Properties of Expectation:

① Expectation of a constant is a constant itself.

Proof:

Let us say that a is the constant and $f(x_i)$ is the probability function when $i = 1, 2 \dots n$ therefore.

$$E(a) = \sum_{i=1}^{n} a\, f(x_i) = a\, f(x_1) + a f(x_2) \dots a f(x_n)$$

$$= a \left[f(x_1) + f(x_2) + \dots f(x_n) \right)$$

$$= a \sum f(x_i) \qquad \therefore \sum f(x) = 1$$

$$E(a) = a(1) = a$$

$$\boxed{E(a) = a}$$ Hence Proved.

② $E(aX+b) = aE(X)+b$

Proof:

If $(aX+b)$ is a r.v then $f(x_i)$ is probability function where $i = 1, 2 \dots n$

$$E(aX+b) = \sum_{i=1}^{n} (ax_i + b) f(x_i)$$

$$= \sum_{i=1}^{n} \left[a x_i f(x_i) + b f(x_i) \right]$$

$$= \sum_{i=1}^{n} a x_i f(x_i) + \sum_{i=1}^{n} b f(x_i)$$

$$= a \sum_{i=1}^{n} x_i f(x_i) + b \sum_{i=1}^{n} f(x_i)$$

$$\boxed{E(aX+b) = a E(X) + b}$$

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③ Expectation of the product of two independent r.v is equal to the product of their expectation.

$$E(XY) = E(X)\,E(Y)$$

Proof:

Let X_i takes the values $x_1, x_2 \cdots x_m$ with corresponding probabilities $g(x_1), g(x_2) \cdots g(x_m)$.

Let X_j takes the values $y_1, y_2 \cdots y_n$ with the corresponding probabilities $h(y_1), h(y_2) \cdots h(y_n)$.

Let $f(x_i, y_j)$ be the probability that X takes values x_i and Y takes values y_j.

According to Expectation.

$$E(XY) = \sum_{i=1}^{m}\sum_{j=1}^{n} (x_i, y_j)\, f(x_i, y_j) \quad — ①$$

Since X and Y are independent therefore for joint happening their probabilities will multiply

$$f(x_i, y_j) = g(x_i) \cdot h(y_j) \quad — ②$$

Put equ ② in ①

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} (x_i, y_j)\, g(x_i)\, h(y_j)$$

$$= \sum_{i=1}^{m} x_i\, g(x_i) \sum_{j=1}^{n} y_j\, h(y_j)$$

$$\boxed{E(XY) = E(X)\, E(Y)}$$

(4) The expectation of the sum of two independent r.v is equal to the sum of their expectations.

$$E(X+Y) = E(X) + E(Y)$$

Proof:

Let us consider that X takes the values x_i where $i = 1, 2 \cdots m$ with the probabilities $g(x_i)$. Let Y takes the values y_j where $j = 1, 2 \cdots n$ with probabilities $h(y_j)$. Let $f(x_i, y_j)$ be the joint probability so that X takes the values x_i and Y takes the values y_j.

Therefore the table of joint probability is as.

	y_1	y_2	y_j	y_n	$g(x_i)$
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$ ..	$f(x_1,y_j)$ ...	$f(x_1,y_n)$	$g(x_1)$
x_2	$f(x_2,y_1)$	$f(x_2,y_2)$ ..	$f(x_2,y_j)$ ..	$f(x_2,y_n)$	$g(x_2)$
$\vdots$	$\vdots$				
x_i	$f(x_i,y_1)$	$f(x_i,y_2)$ ---	$f(x_i,y_j)$ ---	$f(x_i,y_n)$	$g(x_i)$
$\vdots$	$\vdots$				
x_m	$f(x_m,y_1)$	$f(x_m,y_2)$ ---	$f(x_m,y_j)$ ..	$f(x_m,y_n)$	$g(x_m)$
$h(y_j)$	$h(y_1)$	$h(y_2)$ ---	$h(y_j)$ ...	$h(y_n)$	1

Now by defination of expectation

$$E(X+Y) = \sum_{i=1}^{m}\sum_{j=1}^{n}(x_i+y_j)\, f(x_i,y_j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i f(x_i,y_j) + \sum_{i=1}^{m}\sum_{j=1}^{n} y_j f(x_i,y_j) \text{ — } A$$

$$\Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} x_i f(x_i,y_j) = \sum_{i=1}^{m} x_i \sum_{j=1}^{n} f(x_i,y_j)$$

$$= \sum_{i=1}^{m} x_i \left[f(x_i,y_1) + f(x_i,y_2) + \cdots f(x_i,y_n)\right)$$

$\Rightarrow$ According to the table

$$f(x_i,y_1) + f(x_i,y_2) + \cdots f(x_i,y_n) = g(x_i)$$

$$= \sum_{i=1}^{m} x_i\, g(x_i) = E(X) \text{ — (i)}$$

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Likewise

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_j f(x_i, y_j) = \sum_{j=1}^{n} y_j \sum_{i=1}^{m} f(x_i, y_j)$$

$$= \sum_{j=1}^{n} y_j \left\{ f(x_1, y_j) + f(x_2, y_j) + \cdots f(x_m, y_j) \right\}$$

$$= \sum_{j=1}^{n} y_j h(y_j)$$

$$= E(Y) \quad \text{—(ii)}$$

Put (i) & (ii) in (A)

$$\boxed{E(X+Y) = E(X) + E(Y)}$$

(5) The Covariance of two independent random variable is equal to zero.

$$\text{Cov}(X, Y) = E[\{X - E(X)\}\{Y - E(Y)\}]$$

Proof:

$$= E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$

$$= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)$$

$$= E(X)\cancel{E(Y)} - E(X)\cancel{E(Y)} - E(\cancel{Y})E(X) + \cancel{E(X)}E(Y)$$

$$= 0$$

(6) The variance of sum ~~of~~ or differences of two independent r.v is equal to the sum of their expectation.

$$var(x \pm y) = var(x) + var(y)$$

Proof:

$$var(x-y) = E[(x-y) - E(x-y)]^2$$

$$= E[x - y - E(x) + E(y)]^2$$

$$= E[\{x - E(x) - (y - E(y)\}]^2$$

$$= E[(x - E(x)]^2 + (y - E(y))^2$$

$$-2[x - E(x)][y - E(y)]$$

$$= E[[x - E(x)]^2 + [y - E(y)]^2$$

$$-2[x - E(x))(y - E(y))$$

$$= E[x - E(x)]^2 + E[y - E(y)]^2$$

$$-2E[x - E(x)][y - E(y)]$$

$$= E[x - E(x)]^2 + E[y - E(y)]^2 - 0$$

$$v(x \pm y) = var(x) + var(y)$$