partial Y/\partial G_0 = 1$, which is altogether different in magnitude and in concept.

Input-Output Model

The solution of an open input-output model appears as a matrix equation $x^* = (I - A)^{-1}d$. If we denote the inverse matrix $(I - A)^{-1}$ by $V = [v_{ij}]$, then, for instance, the solution for a three-industry economy can be written as $x^* = Vd$, or

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{7.22}$$

What are the rates of change of the solution values x_j^* with respect to the exogenous final demands d_1, d_2, and d_3? The general answer is that

$$\frac{\partial x_j^*}{\partial d_k} = v_{jk} \qquad (j, k = 1, 2, 3) \tag{7.23}$$

To see this, let us multiply out Vd in (7.22) and express the solution as

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} v_{11}d_1 + v_{12}d_2 + v_{13}d_3 \\ v_{21}d_1 + v_{22}d_2 + v_{23}d_3 \\ v_{31}d_1 + v_{32}d_2 + v_{33}d_3 \end{bmatrix}$$

In this system of three equations, each one gives a particular solution value as a function of the exogenous final demands. Partial differentiation of these produces a total of nine comparative-static derivatives:

$$\begin{array}{lll} \dfrac{\partial x_1^*}{\partial d_1} = v_{11} & \dfrac{\partial x_1^*}{\partial d_2} = v_{12} & \dfrac{\partial x_1^*}{\partial d_3} = v_{13} \\[2ex] \dfrac{\partial x_2^*}{\partial d_1} = v_{21} & \dfrac{\partial x_2^*}{\partial d_2} = v_{22} & \dfrac{\partial x_2^*}{\partial d_3} = v_{23} \\[2ex] \dfrac{\partial x_3^*}{\partial d_1} = v_{31} & \dfrac{\partial x_3^*}{\partial d_2} = v_{32} & \dfrac{\partial x_3^*}{\partial d_3} = v_{33} \end{array} \tag{7.23'}$$

This is simply the expanded version of (7.23).

Reading (7.23′) as three distinct columns, we may combine the three derivatives in each column into a matrix (vector) derivative:

$$\frac{\partial x^*}{\partial d_1} \equiv \frac{\partial}{\partial d_1}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} \qquad \frac{\partial x^*}{\partial d_2} = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix} \qquad \frac{\partial x^*}{\partial d_3} = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix} \tag{7.23''}$$

Since the three column vectors in (7.23″) are merely the columns of the matrix V, by further consolidation we can summarize the nine derivatives in a single matrix derivative $\partial x^*/\partial d$. Given $x^* = Vd$, we can simply write

$$\frac{\partial x^*}{\partial d} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} = V \equiv (I - A)^{-1}$$

Thus, $(I - A)^{-1}$, the inverse of the Leontief matrix, gives us an ordered display of all the comparative-static derivatives of our open input-output model. Obviously, this matrix derivative can easily be extended from the present three-industry model to the general n-industry case.

Comparative-static derivatives of the input-output model are useful as tools of economic planning, for they provide the answer to the question: If the planning targets, as reflected in

$(d_1, d_2, \ldots, d_n)$, are revised, and if we wish to take care of all direct and indirect requirements in the economy so as to be completely free of bottlenecks, how must we change the output goals of the n industries?

EXERCISE 7.5

1. Examine the comparative-static properties of the equilibrium quantity in (7.15), and check your results by graphic analysis.
2. On the basis of (7.18), find the partial derivatives $\partial Y^*/\partial I_0$, $\partial Y^*/\partial \alpha$, and $\partial Y^*/\partial \beta$. Interpret their meanings and determine their signs.
3. The numerical input-output model (5.21) was solved in Sec. 5.7.
   (*a*) How many comparative-static derivatives can be derived?
   (*b*) Write out these derivatives in the form of (7.23′) and (7.23″).

7.6 Note on Jacobian Determinants

Our study of partial derivatives was motivated solely by comparative-static considerations. But partial derivatives also provide a means of testing whether there exists functional (linear *or* nonlinear) dependence among a set of n functions in n variables. This is related to the notion of Jacobian determinants (named after Jacobi).

Consider the two functions

$$\begin{aligned} y_1 &= 2x_1 + 3x_2 \\ y_2 &= 4x_1^2 + 12x_1x_2 + 9x_2^2 \end{aligned} \tag{7.24}$$

If we get all the four partial derivatives

$$\frac{\partial y_1}{\partial x_1} = 2 \qquad \frac{\partial y_1}{\partial x_2} = 3 \qquad \frac{\partial y_2}{\partial x_1} = 8x_1 + 12x_2 \qquad \frac{\partial y_2}{\partial x_2} = 12x_1 + 18x_2$$

and arrange them into a square matrix in a prescribed order, called a Jacobian matrix and denoted by J, and then take its determinant, the result will be what is known as a *Jacobian determinant* (or a *Jacobian*, for short), denoted by $|J|$:

$$|J| \equiv \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ (8x_1 + 12x_2) & (12x_1 + 18x_2) \end{vmatrix} \tag{7.25}$$

For economy of space, this Jacobian is sometimes also expressed as

$$|J| \equiv \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right|$$

More generally, if we have n differentiable functions in n variables, not necessarily linear,

$$\begin{aligned} y_1 &= f^1(x_1, x_2, \ldots, x_n) \\ y_2 &= f^2(x_1, x_2, \ldots, x_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= f^n(x_1, x_2, \ldots, x_n) \end{aligned} \tag{7.26}$$

where the symbol f^n denotes the nth function (and *not* the function raised to the nth power), we can derive a total of n^2 partial derivatives. Adopting the notation $f_j^i \equiv \partial y^i/\partial x_j$, we can write the Jacobian

$$|J| \equiv \left|\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}\right| \equiv \begin{vmatrix} \partial y_1/\partial x_1 & \cdots & \partial y_1/\partial x_n \\ \vdots & & \vdots \\ \partial y_n/\partial x_1 & \cdots & \partial y_n/\partial x_n \end{vmatrix} \equiv \begin{vmatrix} f_1^1 & \cdots & f_n^1 \\ \vdots & & \vdots \\ f_1^n & \cdots & f_n^n \end{vmatrix} \tag{7.27}$$

A Jacobian test for the existence of functional dependence among a set of n functions is provided by the following theorem: The Jacobian $|J|$ defined in (7.27) will be identically zero for all values of $x_1, \ldots, x_n$ if and only if the n functions $f^1, \ldots, f^n$ in (7.26) are functionally (linearly or nonlinearly) dependent.

As an example, for the two functions in (7.24) the Jacobian as given in (7.25) has the value

$$|J| = (24x_1 + 36x_2) - (24x_1 + 36x_2) = 0$$

That is, the Jacobian vanishes for all values of x_1 and x_2. Therefore, according to the theorem, the two functions in (7.24) must be dependent. You can verify that y_2 is simply y_1 squared; thus they are indeed functionally dependent—here *non*linearly dependent.

Let us now consider the special case of *linear* functions. We have earlier shown that the rows of the coefficient matrix A of a linear-equation system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n \end{aligned} \tag{7.28}$$

are linearly dependent if and only if the determinant $|A| = 0$. This result can now be interpreted as a special application of the Jacobian criterion of functional dependence.

Take the left side of each equation in (7.28) as a separate function of the n variables $x_1, \ldots, x_n$, and denote these functions by $y_1, \ldots, y_n$. The partial derivatives of these functions will turn out to be $\partial y_1/\partial x_1 = a_{11}$, $\partial y_1/\partial x_2 = a_{12}$, etc., so that we may write, in general, $\partial y_i/\partial x_j = a_{ij}$. In view of this, the elements of the Jacobian of these n functions will be precisely the elements of the coefficient matrix A, already arranged in the correct order. That is, we have $|J| = |A|$, and thus the Jacobian criterion of functional dependence among $y_1, \ldots, y_n$—or, what amounts to the same thing, linear dependence among the rows of the coefficient matrix A—is equivalent to the criterion $|A| = 0$ in the present linear case.

We have discussed the Jacobian in the context of a system of n functions in n variables. It should be pointed out, however, that the Jacobian in (7.27) is defined even if each function in (7.26) contains more than n variables, say, $n + 2$ variables:

$$y_i = f^i(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \qquad (i = 1, 2, \ldots, n)$$

In such a case, if we hold any two of the variables (say, x_{n+1} and x_{n+2}) constant, or treat them as parameters, we will again have n functions in exactly n variables and can form a

Jacobian. Moreover, by holding a different pair of the x variables constant, we can form a different Jacobian. Such a situation will indeed be encountered in Chap. 8 in connection with the discussion of the implicit-function theorem.

EXERCISE 7.6

1. Use Jacobian determinants to test the existence of functional dependence between the paired functions.
   (*a*) $y_1 = 3x_1^2 + x_2$
   $y_2 = 9x_1^4 + 6x_1^2(x_2 + 4) + x_2(x_2 + 8) + 12$
   (*b*) $y_1 = 3x_1^2 + 2x_2^2$
   $y_2 = 5x_1 + 1$
2. Consider (7.22) as a set of three functions $x_i^* = f^i(d_1, d_2, d_3)$ (with $i = 1, 2, 3$).
   (*a*) Write out the 3×3 Jacobian. Does it have some relation to (7.23′)? Can we write $|J| = |V|$?
   (*b*) Since $V \equiv (I - A)^{-1}$, can we conclude that $|V| \neq 0$? What can we infer from this about the three equations in (7.22)?