The choice of the input to be labeled x_1 and x_2 is quite arbitrary. However, if x_2 remains on the vertical axis, the same process could be repeated by drawing vertical lines from the value chosen on the x_1 axis (the assumption with respect to the value for x_1) and finding the points of tangency between the vertical line and its corresponding isoquant. In this case however, the point of tangency will occur at the point where the isoquant assumes an infinite slope. Each point of tangency marks the division between stages II and III for the underlying production function for x_2 with x_1 set at some predetermined level x_1^*. The production function is

$$y = f(x_2; x_1 = x_1^*) \tag{5.8}$$

A line could be drawn that connects all points of zero slope on the isoquant map. This line is called a *ridge line* and marks the division between stages II and III for input x_1, under varying assumptions with regard to the quantity of x_2 that is used. This line is designated as ridge line 1 for x_1.

A second line could be drawn that connects all points of infinite slope on the isoquant map. This is also a ridge line, and marks the division between stages II and III for input x_2, under varying assumptions with regard to the quantity of x_1 that is used. This might be designated as ridge line 2 for x_2.

The two ridge lines intersect at the single point of maximum output. The neoclassical diagram, drawn from an isoquant map that consists of a series of concentric rings, appears not unlike a football. The ridge lines normally assume a positive slope. This is because the level of x_1 that results in maximum output increases as the assumption with regard to the fixed level for x_2 increases. Moreover, the level of x_2 that results in maximum output increases as the assumption with regard to the fixed level for x_1 is increased. The football appearance is the result of the underlying single-input production functions that assume the neoclassical three-stage appearance.

Notice that ridge line 1 connects points where the *MRS* is zero. Ridge line 2 connects points where the *MRS* is infinite. Finally, note that ridge lines can be drawn for only certain types of isoquant patterns or maps. For a ridge line to be drawn, isoquants must assume either a zero or an infinite slope. Look again at figure 5.5. Ridge lines can be drawn only for isoquants appearing in diagram B. For diagrams D, F, L, H and J, there are no points of zero or infinite slope. This suggests that the ridge lines do not exist. Moreover, this implies that the underlying families of production functions for x_1 and x_2 never achieve their respective maxima. Diagram L presents a unique problem. The right angle isoquants have either a zero or an infinite slope everywhere on either side of the angle. This would imply "thick" ridge lines. In this example, the underlying production functions for each input are but a series of points that represent the respective maximum output at each level of input use.

5.4 *MRS* and Marginal Product

The slope or *MRS* of an isoquant and the underlying productivity of the two families of production functions used to derive an isoquant map are closely intertwined. An algebraic relationship can be derived between the *MRS* and the marginal products of the underlying production functions.

Suppose that one wished to determine the change in output (called Δy) that would result if the use of x_1 were changed by some small amount (called Δx_1) and the use of x_2 were also changed by some small amount (called Δx_2). To determine the resulting change in output (Δy), two pieces of information would be needed. First, the exact magnitude of the changes in the use of each of the inputs x_1 and x_2. It is not possible to determine the change in output by merely summing the respective change in the use of the two inputs. An additional piece of information would also be needed. That information is the rate at which each input can be transformed into output. This rate is the marginal physical product of each input x_1 and x_2 (MPP_{x_1} and MPP_{x_2}).

The total change in output can be expressed as

$$(5.9) \qquad \Delta y = MPP_{x_1}\, \Delta x_1 + MPP_{x_2}\, \Delta x_2$$

The total change in output resulting from a given change in the use of two inputs is the change in each input multiplied by its respective *MPP*.

By definition, an isoquant is a line connecting points of equal output. Output does not change along an isoquant. The only way that output can change is to move on the isoquant map from one isoquant to another. Along any isoquant, Δy is exactly equal to zero. The equation for an isoquant can then be written as

$$(5.10) \qquad \Delta y = 0 = MPP_{x_1}\, \Delta x_1 + MPP_{x_2}\, \Delta x_2$$

Equation (5.10) can be rearranged such that

$$(5.11) \qquad MPP_{x_1}\, \Delta x_1 + MPP_{x_2}\, \Delta x_2 = 0$$

$$(5.12) \qquad MPP_{x_2}\, \Delta x_2 = -\, MPP_{x_1}\, \Delta x_1$$

Dividing both sides of equation (5.12) by Δx_1 gives us:

$$(5.13) \qquad MPP_{x_2}\, \Delta x_2/\Delta x_1 = -\, MPP_{x_1}$$

Dividing both sides by MPP_{x_2} yields:

$$(5.14) \qquad \Delta x_2/\Delta x_1 = -\, MPP_{x_1}/MPP_{x_2}$$

or[3]

$$(5.15) \qquad MRS_{x_1x_2} = -\, MPP_{x_1}/MPP_{x_2}$$

The marginal rate of substitution between a pair of inputs is equal to the negative ratio of the marginal products. Thus the slope of an isoquant at any point is equal to the negative ratio of the marginal products at that point, and if the marginal products for both inputs are positive at a point, the slope of the isoquant will be negative at that point. The replacing input (in this example, x_1) is the *MPP* on the top of the ratio. The replaced input (in this example, x_2) is the *MPP* on the bottom of the ratio. By again rearranging, we have

$$(5.16) \qquad MRS_{x_2x_1} = -MPP_{x_2}/MPP_{x_1}$$

The inverse slope of the isoquant is equal to the negative inverse ratio of the marginal products. Thus the slope (or inverse slope) of an isoquant is totally dependent on the *MPP* of each input.

In Section 5.3, a ridge line was defined as a line that connected points of zero or infinite slope on an isoquant map. Consider first a ridge line that connects points of zero slope on an isoquant map. This implies that $MRS_{x_1x_2} = 0$. But $MRS_{x_1x_2} = -MPP_{x_1}/MPP_{x_2}$. The only way for $MRS_{x_1x_2}$ to equal 0 is for MPP_{x_1} to equal zero. If MPP_{x_1} is zero, then the $TPPx_1$ (assuming a given value for x_2 again of x_2^*) must be maximum, and thus the underlying production function for x_1 holding x_2 constant at x_2^* must be at its maximum.

Now consider a ridge line that connects points of infinite slope on an isoquant map. This implies that $MRS_{x_1x_2}$ is infinite. Again $MRS_{x_1x_2} = -MPP_{x_1}/MPP_{x_2}$. $MRS_{x_1x_2}$ will become more and more negative as MPP_{x_2} comes closer and closer to zero. When MPP_{x_2} is exactly equal to zero, the $MRS_{x_1x_2}$ is actually undefined, since any number divided by a zero is undefined. However, note that when $MPP_{x_2} = 0$, then $MRS_{x_2x_1} = 0$, since MPP_{x_2} appears on the top, not the bottom of the ratio. A ridge line connecting points of infinite slope on an isoquant map connects points of zero inverse slope where the inverse slope is defined as $\Delta x_1/\Delta x_2$.

5.5 Partial and Total Derivatives and the Marginal Rate of Substitution

Consider again the Production function

(5.17) $$y = f(x_1,x_2)$$

For many production functions, the marginal product of x_1 (MPP_{x_1}) can be obtained only by making an assumption about the level of x_2. Similarly, the marginal product of x_2 cannot be obtained without making an assumption about the level of x_1. The MPP_{x_1} is defined as

(5.18) $$MPP_{x_1} = \partial f/\partial x_1 \mid x_2 = x_2^*$$

The expression $\partial y/\partial x_1$ is the partial derivative of the production function $y = f(x_1, x_2)$, assuming x_2 to be constant at x_2^*. It is the *MPP* function for the member of the family of production functions for x_1, assuming that x_2 is held constant at some predetermined level x_2^*.

Similarly, the MPP_{x_2}, under the assumption that x_1 is fixed at some predetermined level x_1^*, can be obtained from the expression

(5.19) $$MPP_{x_2} = \partial f/\partial x_2 \mid x_1 = x_1^*$$

In both examples the f refers to output or y.

The big difference between dy/dx_1 and $\partial y/\partial x_1$ is that the dy/dx_1 requires that no assumption be made about the quantity of x_2 that is used. dy/dx_1 might be thought of as the total derivative of the production function with respect to x_1, with no assumptions being made about the value of x_2. The expression $\partial y/\partial x_1$ is the partial derivative of the production function, holding x_2 constant at some predetermined level called x_2^*.

A few examples better illustrate these differences. Suppose that the production function is

(5.20) $$y = x_1^{0.5}x_2^{0.5}$$

Then

(5.21) $$MPP_{x_1} = \partial y/\partial x_1 = 0.5x_1^{-0.5}x_2^{0.5}$$

Since differentiation takes place with respect to x_1, x_2 is treated simply as if it were a constant in the differentiation process, and

(5.22) $$MPP_{x_2} = \partial y/\partial x_2 = 0.5x_2^{-0.5}x_1^{0.5}$$

Since differentiation takes place with respect to x_2, x_1 is treated as if it were a constant in the differentiation process.

Note that in this example, each marginal product contains the other input. An assumption needs to be made with respect to the amount of the other input that is used in order to calculate the respective *MPP* for the input under consideration. Again, the *MPP* of x_1 is conditional on the assumed level of use of x_2. The *MPP* of x_2 is conditional on the assumed level of use of x_1.

Now consider a slightly different production function

$$(5.23) \qquad y = x_1^{0.5} + x_2^{0.5}$$

In this production function, inputs are additive rather than multiplicative. The corresponding *MPP* for each input is

$$(5.24) \qquad MPP_{x_1} = \partial y/\partial x_1 = 0.5x_1^{-0.5}$$

$$(5.25) \qquad MPP_{x_2} = \partial y/\partial x_2 = 0.5x_2^{-0.5}$$

For this production function, MPP_{x_1} does not contain x_2, and MPP_{x_2} does not contain x_1. No assumption needs to be made with respect to the level of use of the other input in order to calculate the respective *MPP* for each input. Since this is true, this is an example where

$$(5.26) \qquad \partial y/\partial x_1 = dy/dx_1$$

and

$$(5.27) \qquad \partial y/\partial x_2 = dy/dx_2$$

The partial and the total derivatives are exactly the same for this particular production function.

Consider again the expression representing the total change in output

$$(5.28) \qquad \Delta y = MPP_{x_1}\,\Delta x_1 + MPP_{x_2}\,\Delta x_2$$

A Δ denotes a finite change, and the respective *MPP*'s for x_1 and x_2 are not exact but rather, merely approximations over the finite range.

Suppose that Δx_1 and Δx_2 become smaller and smaller. At the limit, the changes in x_1 and x_2 become infinitesimally small. If the changes in x_1 and x_2 are no longer assumed to be finite, at the limit, equation (5.28) can be rewritten as

$$(5.29) \qquad dy = MPP_{x_1}\,dx_1 + MPP_{x_2}\,dx_2$$

or

$$(5.30) \qquad dy = \partial y/\partial x_1\,dx_1 + \partial y/\partial x_2\,dx_2.$$

Equation (5.30) is the total differential for the production function $y = f(x_1,x_2)$.

Along an isoquant, there is no change in y, so $dy = 0$. An isoquant by definition connects points representing the exact same level of output. The total differential is equal to zero. The exact $MRS_{x_1x_2}$ at $x_1 = x_1^*$ and $x_2 = x_2^*$ is

$$(5.31) \qquad MRS_{x_1x_2} = dx_2/dx_1 = -\,MPP_{x_1}/MPP_{x_2} = -\,(\partial y/\partial x_1)/(\partial y/\partial x_2)$$

Similarly, the exact $MRS_{x_2x_1}$ is defined as

(5.32) $$MRS_{x_2x_1} = dx_1/dx_2 = -\, MPP_{x_2}/MPP_{x_1} = -\,(\partial y/\partial x_2)/(\partial y/\partial x_1)$$

The total change in the *MPP* for x_1 can be obtained by dividing the total differential of the production function by dx_1. The result is

(5.33) $$dy/dx_1 = \partial y/\partial x_1 + (\partial y/\partial x_2)(dx_2/dx_1)$$

Equation (5.33) is the total derivative of the production function $y = f(x_1,\ x_2)$. It recognizes specifically that the productivity of x_1 is not independent of the level of x_2 that is used.

The total change in output as a result of a change in the use of x_1 is the sum of two effects. The direct effect ($\partial y/\partial x_1$) measures the direct impact of the change in the use of x_1 on output. The indirect effect measures the impact of a change in the use of x_1 on the use of $x_2(dx_2/dx_1)$, which in turn affects y (through $\partial y/\partial x_2$).

The shape of the isoquant is closely linked to the production functions that underlie it. In fact, if the underlying production functions are known, it is possible to determine with certainty the exact shape of the isoquant and its slope and curvature at any particular point. The marginal rate of substitution, or slope of the isoquant at any particular point, is equal to the negative ratio of the marginal products of each input at that particular point. If the marginal product of each input is positive but declining, the isoquant normally will be bowed inward or convex to the origin.

The curvature of an isoquant can be determined by again differentiating the marginal rate of substitution with respect to x_1.[4] If the sign on the derivative is positive, the isoquant is bowed inward and exhibits a diminishing marginal rate of substitution. It is also possible for isoquants to be bowed inward in certain instances where the marginal product of both inputs is positive but not declining. Examples of this exception are contained in Chapter 10.

Diagrams B to D of Figure 5.2 all represent isoquants that are downward sloping, and hence dx_2/dx_1 is negative in each case. In diagram B, $d(dx_2/dx_1)/dx_1$ is positive, which is consistent with a a diminishing marginal rate of substitution. Diagram C illustrates a case in which $d(dx_2/dx_1)/dx_1$ is negative, resulting in isoquants concave to the origin, while for diagram D, $d(dx_2/dx_1)/dx_1$ is zero, and the isoquants have a constant slope with no diminishing or increasing marginal rates of substitution.

The derivative dx_2/dx_1 is positive in diagram *E* and undefined in diagram F. In diagram A, the isoquants have both positive and negative slopes, and the sign on dx_2/dx_1 depends on the particular point being evaluated.

Thus the concept of an isoquant with a particular marginal rate of substitution at any particular point and the concept of a production function with marginal products for each input are not separate and unrelated. Rather the slope, curvature and other characteristics of an isoquant are uniquely determined by the marginal productivity of each input in the underlying production function.

5.6 Concluding Comments

This chapter has been concerned with the physical and technical relationships underlying production in a setting in which two inputs are used in the production of a single output. An isoquant is a line connecting points of equal output on a graph with the axes represented by the two inputs. The slope of an isoquant is referred to as a marginal rate of substitution

(*MRS*). The *MRS* indicates the extent to which one input substitutes for another as one moves from one point to another along an isoquant representing constant output. The marginal rate of substitution is usually diminishing. In other words, when output is maintained at the constant level represented by the isoquant, as units of input x_1 used in the production process are added, each additional unit of x_1 that is added replaces a smaller and smaller quantity of x_2.

A diminishing marginal rate of substitution between two inputs normally occurs if the production function exhibits positive but decreasing marginal product with respect to incremental increases in the use of each input, a condition normally found in stage II of production. Thus the marginal rate of substitution is closely linked to the marginal product functions for the inputs. This chapter has illustrated how the marginal rate of substitution can be calculated if the marginal products for the inputs are known.

Notes

1. Not all textbooks define the marginal rate of substitution as the slope of the isoquant. A number of economics texts define the marginal rate of substitution as the *negative* of the slope of the isoquant. That is, $MRS_{x_1x_2} = -\Delta x_2/\Delta x_1$ (or $-dx_2/dx_1$). Following this definition, a downward-sloping isoquant exibits a positive marginal rate of substitution.

2. or $-\Delta x_2/\Delta x_1$.

3 If the marginal rate of substitution is defined as the negative of the slope of the isoquant, it is equal to the ratio of the marginal products, not the negative ratio of the marginal products.

4 Let the Marginal rate of Substitution (*MRS*) of x_1 for x_2 be defined as dx_2/dx_1. Then the total differential of the *MRS* is defined as

$$dMRS = (\partial MRS/\partial x_1)dx1 + (\partial MRS/\partial x_2)dx_2$$

The total derivative with respect to x_1 is

$$dMRS/dx_1 = (\partial MRS/\partial x_1) + (\partial MRS/\partial x_2)(dx_2/dx_1)$$

or

$$dMRS/dx_1 = (\partial MRS/\partial x_1) + (\partial MRS/\partial x_2)\cdot MRS$$

As units of x_1 are increased, the total change in the marginal rate of substitution ($dMRS/dx_1$) is the sum of the direct effect of the change in the use of x_1 on the *MRS* [$(\partial MRS/\partial x_1)$] plus the indirect effect [$(\partial MRS/\partial x_2)MRS$]. The indirect effect occurs because if output is to remain constant on the isoquant, an increase in x_1 must be compensated with a decrease in x_2.

Problems and Exercises

1. The following combinations of x_1 and x_2 all produce 100 bushels of corn. Calculate the $MRS_{x_1x_2}$ and the $MRS_{x_2x_1}$ at each midpoint.

Combination	Units of x_1	Units of x_2	$MRS_{x_1 x_2}$	$MRS_{x_2 x_1}$
A	10	1		
			________	________
B	5	2		
			________	________
C	3	3		
			________	________
D	2	4		
			________	________
E	1.5	5		

2. For the production function

$$y = 3x_1 + 2x_2$$

find

a. The *MPP* of x_1.
b. The *MPP* of x_2.
c. The marginal rate of substitution of x_1 for x_2.

3. Draw the isoquants for the production function given in Problem 1.

4. Find those items listed in Problem 2 for a production function given by

$$y = ax_1 + bx_2$$

where a and *b* are any constants. Is it possible for such a production function to produce isoquants with a positive slope? Explain.

5. Suppose that the production function is given by

$$y = x_1^{0.5}x_2^{0.333}$$

find

a. The *MPP* of x_1.
b. The *MPP* of x_2.
c. The Marginal rate of substitution of x_1 for x_2.

d. Draw the isoquants for this production function. Do they lie closer to the x_1 or the x_2 axis? Explain. What relationship does the position of the isoquants have relative to the productivity of each input?

6. Suppose that the production function is instead

$$y = 2x_1^{0.5}x_2^{0.333}$$

find

a. The *MPP* of x_1.
b. The *MPP* of x_2.
c. The Marginal rate of substitution of x_1 for x_2.
d. What happens to the position of the isoquants relative to those drawn for Problem 5? Compare your findings with those found for problem 5.