3

Profit Maximization with One Input and One Output

This chapter introduces the fundamental conditions for profit maximization in the single input single output or factor- product case. The concept of the total value of the product and the value of the marginal product is introduced. The value of the marginal product and the marginal factor cost are equal at the point of profit maximization. Profits are normally maximum when the implicit value of the last dollar spent on an input is one dollar. Stages of production are described, and an explanation of why a farmer would choose to operate in stage II is given.

Key terms and definitions:

Total Value of the Product (*TVP*)
Profit
Revenue
Cost Function
Value of the Marginal Product (*VMP*)
Total Factor Cost (*TFC*)
Marginal Factor Cost (*MFC*)
Average Value of the Product (*AVP*)
First Order Condition
Second Order Condition
Necessary Condition
Sufficient Condition
Maximum Profits
Minimum Profits
Stages of Production (I, II, and III)
Rational Stage
Irrational Stage
Implicit Worth
Imputed Value
Shadow Price

3.1 Total Physical Product Versus Total Value of the Product

As indicated in Chapter 2, the output (y) from a production function can be also called total physical product (*TPP*). If a firm such as a farm is operating under the purely competitive conditions, the individual farm firm can sell as little or as much output as desired at the going market price. The market price, p, does not vary. A constant price might be called $p°$. Since

(3.1) $$TPP = y,$$

both sides of equation (3.1) can be multiplied by the constant price $p°$. The result is

(3.2) $$p°TPP = p°y.$$

The expression $p°y$ is the total revenue obtained from the sale of the output y and is the same as $p°TPP$. The expression $p°TPP$ is sometimes referred to as the *total value of the product* (*TVP*). It is a measure of output (*TPP*) transformed into dollar terms by multiplying by $p°$. For a farmer, it represents the revenue obtained from the sale of a single commodity, such as corn or beef cattle. If the output price is constant, the *TVP* function has the same shape as the *TPP* function, and only the units on the vertical axis have changed (Figure 3.1).

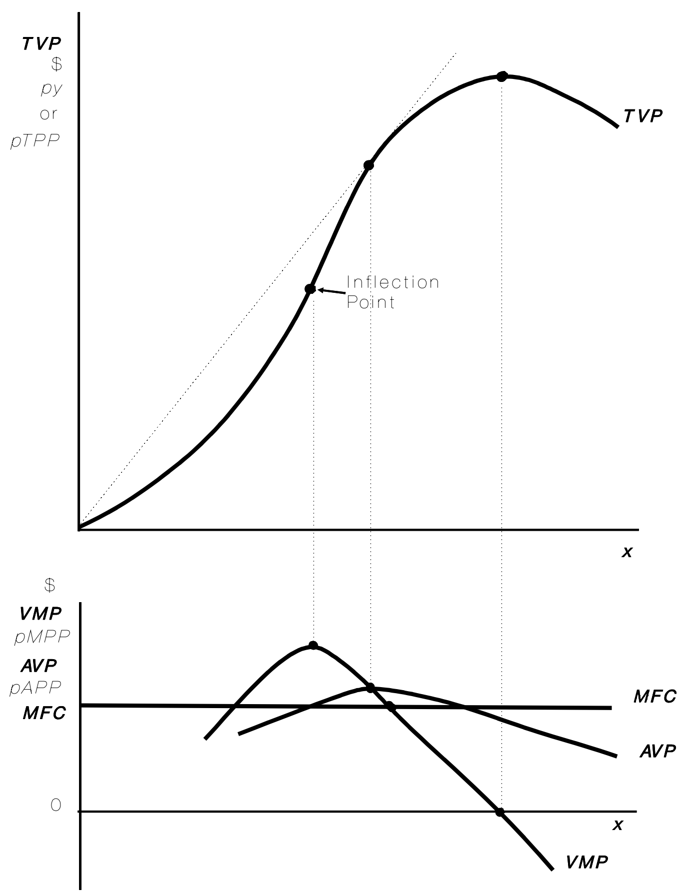

Figure 3.1 The Relationship Between *TVP, VMP, AVP,* and *MFC*

3.2 Total Factor or Resource Cost

Suppose that production requires only one input. Suppose also that a farmer can purchase as much of this input as is needed at the going market price v. The purely competitive environment is again assumed to exist. The market price for the input, factor, or resource does not vary with the amount that an individual farmer purchases. Thus the market price might be designated as $v°$. The term $v°x$ can be referred to as *total factor cost* or *total resource cost*. These terms are sometimes abbreviated as *TFC* or *TRC*. Hence

$$(3.3) \qquad TRC = TFC = v°x.$$

The *TFC* function has a constant slope, in this case equal to $v°$. Another way of looking at $v°$ is that it is the increase in cost associated with the purchase of an additional unit of the input. The increase in cost is equal to the price of the input $v°$.

3.3 Maximizing the Difference between Returns and Costs

A farmer might be interested in maximizing net returns or profit. Profit (Π) is the total value of the product (*TVP*) less the total factor cost (*TFC*). The profit function for the farmer can be written as

$$(3.4) \qquad \Pi = TVP - TFC.$$

Or, equation (3.4) might be written as

$$(3.5) \qquad \Pi = p°y - v°x$$

Figure 3.2 illustrates the *TVP* function, the *TFC* function, and the profit function, assuming that the underlying production function is of the neoclassical form as described in detail in chapter 2. The profit function is easily drawn, since it is a graph representing the vertical difference between *TVP* and *TFC*. If *TFC* is greater than *TVP*, profits are negative and the profit function lies below the horizontal axis. These conditions hold at both the very early stages as well as the late stages of input use. Profits are zero when *TVP* = *TFC*. This condition occurs at two points on the graph, where the profit function cuts the horizontal axis. The profit function has a zero slope at two points. Both of these points correspond to points where the slope of the *TVP* curve equals the slope of the *TFC* curve. The first of these points corresponds to a point of profit minimization, and the second is the point of profit maximization, which is the desired level of input use.

The slope of the profit function can be expressed (using Δ notation) as $\Delta\Pi/\Delta x$. Hence

$$(3.7) \qquad \Delta\Pi/\Delta x = \Delta TVP/\Delta x - \Delta TFC/\Delta x$$

The slope of the function is equal to zero at the point of profit maximization (and at the point of profit minimization−more about this later). Therefore, the slope of the *TVP* function ($\Delta TVP/\Delta x$) must equal the slope of the *TFC* function ($\Delta TFC/\Delta x$) at the point of profit maximization.

3.3 Value of the Marginal Product and Marginal Factor Cost

The value of the marginal product (*VMP*) is defined as the value of the incremental unit of output resulting from an additional unit of x, when y sells for a constant market price $p°$.

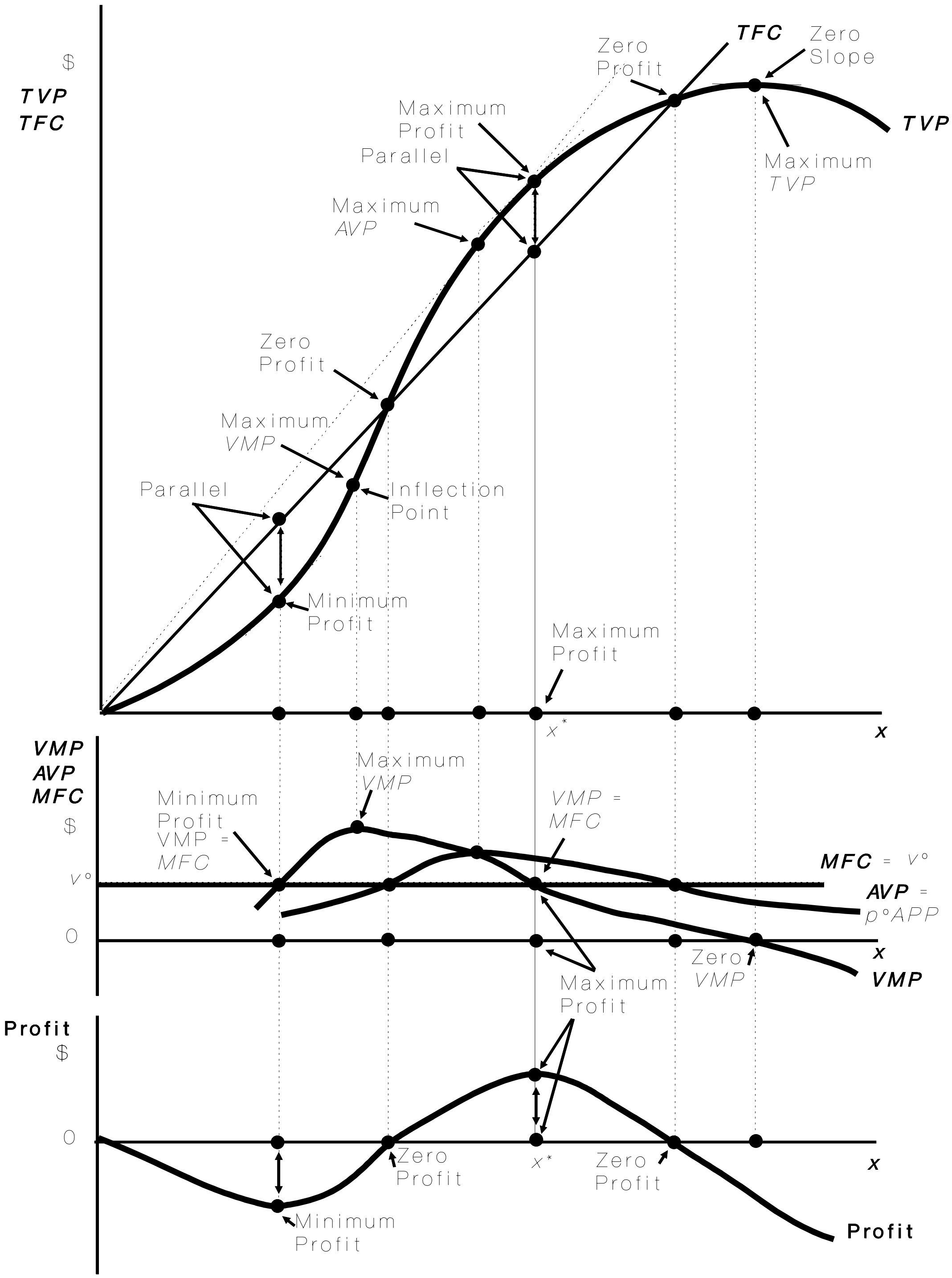

Figure 3.2 ***TVP*, *TFC*, *VMP*, *MFC*, and Profit**

The *VMP* is another term for the slope of the *TVP* function under a constant product price assumption. In other words, *VMP* is another name for $\Delta TVP/\Delta x$. Since $TVP = p°TPP$, the *VMP* must equal $p°\ \Delta TPP/\Delta x$. But $\Delta TPP/\Delta x = MPP$. Therefore, *VMP* must be equal to $p°MPP$.

The *marginal factor cost* (*MFC*), sometimes called *marginal resource cost* (*MRC*), is defined as the increase in the cost of inputs associated with the purchase of an additional unit of the input. The *MFC* is another name for the slope of the *TFC* function. Note that if the input price is assumed to be constant at $v°$, then $MFC = v°$These relationships might also be expressed by

(3.6) $$\Pi = TVP - TFC$$

3.4 Equating *VMP* and *MFC*

The points where the slope of *TVP* equals the slope of *TFC* corresponds either to a point of profit minimization or a point of profit maximization. These points are also defined by

(3.8) $$p°\ MPP = VMP = MFC = v°$$

Figure 3.2 also illustrates these relationships. *MFC*, being equal to a constant $v°$, is a straight line. Notice that *APP* can be multiplied by the price of the product $p°$, and is sometimes referred to as average value of the product (*AVP*). It is equal to $p°APP$ or $p°y/x$, or in this case \$4.00·(*APP*).

There are many ways of rearranging the equation $p°\ MPP = v°$. One possibility is to divide both sides of the equation by the output price $p°$. Then at the point of maximum profit, *MPP* must be equal to $v°/p°$, the factor/product price ratio. Another possibility is to divide both sides of the equation by average physical product (*APP*) or y/x. The profit maximizing condition would then be given by

(3.9) $$MPP/APP = (v°x)/(p°y)$$

However, MPP/APP is the elasticity of production for x. The term $v°x$ represents total factor cost. The term $p°y$ represents total revenue to the farm, since it is the price of the output times output. At the point of profit maximization, the elasticity of production will be exactly equal to the ratio of total factor cost to total revenue for the farm.

The data contained in Table 2.5 can be used to determine how much nitrogen fertilizer should be applied to the corn. To do this, prices must be assigned both to corn and to the nitrogen fertilizer. Assume that the price of corn is \$4.00 per bushel and that nitrogen costs \$0.15 per pound. These data are presented in Table 3.1.

Several comments can be made with regard to the data contained in Table 3.1. First, at a nitrogen application level of 180 pounds per acre, the *MPP* of nitrogen is calculated to be 0.0264. The number is very close to zero and suggests that maximum yield is at very close to an application rate of 180 pounds per acre. The *MPP* is calculated by first differentiating the *TPP* or production function to find the corresponding *MPP* function

(3.10) $$y = 0.75x + 0.0042x^2 - 0.000023x^3$$

(3.11) $$dy/dx = 0.75 + 0.0084x - 0.000069x^2$$

Table 3.1 Profit Maximization in the Application of Nitrogen to Corn

Quantity of Nitrogen	Corn Yield (bu/acre)	*MPP* of Nitrogen	$p°$ ($)	*VMP* ($p°MPP$)	*MFC* ($v°$) ($)	Profit (π) ($)
0	0.0	0.7500	4.00	3.0000	0.15	0.0
20	16.496	0.8904	4.00	3.5616	0.15	62.98
40	35.248	0.9756	4.00	3.9024	0.15	134.99
60	55.152	1.0056	4.00	4.0224	0.15	211.61
80	75.104	0.9804	4.00	3.9216	0.15	288.42
100	94.000	0.9000	4.00	3.6000	0.15	361.00
120	110.736	0.7644	4.00	3.0576	0.15	424.94
140	124.208	0.5736	4.00	2.2944	0.15	475.83
160	133.312	0.3276	4.00	1.3104	0.15	509.25
180	136.944	0.0264	4.00	0.1056	0.15	520.78
200	134.000	−0.3300	4.00	−1.3200	0.15	506.00
220	123.376	−0.7416	4.00	−2.9664	0.15	460.50
240	103.968	−1.2084	4.00	−4.8336	0.15	379.87

Then the *MPP* at $x = 180$ is

$$MPP = 0.75 + 0.0084(180) - 0.000069(180)^2 = 0.0264$$

However, since at the point where $x = 180$, *MPP* is still positive, the true yield maximum must be at a nitrogen application level of slightly greater than 180 pounds per acre, where $dy/dx = MPP = 0$.

Profits appear to be greatest at a nitrogen application rate of 180 pounds per acre. However, at 180 pounds per acre, the return from the incremental unit of nitrogen (the *VMP* of *x*) is $0.1056, whereas its cost is $0.15. The results suggest that the last unit of nitrogen that was used returned less than it cost. The profit-maximizing level of nitrogen use must be at slightly less than 180 pounds per acre. If the input is not free, the profit-maximizing level of input use will always be somewhat less than the level of input use that maximizes the production function. In many instances, however, the difference between the profit-maximizing level of input use and the yield-maximization level of input use may not be very large. In this case the incremental pound of nitrogen must return corn worth only $0.15 in order to cover its cost. If corn sells for $4.00 per bushel, this is but $0.15/$4.00 = 0.0375 bushel of corn from the incremental pound of nitrogen.

The difference between the level of nitrogen needed to maximize profits versus the amount needed to maximize output and total revenue does not appear to be very great. If nitrogen were free, there would be no difference at all. As the price of nitrogen increases, the level of nitrogen required to maximize profits is reduced. For example, if nitrogen sold for $1.00 per pound, the last pound of nitrogen applied would need to produce 0.25 bushel of corn at $4.00 per bushel. In general, the distinction between the point representing maximum profit and the point representing maximum revenue becomes more and more important as input prices increase.

If the price of fertilizer is very cheap, the farmer will lose little by fertilizing at a level consistent with maximum yield rather than maximum profit. However, if fertilizer is expensive, the farmer needs to pay close attention to the level of input use that maximizes profits. The same analysis holds true for other inputs used in agricultural production processes for both livestock and crops.

Profits per acre of corn in this example appear to be extraordinarily high, but remember that the production function describing corn yield response to the application of nitrogen assumes that all other inputs are fixed and given. The cost per acre for these inputs could be calculated. Suppose that this turns out to be $450 per acre. This value could be subtracted from each value in the profit column. Conclusions with regard to the profit maximizing level of nitrogen use would in no way be altered by doing this.

3.5 Calculating the Exact Level of Input Use to Maximize Output or Profits

The exact level of input use required to maximize output (y) or yield can sometimes be calculated. Several examples will be used to illustrate problems in doing this with various production functions. From the earlier discussion it is apparent that if output is to be at its maximum, the *MPP* of the function must be equal to zero. The last unit of input use resulted in no change in the output level and requires that $MPP = dy/dx = 0$ at the point of output maximization.

Suppose the production function

$$y = 2x \tag{3.12}$$

In this case

$$MPP = dy/dx = 2 \text{ (and not zero!)} \tag{3.13}$$

The *MPP* is always 2, and 2 cannot be equal to zero, and the production function has no maximum. A more general case might be the production function

$$y = bx \tag{3.14}$$

$$MPP = dy/dx = b = 0\ ? \tag{3.15}$$

If b were zero, regardless of the amount of x that was produced, no y would result. For any positive value for b, the function has no maximum. Now suppose the production function

$$y = x^{0.5} \tag{3.16}$$

$$MPP = dy/dx = 0.5\,x^{-0.5} = 0\ ? \tag{3.17}$$

The only value for x is zero for which the *MPP* would also be equal to 0. Again, this function has no maximum. In general, any function of the form

$$y = ax^b \tag{3.18}$$

where a and b are positive numbers, has no maximum.

Now suppose a production function

$$y = 10 + 8x - 2x^2 \tag{3.19}$$

$$dy/dx = 8 - 4x = 0 \tag{3.20}$$

$$4x = 8 \tag{3.21}$$

$$x = 2 \tag{3.22}$$