As a result, another type of saddle point occurs, as illustrated in panels I and J of Figure 6.1, which appears somewhat like a bird with wings outstretched. Like the earlier saddle points, a minimum exists in one direction and a maximum in another direction at a value for x_1 and x_2 of 0.25, but the saddle no longer is parallel to one of the axes, but rather lies along a line running between the two axes. This is the result of the product of the second cross partials being greater than the second direct partials. By changing the function only slightly and making the coefficient 10 on the product of x_1 and x_2 a -10 results in the surface and contour lines illustrated in panels K and L of Figure 16.1. Compare these with panels I and J.

In the preceding examples, care was taken to develop polynomial functions that had potential maxima or minima at levels for x_1 and x_2 at positive but finite amounts. If a true maximum exists, the resultant isoquant map will consist of a series of concentric rings centered on the maximum with ridge lines intersecting at the maximum.

One is sometimes tempted to attempt the same approach for other types of functions. For example, consider a function such as

$$(6.49) \qquad y = 10x_1^{0.5}x_2^{0.5}$$

In this instance

$$(6.50) \qquad f_1 = 5x_1^{-0.5}x_2^{0.5}$$

And

$$(6.51) \qquad f_2 = 5x_1^{0.5}x_2^{-0.5}$$

These first partial derivatives of equation (6.49) could be set equal to zero, but they would each assume a value of zero only at $x_1 = 0$ and $x_2 = 0$. There is no possibility that f_1 and f_2 could be zero for any combination of positive values for x_1 and x_2. Hence the function never achieves a maximum.

6.4 Some Matrix Algebra Principles

Matrix algebra is a useful tool for determining if a function has achieved a maximum or minimum.A *matrix* consists of a series of numbers (also called *values* or *elements*) organized into rows and columns. The matrix

$$(6.52) \qquad \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

is a square 3 *x* 3 matrix, since it has the same number of rows and columns. For each element, the first subscript indicates its row, the second subscript its column. For example a_{23} refers to the element or value located in the second row and third column.

Every square matrix has a number associated with it called its *determinant*. For a 1 x 1 matrix with only one value or element, its determinant is a_{11}. The determinant of a 2 x 2 matrix is $a_{11}a_{22} - a_{12}a_{21}$. The determinant of a 3 x 3 matrix is $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{33}a_{21}a_{12}$. Determinants for matrices larger than 3 x 3 are very difficult to calculate, and a computer routine is usually used to calculate them.

The *principal minors* of a matrix are obtained by deleting first all rows and columns of the matrix except the element located in the first row and column (a_{11}) and finding the resultant determinant. In this example, the first principal minor is a_{11}. Next, all rows and columns except the first two rows and columns are deleted, and the determinant for the remaining 2 x 2 matrix is calculated. In this example, the second principal minor is $a_{11}a_{22} - a_{12}a_{21}$. The third principal minor would be obtained by deleting all rows and columns with row or column subscripts larger than 3, and then again finding the resultant determinant.

The second order conditions can better be explained with the aid of matrix algebra. The second direct and cross partial derivatives of a two input production function could form the square 2 x 2 matrix

(6.53) $$\begin{matrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{matrix}$$

The principal minors of equation (6.53) are

(6.54) $$H_1 = f_{11}$$

$$H_2 = f_{11}f_{22} - f_{12}f_{21}$$

Assuming that the first-order conditions have been met, The second-order condition for a maximum requires that the principal minors H_1 and H_2 alternate in sign, starting with a negative sign. In other words, $H_1 < 0; H_2 > 0$.

For a minimum, all principal minors must be positive. That is, $H_1, H_2 > 0$.

A saddle point results for either of the remaining conditions

$$H_1 > 0; H_2 < 0$$

or, $$H_1 < 0; H_2 < 0$$

6.5 A Further Illustration

A further illustration of second-order conditions is obtained from the two input polynomial

(6.55) $$y = 40x_1 - 12x_1^2 + 1.2x_1^3 - 0.035x_1^4 + 40x_2 - 12x_2^2 + 1.2x_2^3 - 0.035x_2^4$$

This function has nine values where the first derivatives are equal to zero. Each of these values, called *critical values*, represents a maximum, a minimum, or a saddle point. Figure 6.2 illustrates the function. Table 6.1 illustrates the corresponding second order conditions. In this example, H_1 is f_{11} and H_2 is $f_{11}f_{22} - f_{12}f_{21}$.

This function differs from the previous functions in that there are several combinations of x_1 and x_2 that generate critical values where the slope of the function is equal to zero. There is but one global maximum for the function, but several local maxima. A global maximum might be thought of as the top of the highest mountain, whereas a local maximum might be considered the top of a nearby hill. There are also numerous saddle points. The second-order conditions can be verified by carefully studying figure 6.2.

6.6 Maximizing a Profit Function with Two Inputs

The usefulness of the criteria for maximizing a function can be further illustrated with an agricultural example using a profit function for corn. Suppose that the production function for corn is given by

$$y = f(x_1, x_2) \tag{6.56}$$

where y = corn yield in bushels per acre
x_1 = pounds of potash applied per acre
x_2 = pounds of phosphate applied per acre

Table 6.1 Critical Values for the Polynomial $y = 40x_1 - 12x_1^2 + 1.2x_1^3 - 0.035x_1^4 + 40x_2 - 12x_2^2 + 1.2x_2^3 - 0.035x_2^4$

x_2 \ x_1	2.54	6.93	16.24
16.24	local maximum $y = 232.3$ $H_1 < 0$ $H_2 > 0$	saddle point $y = 209.5$ $H_1 > 0$ $H_2 < 0$	global maximum $y = 378.8$ $H_1 < 0$ $H_2 > 0$
6.93	saddle point $y = 61.9$ $H_1 < 0$ $H_2 < 0$	local minimum $y = 39.1$ $H_1 > 0$ $H_2 > 0$	saddle point $y = 209.5$ $H_1 < 0$ $H_2 < 0$
2.54	local maximum $y = 84.8$ $H_1 < 0$ $H_2 > 0$	saddle point $y = 61.9$ $H_1 > 0$ $H_2 < 0$	local maximum $y = 232.3$ $H_1 < 0$ $H_2 > 0$

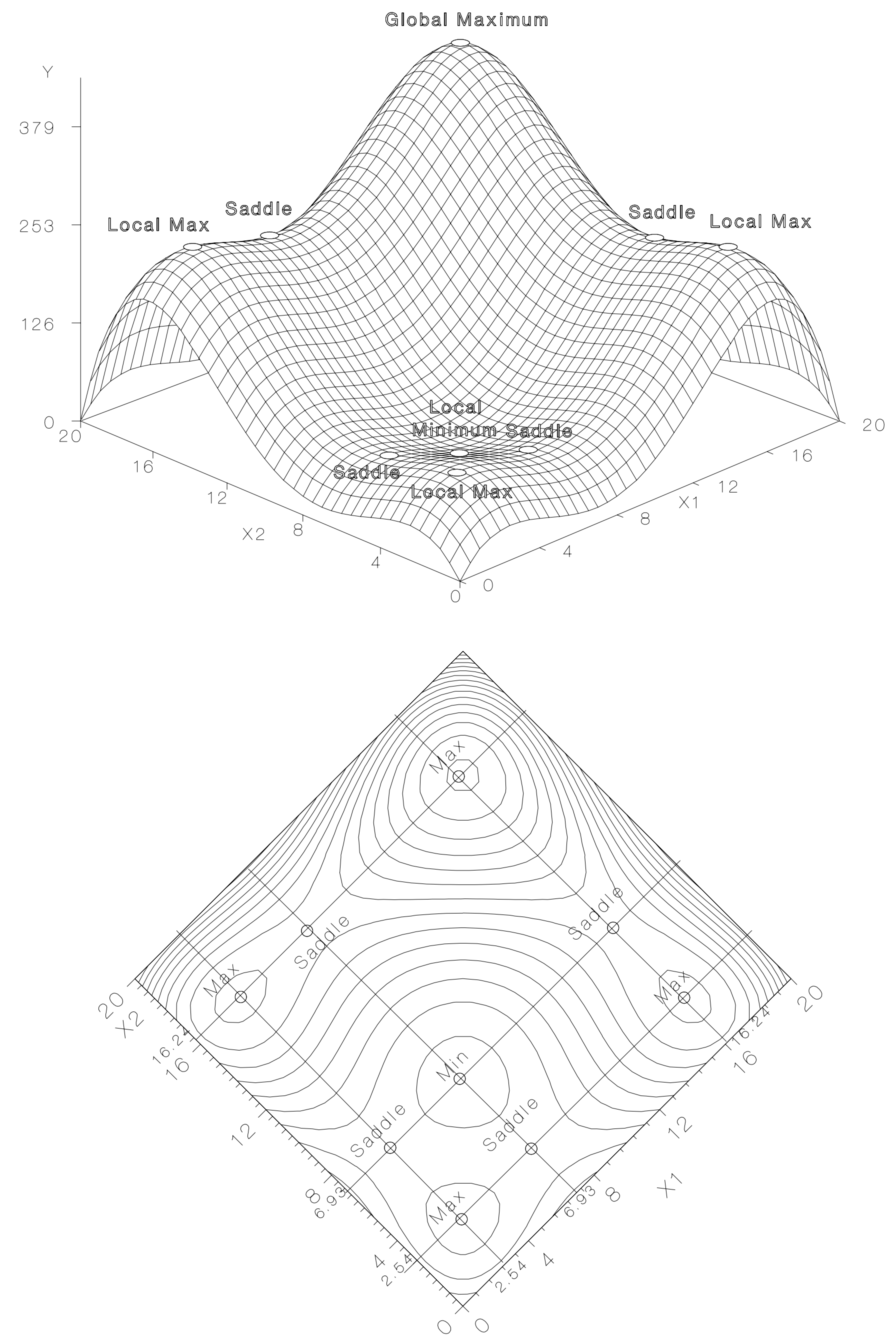

Figure 6.2 Critical Values for the Polynomial $y = 40x_1 - 12x_1^2 + 1.2x_1^3 - 0.035x_1^4 + 40x_2 - 12x_2^2 + 1.2x_2^3 - 0.035x_2^4$

All other inputs are presumed to be fixed and given, or already owned by the farm manager. The decision faced by the farm manager is how much of the two fertilizer inputs or factors of production to apply to maximize profits to the farm firm. The total revenue or total value of the product from the sale of the corn from 1 acre of land is

$$(6.57) \qquad TVP = py$$

where
p = price of corn per bushel
y = corn yield in bushels per acre

The total input or factor cost is

$$(6.58) \qquad TFC = v_1x_1 + v_2x_2$$

where v_1 and v_2 are the prices on potash and phosphate respectively in cents per pound. The profit function is

$$(6.59) \qquad \Pi = TVP - TFC$$

Equation (6.59) can also be expressed as

$$(6.60) \qquad \Pi = py - v_1x_1 - v_2x_2 \text{ , or}$$

$$(6.61) \qquad \Pi = pf(x_1,x_2) - v_1x_1 - v_2x_2$$

The first order, or necessary conditions for a maximum are

$$(6.62) \qquad \Pi_1 = pf_1 - v_1 = 0$$

$$(6.63) \qquad \Pi_2 = pf_2 - v_2 = 0$$

Equations (6.62) and (6.63) require that the slope of the *TVP* function with respect to each input equal the slope of the *TFC* function for each input, or that the difference between the slopes of the two functions be zero for both inputs, or as

$$(6.64) \qquad pf_1 = v_1$$

$$(6.65) \qquad pf_2 = v_2$$

The value of the marginal product must equal the marginal factor cost for each input. If the farmer is able to purchase as much of each type of fertilizer as he or she wishes at the going market price, the marginal factor cost is the price of the input, v_1 or v_2. This also implies that at the point of profit maximization the ratio of *VMP* to *MFC* for each input is 1. In other words

$$(6.66) \qquad pf_1/v_1 = pf_2/v_2 = 1$$

The last dollar spent on each input must return exactly $1, and most if not all previous units will have given back more than a dollar. The accumulation of the excess dollars in returns over costs represents the profits or net revenues accruing to the farm firm.

Moreover, the equations representing the first order conditions can be divided by each other:

(6.67) $pf_1/pf_2 = v_1/v_2$.

Note that the output price cancels in equation (6.67) such that

(6.68) $f_1/f_2 = v_1/v_2$

Recall from Chapter 5 that f_1 is the *MPP* of x_1 and f_2 is the *MPP* of x_2. The negative ratio of the respective marginal products is one definition of the marginal rate of substitution of x_1 for x_2 or $MRS_{x_1x_2}$. Then at the point of profit maximization

(6.69) $MRS_{x_1x_2} = v_1/v_2$. or

(6.70) $dx_2/dx_1 = v_1/v_2$

As will be seen later, equation (6.70) holds at other points on the isoquant map in addition to the point of profit maximization.

The second order conditions also play a role. Assuming fixed input prices (v_1 and v_2), the second order conditions for the profit function are

(6.71) $\Pi_{11} = pf_{11}$

(6.72) $\Pi_{22} = pf_{22}$

(6.73) $\Pi_{12} = \Pi_{21} = pf_{12} = pf_{21}$ (by Young's theorem)

Or in the form of a matrix

(6.74)

$$\begin{matrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{matrix}$$

For a maximum

(6.75) $pf_{11} < 0$, and

(6.76) $pf_{11}pf_{22} - pf_{12}pf_{21} > 0$

The principal minors must alternate in sign starting with a minus. Equations (6.75) and (6.76) require that the *VMP* functions for both x_1 and x_2 be downsloping. With fixed input prices, the input cost function will have a constant slope, or the slope of *MFC* will be zero.

The conditions that have been outlined determine a single point of global profit maximization, assuming that the underlying production function itself has but a single maximum. This single profit-maximization point will require less of both x_1 and x_2 than would be required to maximize output, unless one or both of the inputs were free.

6.7 A Comparison with Output- or Yield-Maximization Criteria

A comparison can be made of the criteria for profit maximization versus the criteria for yield maximization. If the production function is

(6.77) $y = f(x_1,x_2)$

Maximum yield occurs where

(6.78) $f_1 = MPP_{x_1} = 0$

(6.79) $f_2 = MPP_{x_2} = 0$, or

(6.80) $f_1 = f_2 = 0$

The second-order conditions for maximum output require that $f_{11} < 0$; and $f_{11}f_{22} > f_{12}f_{21}$. The *MPP* for both inputs must be downward sloping.

The first- and second-order conditions comprise the necessary and sufficient conditions for the maximization of output or yield and are the mathematical conditions that define the center of an isoquant map that consists of a series of concentric rings.

Since zero can be multiplied or divided by any number other than zero, and zero would still result, when *MPP* for x_1 and x_2 is zero,

(6.81) $pf_1/v_1 = pf_2/v_2 = 0$

To be at maximum output, the last dollar spent on each input must produce no additional output, yield, or revenue.

Recall that the first-order, or necessary conditions for maximum profit occur at the point where

(6.82) $pf_1 - v_1 = 0$

(6.83) $pf_2 - v_2 = 0$

(6.84) $pf_1/v_1 = pf_2/v_2 = 1$

and the corresponding second order conditions for maximum profit require that

(6.85) $pf_{11} < 0$

(6.86) $pf_{11}pf_{22} - pf_{12}pf_{21} > 0$

(6.87) $p^2(f_{11}f_{22} - f_{12}f_{21}) > 0$

Since p^2 is positive, the required signs on the second-order conditions are the same for both profit and yield maximization.

6.8 Concluding Comments

This chapter has developed some of the fundamental rules for determining if a function is at a maximum or a minimum. The rules developed here are useful in finding a solution to the unconstrained maximization problem. These rules also provide the basis for finding the solution to the problem of constrained maximization or minimization. The constrained maximization or minimization problem makes it possible to determine the combination of inputs that is required to produce a given level of output for the least cost, or to maximize the level of output for a given cost. The constrained maximization problem is presented in further detail in Chapters 7 and 8.

Notes

[1.] A simple example can be used to illustrate that Young's theorem does indeed hold in a specific case. Suppose that a production function

$y = x_1^2 x_2^3$. Then

$f_1 = 2x_1x_2^3$
$f_2 = 3x_1^2x_2^2$
$f_{12} = 6x_1x_2^2$
$f_{21} = 6x_1x_2^2$

A formal proof of Young's theorem in the general case can be found in most intermediate calculus texts.

Problems and Exercises

1. Does the function $y = x_1x_2$ ever achieve a maximum? Explain.

2. Does the function $y = x_1^2 - 2x_2^2$ ever achieve a maximum? Explain.

3. Does the function $y = x_1 + 0.1x_1^2 - 0.05x_1^3 + x_2 + 0.1x_2^2 - 0.05x_2^3$ ever achieve a maximum? If so, at what level of input use is output maximized.

4. Suppose that price of the output is $2. For the function given in Problem 3, what level of input use will maximize the total value of the product?

5. Assume that the following conditions exist

$f_1 = 0$

$f_2 = 0$

Does a maximum, minimum, or saddle point exist in each case?

a. $f_{11} > 0$

$f_{11} \cdot f_{22} - f_{12} \cdot f_{21} < 0$

b. $f_{11} < 0$

$f_{11} \cdot f_{22} - f_{12} \cdot f_{21} > 0$

c. $f_{11} > 0$

$f_{11} \cdot f_{22} - f_{12} \cdot f_{21} > 0$

d. $f_{11} < 0$

$f_{11} \cdot f_{22} - f_{12} \cdot f_{21} < 0$

6. Suppose that the price of the output is $3, the price of the input x_1 is $5, and the price of input x_2 is $4. Is it possible to produce and achieve a profit? Explain. What are the necessary and sufficient conditions for profit maximization?