6

Maximization in the Two-Input Case

This chapter develops the fundamental mathematics for the maximization or minimization of a function with two or more inputs and a single output. The necessary and sufficient conditions for the maximization or minimization of a function are derived in detail. Illustrations are used to show why certain conditions are required if a function is to be maximized or minimized. Examples of functions that fulfill and violate the rules are illustrated. An application of the rules is made using the yield maximization problem.

Key terms and definitions:

Maximization
Minimization
First-Order Conditions
Second-Order Conditions
Young's Theorem
Necessary Conditions
Sufficient Conditions
Matrix
Matrix of Partial Derivatives
Principal Minors
Local Maximum
Global Maximum
Saddle Point
Determinant
Critical Value
Unconstrained Maximization and Minimization
Constrained Maximization and Minimization

6.1 An Introduction to Maximization

An isoquant map might be thought of as a contour map of a hill. The height of the hill at any point is measured by the amount of output that is produced. An *isoquant* connects all points producing the same quantity of output, or having the same elevation on the hill. In general, isoquants consist of concentric rings, just as there are points on all sides of the hill that have the same elevation. Similarly, there are many different combinations of two inputs that would all produce exactly the same amount of output.

An infinite number of isoquants can be drawn. Each isoquant represents a slightly different output level or elevation on the hill. Isoquants never intersect or cross each other, for this would imply that the same combination of two inputs could produce two different levels of output. The quantity of output produced from each combination of the two inputs is unique. If one is standing at a particular point on a side of a hill, that particular point has one and only one elevation.

If the isoquants are concentric rings, any isoquant drawn inside another isoquant will always represent a slightly greater output level than the one on the outside (Figure 5.1, diagram A). If the isoquants are not rings, the greatest output is normally associated with the isoquant at the greatest distance from the origin of the graph. No two isoquants can represent exactly the same level of output. Each isoquant by definition represents a slightly different quantity of output from any other isoquant.

If an isoquant map is drawn as a series of concentric rings, these rings become smaller and smaller as one moves toward the center of the diagram. At comparatively low levels of output, the possible combinations of the two inputs x_1 and x_2 suggest a wide range of options: a large quantity of x_2 and a small quantity of x_1: a small quantity of x_2 and a large quantity of x_1, or something in between. At higher levels of output, the isoquant rings become smaller and smaller, suggesting that the range of options becomes more restricted, but there remains an infinite number of possible combinations on a particular isoquant within the restricted range, each representing a slightly different combination of x_1 and x_2.

The concentric rings finally become a single point. This is the global point of maximum output and would be the position where the farm manager would prefer to operate a farm if inputs were free and there were no other restrictions on the use of the inputs. This single point is the point where the two ridge lines intersect. The *MRS* for an isoquant consisting of a single point is undefined, but this point represents the maximum amount of output that can be produced from any combination of the two inputs x_1 and x_2.

If one were standing on the top of a hill, at the very top, the place where one would be standing would be level. Moreover, regardless of the direction that one looked from the top of a hill, the hill would slope downward from its level top. If one were standing on the hilltop, no other point on the hill would slope upward. If it did, one would not be on the top of the hill. Every other point on the hill would be at a somewhat lower elevation.

The top of the highest hill represents the greatest possible elevation, or global maximum. However, hills that are not as high are also level at the top. The tops of these hills represent local, but not global maxima.

Minimum points can be defined similarly. The bottom of a valley is also level. The bottom of the deepest valley represents a global minimum, while the bottom of other valleys not as deep represent local but not global minima. If one were to draw contour lines for a valley, they would be indistinguishable from the contour lines for a hill.

The slope at both the bottom of a valley and at the top of the hill is zero in all directions. It is not possible to distinguish the bottom of a valley from the top of a hill simply by looking at the slope at that point, because the slope for both is zero. Much of the mathematics of maximization and minimization is concerned with the problem of distinguishing bottoms of valleys from tops of hills based on second derivative tests or second order conditions.

6.2 The Maximum of a Function

The problem of finding the combination of inputs x_1 and x_2 that results in the true maximum output from a two-input production function is the mathematical equivalent of finding the top of the hill, or the point on a hill with the greatest elevation. Two conditions need to be checked. First, the point under consideration must be level, or have a zero slope, which is a necessary condition, but level points are found not only at the top of hills but at the bottom of valleys.

The saddle for a horse provides another example and problem for the mathematician. The saddle is level in the middle, but it slopes upward at both ends and downward at both sides. A saddle looks like neither a hill nor a valley, but is a combination of both. So an approach needs to be taken that will separate the true hill from the valley and the saddle point.

Suppose again the general production function

(6.1) $$y = f(x_1, x_2)$$

The first-order or necessary conditions for the maximization of output are

(6.2) $$\partial y/\partial x_1 = 0, \text{ or } f_1 = 0$$

and

(6.3) $$\partial y/\partial x_2 = 0 \text{ or } f_2 = 0$$

Equations (6.2) and (6.3) ensure that the point is level relative to both the x_1 and the x_2 axes.

The second order conditions for the maximization of output require that the partial derivatives be obtained from the first order conditions. There are four possible second derivatives obtained by differentiating the first equation with respect to x_1 and then with respect to x_2. The second equation can also be differentiated with respect to both x_1 and x_2.

These four second partial derivatives are

(6.4) $$\partial(\partial y/\partial x_1)/\partial x_1 = \partial^2 y/\partial x_1^2 = f_{11}$$

(6.5) $$\partial(\partial y/\partial x_1)/\partial x_2 = \partial^2 y/\partial x_1 \partial x_2 = f_{12}$$

(6.6) $$\partial(\partial y/\partial x_2)/\partial x_1 = \partial^2 y/\partial x_2 \partial x_1 = f_{21}$$

(6.7) $$\partial(\partial y/\partial x_2)/\partial x_2 = \partial^2 y/\partial x_2^2 = f_{22}$$

Young's theorem states that the order of the partial differentiation makes no difference and that $f_{12} = f_{21}$. [1]

The second order conditions for a maximum require that

(6.8) $$f_{11} < 0$$

and

(6.9) $f_{11}f_{22} > f_{12}f_{21}$.

Since $f_{12}f_{21}$ is non−negative, $f_{11}f_{22}$ must be positive for equation (6.9) to hold, and $f_{11}f_{22}$ can be positive only if f_{22} is also negative. Taken together, these first- and second-order conditions provide the necessary and sufficient conditions for the maximization of a two-input production function that has one maximum.

6.3 Some Illustrative Examples

Some specific examples will further illustrate these points. Suppose that the production function is

(6.10) $y = 10x_1 + 10x_2 - x_1^2 - x_2^2$

The first order or necessary conditions for a maximum are

(6.11) $f_1 = 10 - 2x_1 = 0$

(6.12) $x_1 = 5$

(6.13) $f_2 = 10 - 2x_2 = 0$

(6.14) $x_2 = 5$

The critical values for a function is a point where the slope of the function is equal to zero. The critical values for this function occur at the point where $x_1 = 5$, and $x_2 = 5$. This point could be a maximum, a minimum or a saddle point.

For a maximum, the second order conditions require that

(6.15) $f_{11} < 0$ and $f_{11}f_{22} > f_{12}f_{21}$

For equation (6.10)

(6.16) $f_{11} = -2 < 0$

(6.17) $f_{22} = -2$

(6.18) $f_{12} = f_{21} = 0$, since x_2 does not appear in f_1, nor x_1 in f_2.

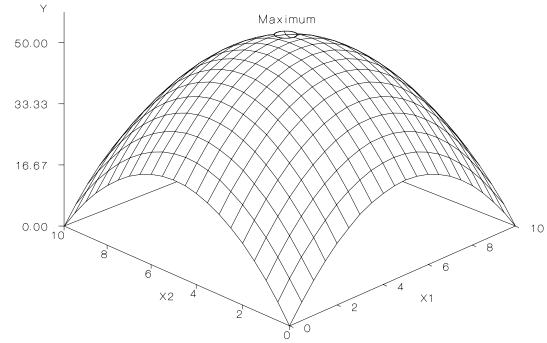
Hence

(6.19) $f_{11}f_{22} - f_{12}f_{21} = 4 > 0$

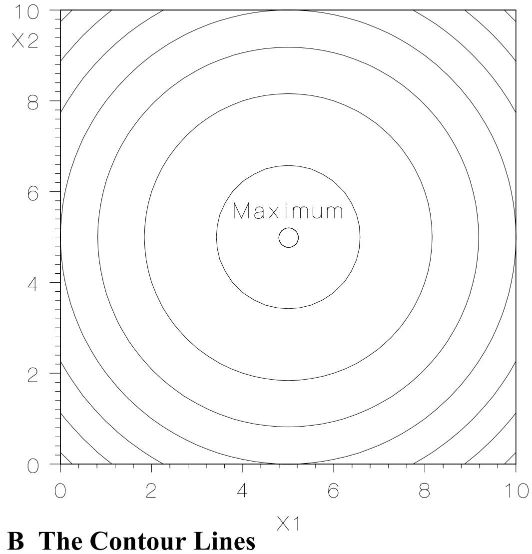
The necessary and sufficient conditions have been met for the maximization of equation (6.10) at $x_1 = 5$, $x_2 = 5$. This function and its contour lines are illustrated in panels A and B of Figure 6.1.

Now consider a production function

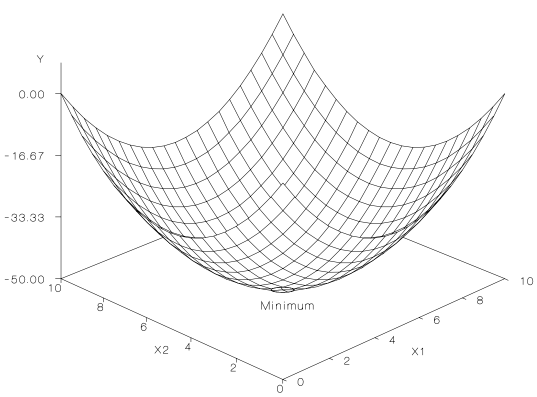
(6.20) $y = -10x_1 - 10x_2 + x_1^2 + x_2^2$



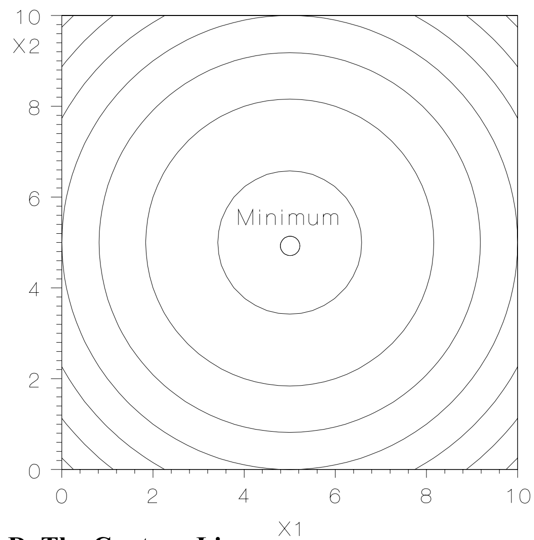
A The Surface



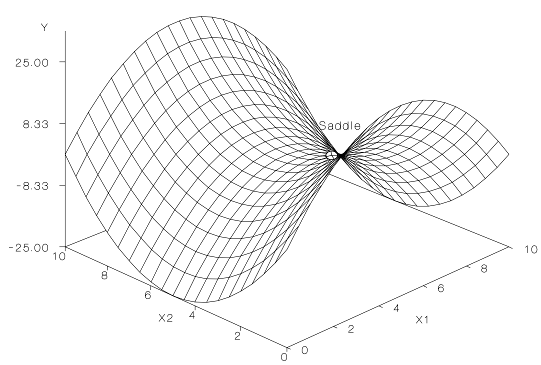
B The Contour Lines



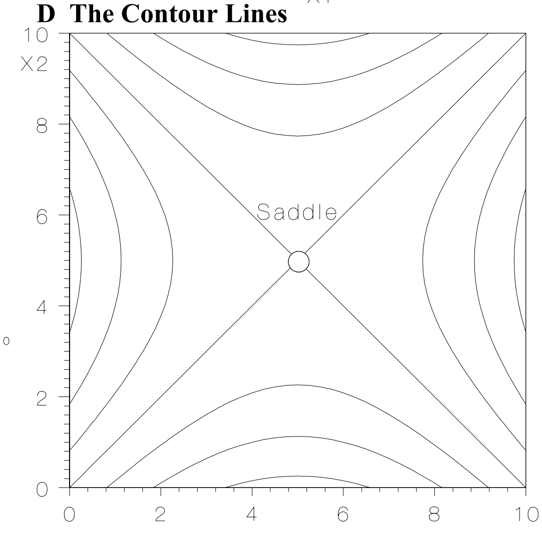
C The Surface



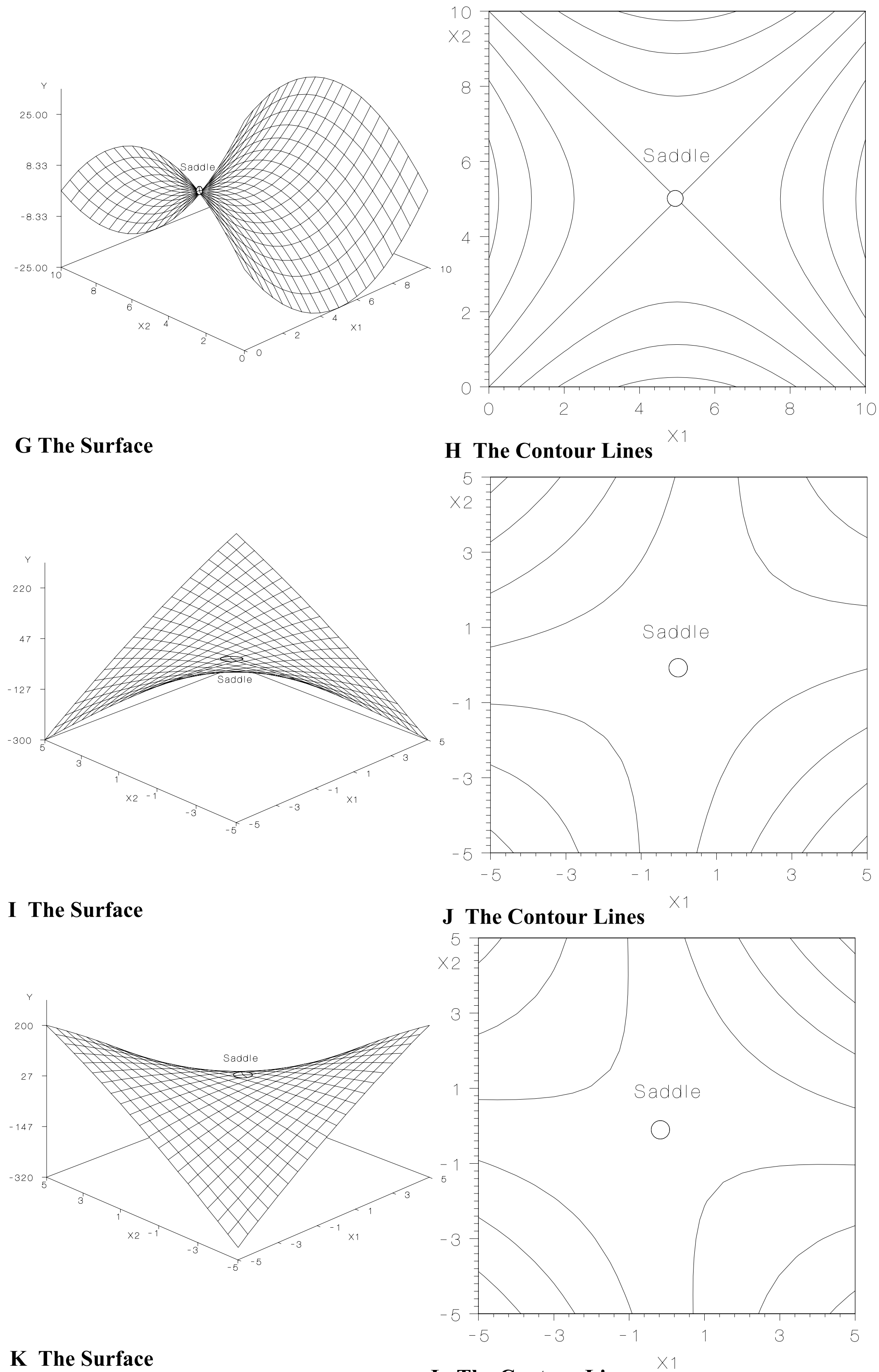
D The Contour Lines



E The Surface



F The Contour Lines



G The Surface

H The Contour Lines

I The Surface

J The Contour Lines

K The Surface

L The Contour Lines

Figure 6.1 Alternative Surfaces and Contours Illustrating Second-Order Conditions

The first-order conditions are

(6.21) $f_1 = -10 + 2x_1 = 0$

(6.22) $x_1 = 5$

(6.23) $f_2 = -10 + 2x_2 = 0$

(6.24) $x_2 = 5$

The second order conditions for a minimum require that

(6.25) $f_{11} > 0$

(6.26) $f_{11} f_{22} > f_{12} f_{21}$

For equation (6.20) the second order conditions are

(6.27) $f_{11} = 2 > 0$

(6.28) $f_{22} = 2$

Moreover

(6.29) $f_{11}f_{22} - f_{12}f_{21} = 4 > 0$

The necessary and sufficient conditions have been met for the minimization of equation (6.20) at $x_1 = 5$, $x_2 = 5$. This function and its contour lines are illustrated in panels C and D of Figure 6.1.

Now consider a function

(6.30) $y = 10x_1 - 10x_2 - x_1^2 + x_2^2$

The first order conditions are

(6.31) $f_1 = 10 - 2x_1 = 0$

(6.32) $x_1 = 5$

(6.33) $f_2 = -10 + 2x_2 = 0$

(6.34) $x_2 = 5$

For equation (6.30), the second order conditions are

(6.35) $f_{11} = -2 < 0$

(6.36) $f_{22} = 2$

Moreover

(6.37) $f_{11}f_{22} - f_{12}f_{21} = -4 < 0$

The necessary and sufficient conditions have not been met for the minimization or maximization of equation (6.30) at $x_1 = 5$, $x_2 = 5$. This function is the unique saddle point illustrated panels E and F of Figure 6.1 that represents a maximum in the direction parallel to the x_1 axis, but a minimum in the direction parallel to the x_2 axis.

The function

(6.38) $$y = -\ 10x_1 + 10x_2 + x_1^2 - x_2^2$$

results in a very similar saddle point with the axes reversed. That is, a minimum occurs parallel to the x_1 axis, but a maximum occurs parallel to the x_2 axis. The surface of this function is illustrated in panels G and H of Figure 6.1. Now consider a function

(6.39) $$y = -2x_1 - 2x_2 - x_1^2 - x_2^2 + 10x_1x_2$$

The first order conditions are

(6.40) $$f_1 = -2 - 2x_1 + 10x_2 = 0$$

(6.41) $$f_2 = -2 - 2x_2 + 10x_1 = 0$$

Solving for x_2 in equation (6.41) for f_2 gives us

(6.42) $$-2x_2 = 2 - 10x_1$$

(6.43) $$x_2 = 5x_1 - 1$$

Inserting equation (6.43) x_2 into equation (6.40) for f_1 results in

(6.44) $$x_1 = 0.25$$

Since $x_2 = 5x_1 - 5$, x_2 also equals 0.25.

In this instance the second order conditions are

(6.45) $$f_{11} = -2 < 0$$

(6.46) $$f_{22} = -2 < 0$$

However

(6.47) $$f_{12} = f_{21} = 10$$

Thus

(6.48) $$f_{11}f_{22} - f_{12}f_{21} = 4 - 100 = -96 < 0$$

Although these conditions may at first appear to be sufficient for a maximum at $x_1 = x_2 = 0.25$, the second order conditions have not been fully met. In this example, the product of the direct second partial derivatives $f_{11}f_{22}$ is less than the product of the second cross partial derivatives $f_{12}f_{21}$, and therefore $f_{11}f_{22} - f_{12}f_{21}$ is less than zero. In the earlier examples, the second cross partial derivatives were always zero, since an interaction term such as $10x_1x_2$ did not appear in the original production function.