The reason the variable cost function appears to be a mirror- image production function with its axes reversed now becomes clear. The production function reflects the fact that each incremental unit of input produces less and less additional output. The corresponding variable cost function reflects the fact that incremental units of output become more and more costly in terms of input requirements.

The fertilizer response data contained in table 2.5 in chapter 2 is presented in a manner in which this dual relationship can be readily observed (Table 4.2).

Table 4.2 Corn Response to Nitrogen Fertilizer

Nitrogen x	Corn y	Exact MPP	$1/MPP$	$v°$	$v°/MPP$ (MC)	Exact APP	$1/APP$	$v°/APP$ (AVC)
0	0.0	0.7500	1.33	0.15	0.200	a	a	a
20	16.496	0.8904	1.12	0.15	0.168	0.8248	1.21	0.182
40	35.248	0.9756	1.03	0.15	0.154	0.8812	1.13	0.170
60	55.152	1.0056	0.99	0.15	0.149	0.9192	1.09	0.163
80	75.104	0.9804	1.02	0.15	0.153	0.9388	1.07	0.160
100	94.000	0.9000	1.11	0.15	0.167	0.9400	1.06	0.160
120	110.736	0.7644	1.31	0.15	0.196	0.9228	1.08	0.163
140	124.208	0.5736	1.74	0.15	0.262	0.8872	1.13	0.169
160	133.312	0.3276	3.05	0.15	0.458	0.8332	1.20	0.180
180	136.944	0.0264	37.88	0.15	5.682	0.7608	1.31	0.197
200	134.000	-0.3300	-3.03	0.15	-0.454	0.6700	1.49	0.224
220	123.376	-0.7416	-1.35	0.15	-0.202	0.5608	1.78	0.267
240	103.968	-1.2084	-0.83	0.15	-0.124	0.4332	2.31	0.346

[a]Undefined. Errors due to rounding.

Compared with Table 2.5, the data appear inverted. In Chapter 2, average physical product was defined as y/x, and marginal physical product was defined as $\Delta y/\Delta x$. Now x/y and $\Delta x/\Delta y$ have been calculated.

If $y/x = APP$, then x/y must be $1/APP$. The expression x/y represents the average cost for nitrogen to produce the incremental unit of output, but the cost is expressed in terms of physical units of the input, not in dollar terms. This cost is equal to $1/APP$. This cost can be converted to dollar units by multiplying by the price of nitrogen, earlier called $v°$. The result is the average variable cost for nitrogen per unit of output $AVC_n = v°/APP$.

If $\Delta y/\Delta x = MPP$, then $\Delta x/\Delta y$ must be $1/MPP$. The expression $\Delta x/\Delta y$ represents the marginal cost for nitrogen to produce the incremental unit of output, but again the cost is represented in physical terms, not in dollar terms. This cost is equal to $1/MPP$. This cost can again be converted to dollar units by multiplying by the price of nitrogen or $v°$. The result is the marginal cost for nitrogen per unit of output $MC_n = v°/MPP$.

At a nitrogen application rate of 180 pounds per acre, marginal cost is $5.68 per bushel of corn produced. If corn is selling for $4.00 per bushel, the incremental bushel of corn costs $5.68 but returns only $4.00. However, at a nitrogen application rate of 160 pounds per acre, the marginal cost of the incremental bushel of corn is but $0.458. If corn is selling for $4.00 per bushel, the difference of $3.54 is profit to the farmer. The farmer could increase profits by increasing the use of nitrogen fertilizer until the marginal cost associated with the production of the incremental bushel of corn just equals marginal revenue. This should be at a nitrogen application level of slightly less than 180 pounds per acre- 179.322 pounds to be exact. That is exactly the solution found in Chapter 3. It makes no difference whether *VMP* is equated to *MFC* or *MR* is equated to *MC*. The solution provides the farmer with exactly the same conclusion with regard to how much input should be used. The solution to the profit-maximization problem is the same regardless of whether it is done on the output or input side.

4.4 The Inverse of a Production Function

Any production function has an underlying dual cost function or correspondence (Figure 4.6). The production function has input (nitrogen or x) on the horizontal axis and output (corn or y) on the vertical axis. The corresponding cost function expressed in physical terms is the production function with the axes reversed. The result is the inverse production function, or cost function expressed in physical terms. This cost function is dual to the production function.

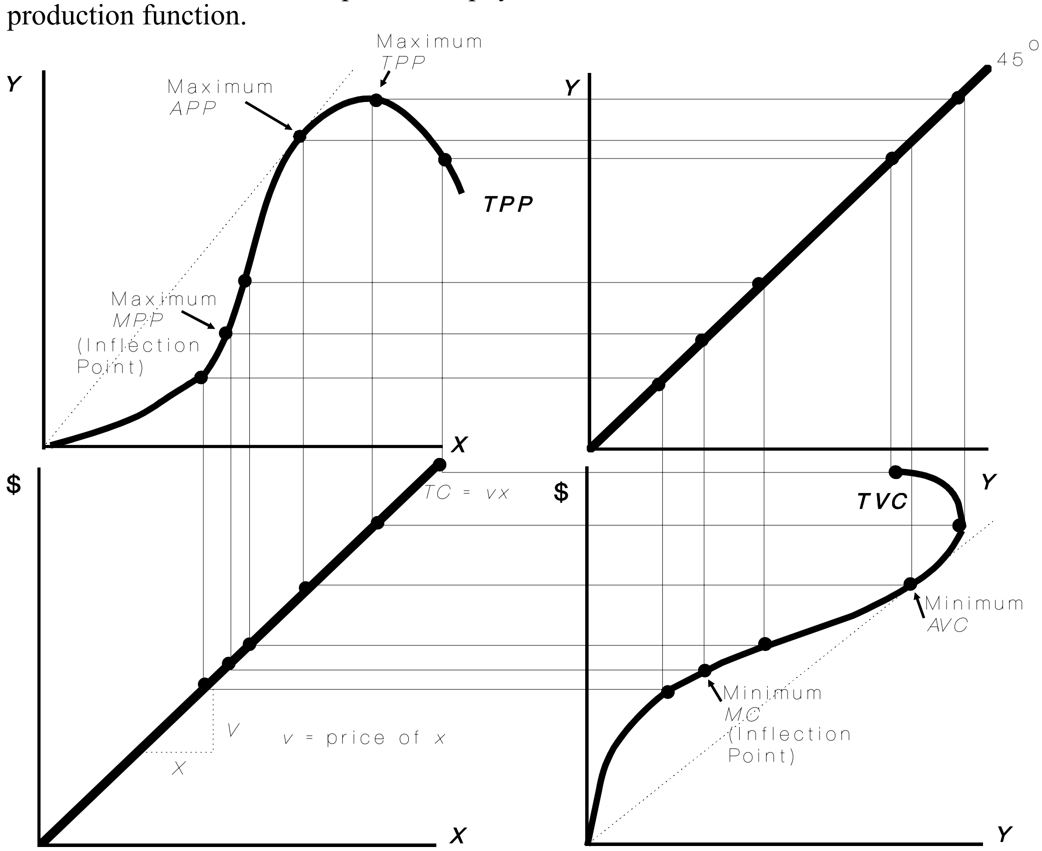

Figure 4.6 A Cost Function as an Inverse Production Function

Note that this function is in many ways the mirror image of the underlying production function. If the production function is increasing at an increasing rate, the inverse production function increases at a decreasing rate. If the production function is increasing at a decreasing rate, the inverse production function increases at an increasing rate.

Inverses to production functions for some simple functions might readily be calculated. All that is required is to solve the function in terms of the *x* instead of y. For example, suppose that the production function is

(4.24) $y = 2x$

The corresponding inverse production function is

(4.25) $x = y/2 = 0.5y$

If the production function is

(4.26) $y = bx$ where b is any number

the corresponding inverse production function is

(4.27) $x = y/b$.

Suppose that the production function is

(4.28) $y = x^{0.5}$

The corresponding inverse production function is

(4.29) $x = y^{1/0.5} = y^2$

And if the production function is

(4.30) $x = y^{1/0.5} = y^2$

The inverse production function is

(4.31) $x = y^{1/2} = y^{0.5}$

For the production function

(4.32) $y = ax^b$

The corresponding inverse function is

(4.33) $x = (y/a)^{1/b}$

In each of these examples, the inverse function contains all the coefficients contained in the original production function and can be converted into true variable cost functions by multiplying by the constant price ($v°$) of the input x. If these functions were drawn, the vertical axis would then be in terms of dollars rather than physical units of the input x.

It is therefore not necessary to know the physical quantities of the inputs that are used in the production process in order to determine the coefficients of the production function. If the cost function is known, it is frequently possible to determine the underlying production function.

A general rule is that if the production function is

(4.34) $y = f(x)$

then the corresponding inverse production function is

(4.35) $x = f^{-1}(y)$

Not all production functions can be inverted into another function to obtain the corresponding dual cost function. Any production function that includes both increasing and decreasing *TPP* will not have a inverse function, but only an inverse correspondence. The neoclassical production function is an example. The inverse in Figure 4.6 is actually a correspondence, but not a function.

The total cost for the input expressed in terms of units of output is obtained by multiplying the inverse function times the input price. Suppose that

(4.36) $y = f(x)$

Then

(4.37) $x = f^{-1}(y)$.

Multiplying by $v°$ results in the total cost (TC_x) for the input (x or nitrogen) from the production function for corn [$y = f(x)$]

(4.38) $v°x = TFC = TC_x = v°f^{-1}(y)$

4.5 Linkages between Cost and Production Functions

Suppose that the price of the input is $v°$ and the production function is

(4.39) $y = 2x$

Then $MPP = APP = 2$, and $MC_x = AVC_x = v°/2$.

If the production function is

(4.40) $y = bx$

Then $MPP = APP = b$, and $MC_x = AVC_x = v°/b$.

If the production function is

(4.41) $y = x^{0.5}$

then $MPP = 0.5/x^{0.5}$, $APP = x^{0.5}/x = x^{0.5}x^{-1} = x^{-0.5} = 1/x^{0.5}$,

$MC_x = (v°x^{0.5})/0.5 = 2v°x^{0.5}$, and $AVC_x = v°x^{0.5}$.

If *MPP* is precisely one half of *APP*, then MC_x will be precisely twice AVC_x. If the elasticity of production (E_p) is defined as the ratio *MPP*/*APP*, then $1/E_p$ is the ratio of MC_x/AVC_x.

If the production function is

(4.42) $y = ax^b$

then the inverse production function is

(4.43) $x = (y/a)^{1/b}$

(4.44) $MPP = abx^{b-1}$

(4.45) $APP = ax^{b-1}$

(4.46) $E_p = b$

(4.47) $MC_x = v°/abx^{b-1}$

(4.48) $AVC_x = v°/ax^{b-1}$

(4.49) ratio of $MC_x/AVC_x = 1/b$

Some important relationships between *APP*, *MPP*, *MC*, and *AVC* become clear. In stage I, *MPP* is greater than *APP* and E_p is greater than 1. As a result, MC_x must be less than AVC_x in stage I. The exact proportion is defined by $1/E_p$. In stages II and III, *MPP* is less than *APP*, and as a result, E_p is less than 1. Therefore, MC_x must be greater than AVC_x. The exact proportion is again defined by $1/E_p$. At the dividing point between stages I and II, $MPP = APP$ and $E_p = 1$. $1/E_p = 1$ and $MC_x = AC_x$, and at the dividing point between stages II and III, $MPP = 0$, $E_p = 0$, $1/E_p$ is undefined, and MC_x is undefined.

4.6 The Supply Function for the Firm

The profit-maximizing firm will equate marginal cost with marginal revenue. If the firm operates under conditions of pure competition, marginal revenue will be the same as the constant price of the output. If the farmer produces but one output, the marginal cost curve that lies above average variable cost will be the supply curve for the farm. Each point on the marginal cost curve above average variable cost consists of a point of profit maximization if the output sells for the price associated with the point. The supply curve or function for the farm will consist of the series of profit maximizing points under alternative assumptions with respect to marginal revenue or the price of the product.

Consider, for example, the production function

(4.50) $y = ax^b$

The inverse production function is

(4.51) $x = (y/a)^{1/b}$

Variable cost is defined as

(4.52) $VC = vx = v(y/a)^{1/b}$

Marginal cost can be found by differentiating equation (4.52) with respect to *y*

(4.53) $MC = d(vx)/dy = (1/b)vy^{(1/b)-1}a^{-1/b}$

(4.54) $MC = (1/b)vy^{(1-b)/b}a^{-1/b}$

Equating marginal cost with marginal revenue or the price (*p*) of the product yields

(4.55) $p = (1/b)vy^{(1-b)/b}a^{-1/b}$

$MR = MC$

Solving equation (4.55) for *y* yields the supply function for the firm

(4.56) $y = (bp)^{b/(1-b)} v^{-b/(1-b)} a^{(1/b)(b/(1-b))}$

The elasticity of supply with respect to the product price is

(4.57) $(dy/dp)(p/y) = b/(1 - b)$

The elasticity of supply is positive when b is less than 1.

The elasticity of supply with respect to the input price is

(4.58) $(dy/dv)(v/y) = -b/(1 - b)$

The elasticity of supply with respect to the input price is negative if b is less than 1.

Average (variable) cost is

(4.59) $AC = vx/y = [v(y/a)^{1/b}]/y = vy^{(1-b)/b}\, a^{-1/b}$

Since marginal cost is

(4.60) $MC = (1/b)vy^{(1-b)/b}a^{-1/b}$

The ratio of marginal to average cost is

(4.61) $MC/AC = 1/b = 1/E_p$

In this example, the marginal and average cost functions must remain in fixed proportion to each other. The proportion is equal to 1 over the elasticity of production for the production function. Figure 4.7 illustrates the aggregate supply function derived for a production function in which b is less than 1, and the product price is set at alternative levels. The supply function is the portion of the marginal cost function above average variable cost. However, in this example marginal cost lies above average variable cost everywhere and is at the fixed ratio to average variable cost of $1/b$.

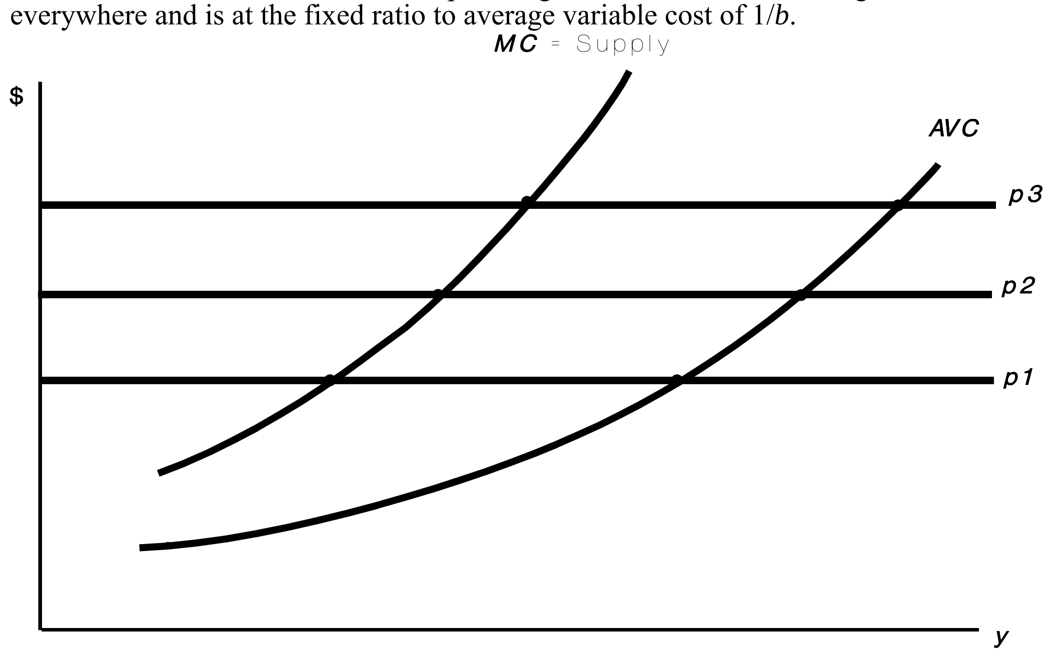

Figure 4.7 Aggregate Supply When the Ratio *MC/AC* = 1/*b* and *b* is Less Than 1

4.7 Concluding Comments

Profit maximizing conditions for the firm have been derived. Profits are maximum when the level of output chosen is where marginal cost equals marginal revenue. The cost function is the inverse of the production function that underlies it multiplied by the price of the input. A close linkage thus exists between the coefficients of the production function and those of the underlying cost function. The firm's supply curve can be derived from the equilibrium $MC = MR$ conditions and is represented by the marginal cost curve above average variable cost. Expressions for elasticities of supply with respect to product and input prices can be obtained from the equilibrium conditions.

Problems and Exercises

1. Explain the difference between total value of the product (*TVP*) and total revenue (*TR*).

2. Explain the difference between total cost (*TC*) and total factor cost (*TFC*).

3. Suppose that the price of the input x is \$3. Total fixed costs are \$200. Fill in the blanks.

x (Input)	y (Output)	*TVC*	*TC*	*MC*	*AVC*	*AC*
0	0	___	___		___	___
				___		
10	50	___	___		___	___
				___		
25	75	___	___		___	___
				___		
40	80	___	___		___	___
				___		
50	85	___	___		___	___

4. Suppose that the production function is

$$y = 3x^{0.5}$$

The price of the input is \$3. per unit, and total fixed costs are \$50. Find and graph the functions that represent.

a. *MPP*
b. *APP*
c. *AVC*
d. *AC* (or *ATC*)
e. *MC*

Suppose that the output price is \$5. Find:

f. *AVP*
g. *VMP*
h. *MFC*

5. Using the data contained in Problem 4, find the profit- maximizing level of input use by equating *VMP* and *MFC*.

6. Using the data contained in Problem 4, find the profit-maximizing output level by equating *MR* and *MC*. What is the relationship between the profit- maximizing output level and the profit-maximizing input level?

7. Draw a three-stage production function on a sheet of paper. Now turn the paper so that the input x is on the vertical axis and output y is on the horizontal axis. Now turn the sheet of paper over and hold the sheet of paper up to a light. Look at the production function through the back side. What you see is the cost function that underlies the production function, with costs expressed in physical units of input use rather than dollars. If input prices are constant, the vertical axis can be converted to dollars by multiplying the physical units of input by the corresponding input price.

8. Draw a graph of the corresponding total cost correspondence when fixed costs are zero, the input costs $2 per unit, and the production function is given by

$$y = 0.4x + 0.09x^2 - 0.003x^3$$

Reference

Viner, Jacob, "Cost Curves and Supply Curves," *Zeitschrift fur Nationalokonomie* III (1931) pp. 23-46. Also in American Economics Association, *Readings in Price Theory*, K. E. Boulding and G. J. Stigler eds. Homewood, Ill.: Richard D. Irwin, 1952.