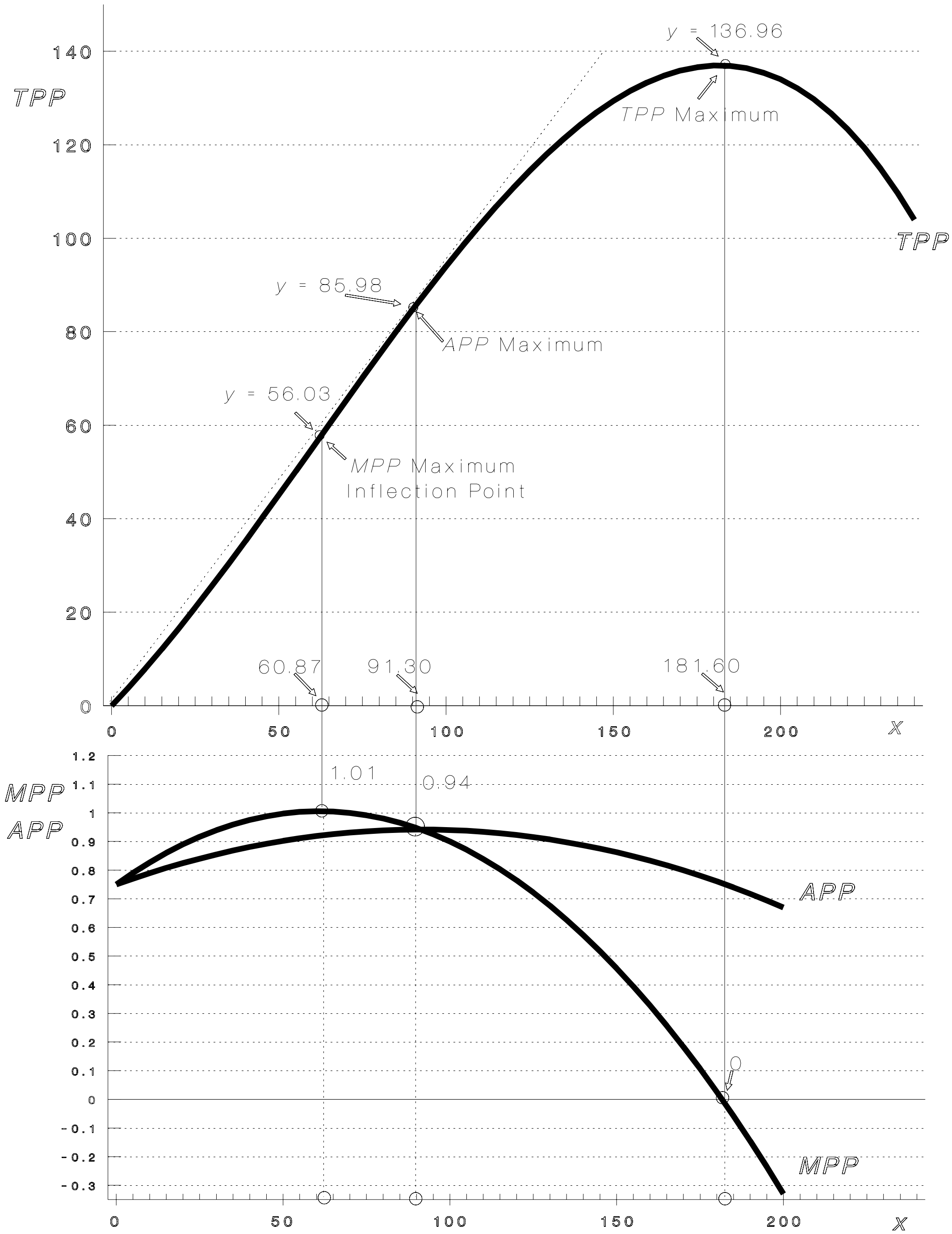

Figure 2.4 ***TPP, MPP*****, and** ***APP*** **For Corn (*****y*****) Response to Nitrogen (*****x*****)**
Based on Data Contained in Table 2.5

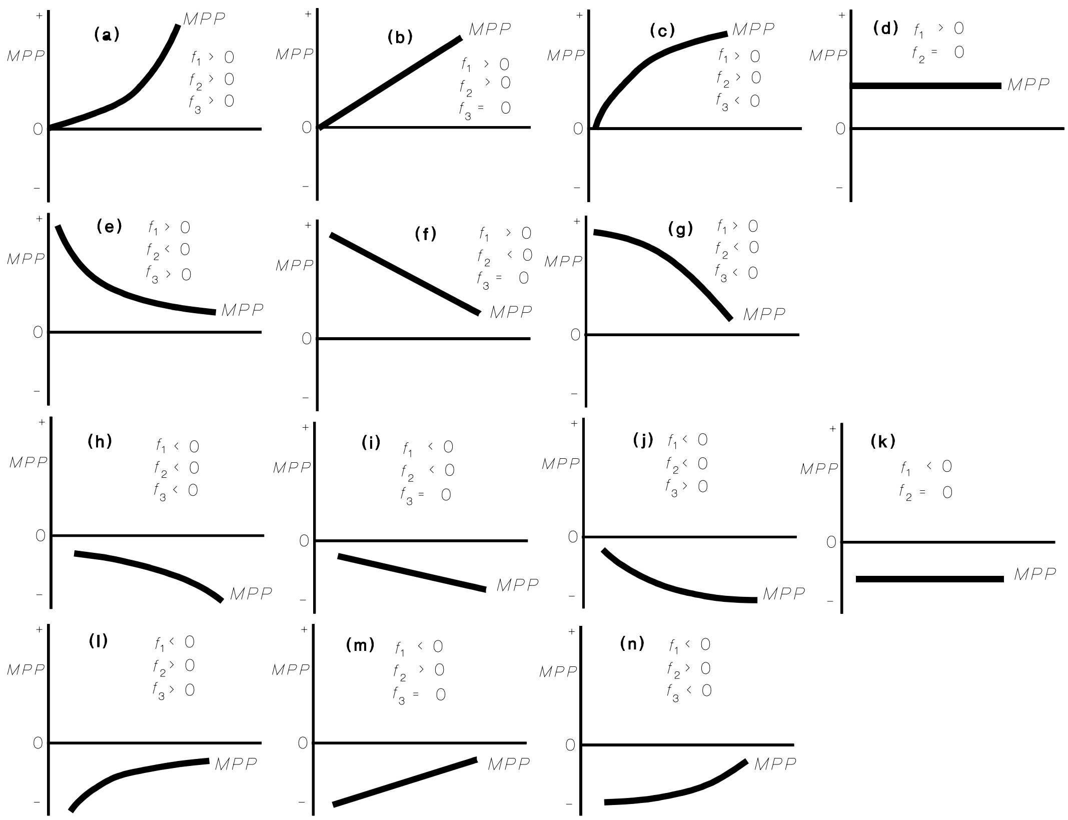

Figure 2.5 ***MPP*'s for the Production Function $y = f(x)$**

$f_1 = MPP$; f_2 = slope of MPP; f_3 = curvature of MPP

The first derivative of the *TPP* function could also be zero at the point where the *TPP* function is minimum. The sign on the second derivative of the *TPP* function is used to determine if the *TPP* function is at a maximum or a minimum. If the first derivative of the *TPP* function is zero and the second derivative is negative, the production function is at its maximum. If the first derivative of the *TPP* function is zero, and the second derivative is positive, the production function is at its minimum point. If both the first and second derivatives are zero, the function is at an inflection point, or changing from convex to the horizontal axis to concave to the horizontal axis. However, all inflection points do not necessarily have first derivatives of zero. Finally, if the first derivative is zero and the second derivative does not exist, the production function is constant.

The second derivative of the production function is the first derivative of the *MPP* function, or slope of the *MPP* function. The second derivative (d^2y/dx^2 or $f''(x)$ or f_2) is obtained by again differentiating the production function.

$$d^2y/dx^2 = f''(x) = f_2 = dMPP/dx \tag{2.37}$$

If equation (2.37) is positive for a particular value of x, then *MPP* is increasing at that particular point. A negative sign indicates that *MPP* is decreasing at that particular point. If $f''(x)$ is zero, *MPP* is likely at a maximum at that point. In figure 2.4, the first derivative of the *MPP* function (second derivative of the *TPP* function) is positive in (a), (b), and (c), (l), (m),and (n); negative in (e), (f), (g), (h), (i), and (j), and zero in (d) and (k).

The second derivative of the *MPP* function represents the curvature of *MPP* and is the third derivative of the original production (or *TPP*) function. It is obtained by again differentiating the original production function

(2.38) $$d^3y/dx^3 = f'''(x) = f_3 = d^2MPP/dx^2$$

The sign on $f'''(x)$ for a particular value of x indicates the rate of change in *MPP* at that particular point. If *MPP* is in the postive quadrant and $f'''(x)$ is positive, *MPP* is increasing at an increasing rate [(a) in Figure 2.5] or decreasing at a decreasing rate (e). If *MPP* is in the negative quadrant, a positive $f'''(x)$ indicates that MPP is either decreasing at a decreasing rate (j) or increasing at a decreasing rate (l).

When *MPP* is in the positive quadrant, a negative sign on $f'''(x)$ indicates that *MPP* is either increasing at a decreasing rate (c), or decreasing at an increasing rate (g). When *MPP* is in the negative quadrant, a negative sign on $f'''(x)$ indicates that *MPP* is decreasing at an increasing rate (h) or increasing at an increasing rate (n).

If $f'''(x)$ is zero, *MPP* has a constant slope with no curvature as is the case in (f), (l), and (m). If *MPP* is constant, $f'''(x)$ does not exist.

A similar approach might be used for *APP*. *APP* equals y/x, and if y and x are positive, then *APP* must also be positive. As indicated earlier, the slope of *APP* is

(2.39) $$d(y/x)/dx = f'(y/x) = dAPP/dx$$

For a particular value of x, a positive sign indicates a positive slope and a negative sign a negative slope.

The curvature of *APP* can be represented by

(2.40) $$d^2(y/x)/dx^2 = f''(y/x) = d^2APP/dx^2$$

For a particular value of x, a positive sign indicates that *APP* is increasing at an increasing rate, or decreasing at a decreasing rate. A negative sign on equation (2.40) indicates that *APP* is increasing at a decreasing rate, or decreasing at an increasing rate. A zero indicates an *APP* of constant slope. The third derivative of *APP* would represent the rate of change in the curvature of *APP*.

Here are some examples of how these rules can be applied to a specific production function representing corn yield response to nitrogen fertilizer. Suppose the production function

(2.41) $$y = 50 + 5.93\, x^{0.5}$$

where

y = corn yield in bushels per acre
x = pounds of nitrogen applied per acre

(2.42) $$MPP = f'(x) = 2.965\, x^{-0.5} > 0$$

For equation (2.41), *MPP* is always positive for any positive level of input use, as indicated by the sign on equation (2.42). If additional nitrogen is applied, some additional response in terms of increased yield will always result. If x is positive, *MPP* is positive and the production function has not reached a maximum.

(2.43) $$dMPP/dx = f''(x) = -1.48\, x^{-1.5} < 0$$

If equation (2.43) is negative, *MPP* is slopes downward. Each additional pound of nitrogen that is applied will produce less and less additional corn yield. Thus the law of diminishing (MARGINAL) returns holds for this production function throughout its range.

(2.44) $$d^2MPP/dx^2 = f'''(x) = 2.22\, x^{-2.5} > 0$$

If equation (2.44) holds, the *MPP* function is decreasing at a decreasing rate, coming closer and closer to the horizontal axis but never reaching or intersecting it. This is not surprising, given that incremental pounds of nitrogen always produce a positive response in terms of additional corn.

(2.45) $$APP = y/x = 50/x + 5.93x^{-0.5}$$

$$= 50\, x^{-1} + 5.93x^{-0.5} > 0$$

If x is positive, *APP* is positive. Corn produced per pound of nitrogen fertilizer is always positive [equation (2.45)].

(2.46) $$dAPP/dx = \mathrm{d}(y/x)/dx = -50\, x^{-2} - 2.97\, x^{-1.5} < 0$$

If x is positive, *APP* is sloped downward. As the use of nitrogen increases, the average product per unit of nitrogen declines [Equation (2.46)].

(2.47) $$d^2APP/dx^2 = d^2(y/x)/dx^2 = 100x^{-3} + 4.45\, x^{-2.5} > 0$$

If x is positive, *APP* is also decreasing at a decreasing rate. As the use of nitrogen increases, the average product per unit of nitrogen decreases but at a decreasing rate [equation (2.47)].

2.9 A Single-Input Production Elasticity

The term *elasticity* is used by economists when discussing relationships between two variables. An elasticity is a number that represents the ratio of two percentages. Any elasticity is a pure number in that it has no units.

The elasticity of production is defined as the percentage change in output divided by the percentage change in input, as the level of input use is changed. Suppose that x' represents some original level of input use that produces y' units of output. The use of x is then increased to some new amount called x'', which in turn produces y'' units of output. The elasticity of production (E_p) is defined by the formula

(2.48) $$E_p = [(y' - y'')/y]/[(x' - x'')/x].$$

where y, y'', x, and x'' are as defined previously, and x and y represent mid values between the old and new levels of inputs and outputs. Thus

(2.49) $$x = (x' + x'')/2$$

and $$y = (y' + y'')/2$$

Since the elasticity of production is the ratio of two percentages, it does not depend on the specific units in which the input and output are measured. For example, suppose that *y* represents corn yield in bushels per acre, and *x* represents nitrogen in pounds per acre. Then suppose that corn yield is instead measured in terms of liters per hectare, and nitrogen was measured in terms of kilograms per hectare. If the same amount of nitrogen is applied in both instances, the calculated value for the elasticity of production will be the same, regardless of the units in which *y* and *x* are measured.

Another way of expressing the elasticity of production is

(2.50) $$E_p = (\Delta y/y)/(\Delta x/x)$$

where $\Delta y = y' - y''$

and $\Delta x = x' - x''$

The elasticity of production is one way of measuring how responsive the production function is to changes in the use of the input. A large elasticity (for example, an elasticity of production greater than 1) implies that the output responds strongly to increases in the use of the input. An elasticity of production of between zero and 1 suggests that output will increase as a result of the use of *x*, but the smaller the elasticity, the less the response in terms of increased output. A negative elasticity of production implies that as the level of input use increases, output will actually decline, not increase.

The elasticity of production can also be defined in terms of the relationship between *MPP* and *APP*. The following relationships hold. First

(2.51) $$E_p = (\Delta y/y)/(\Delta x/x)$$

Equation (2.51) might also be written as

(2.52) $$E_p = (\Delta y/\Delta x)\cdot(x/y)$$

Notice that

(2.53) $$\Delta y/\Delta x = MPP$$

and that

(2.54) $$x/y = 1/APP$$

Thus

(2.55) $$E_p = MPP/APP$$

Notice that a large elasticity of production indicates that *MPP* is very large relative to *APP*. In other words, output occurring from the last incremental unit of fertilizer is very great relative to the average output obtained from all units of fertilizer. If the elasticity of production is very small, output from the last incremental unit of fertilizer is small relative to the average productivity of all units of fertilizer.

2.10 Elasticities of Production for a Neoclassical Production Function

A unique series of elasticities of production exist for the neoclassical production function, as a result of the relationships that exist between *MPP* and *APP*. These are illustrated in Figure 2.6 and can be summarized as follows

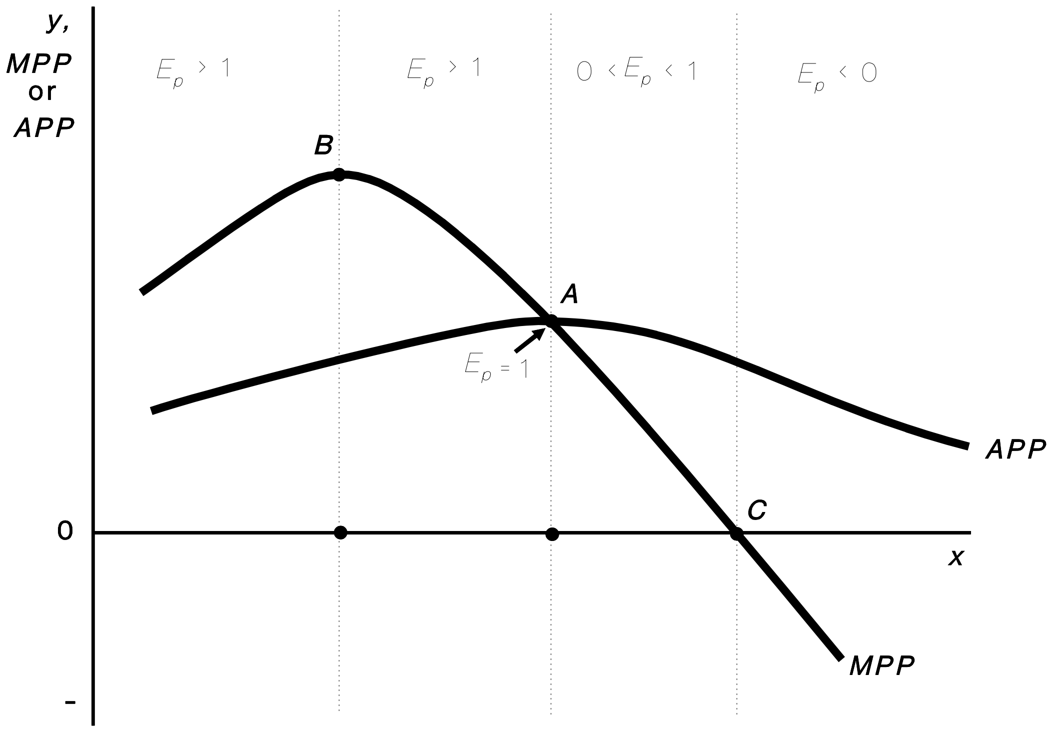

Figure 2.6 *MPP, APP* and the Elasticity of Production

1. The elasticity of production is greater than 1 until the point is reached where *MPP* = *APP* (point *A*).

2. The elasticity of production is greatest when the ratio of *MPP* to *APP* is greatest. For the neoclassical production function, this normally occurs when *MPP* reaches its maximum at the inflection point of the production function (point *B*).

3. The elasticity of production is less than 1 beyond the point where *MPP* = *APP* (point *A*).

4. The elasticity of production is zero when *MPP* is zero. Note that *APP* must always be positive (point *C*).

5. The elasticity of production is negative when *MPP* is negative and, of course, output is declining (beyond point *C*). If the production function is decreasing, *MPP* and the elasticity of production are negative. Again, *APP* must always be positive.

6. A unique characteristic of the neoclassical production function is that as the level of input use is increased, the relationship between *MPP* and *APP* is continually changing, and therefore the ratio of *MPP* to *APP* must also vary. Since $E_p = MPP/APP$, the elasticity of production too must vary continually as the use of the input increases. This is a characteristic of the neoclassical production function, which in general is not true for some other production functions.

2.11 Further Topics on the Elasticity of Production.

The expression $\Delta y/\Delta x$ is only an approximation of the true *MPP* of the production function for a specific amount of the input *x*. The actual *MPP* at a specific point is better represented by inserting the value of *x* into the marginal product function *dy*/*dx*.

The elasticity of production for a specific level of *x* might be obtained by determining the value for *dy*/*dx* for that level of *x* and then obtaining the elasticity of production from the expression

(2.56) $$E_p = (dy/dx)\cdot x/y$$

Now suppose that instead of the neoclassical production function, a simple linear relationship exists between *y* and *x*. Thus

(2.57) $$TPP = y = bx$$

where *b* is some positive number. Then $dy/dx = b$, but note also that since $y = bx$, then $y/x = bx/x = b$. Thus MPP $(dy/dx) = APP$ $(y/x) = b$. Hence, $MPP/APP = b/b = 1$.

The elasticity of production for any such function is 1. This means that a given percentage increase in the use of the input *x* will result in exactly the same percentage increase in the output *y*. Moreover, any production function in which the returns to the variable input are equal to some constant number will have an elasticity of production equal to 1.

Now suppose a slightly different production function

(2.58) $$y = a\sqrt{x}$$

Another way of writing equtation (2.58) is

(2.59) $$y = ax^{0.5}$$

In this case

(2.60) $$dy/dx = 0.5\ ax^{-0.5}$$

And

(2.61) $$y/x = ax^{-0.5}$$

Thus, $(dy/dx)/(y/x) = 0.5$

Hence the elasticity of production is 0.5. This means that for any level of input use *MPP* will be precisely one half of *APP*. In general, the elasticity of production will be *b* for any production function of the form

(2.62) $$y = ax^b$$

where *a* and *b* are any numbers. Notice that

(2.63) $$dy/dx = bax^{b-1}$$

and that

(2.64) $$y/x = ax^b/x = ax^bx^{-1} = ax^{b-1}.$$

(Another way of writing the expression $1/x$ is x^{-1}. Therefore, $y/x = yx^{-1}$. But $y = ax^b$, and, as a result, $x^b x^{-1} = x^{b-1}$.)

Thus the ratio of *MPP* to *APP*− the elasticity of production− for such a function is always equal to the constant *b*. This is not the same as the relationship that exists between *MPP* and *APP* for the neoclassical production function in which the ratio is not constant but continually changing as the use of *x* increases.

2.12 Concluding Comments

This chapter has outlined in considerable detail the physical or technical relationships underlying the factor-product model. A production function was developed using tabular, graphical, and mathematical tools, with illustrations from agriculture. The law of diminishing *MARGINAL* returns was introduced. Marginal and average physical product concepts were developed. The rules of calculus for determining if a function is at a maximum or minimum were outlined, using a total physical product and marginal physical product concepts to illustrate the application. Finally, the concept of an elasticity of production was introduced, and the elasticity of production was linked to the marginal and average product functions.

Problems and Exercises

1. Suppose the following production function data. Fill in the blanks.

x (Input)	*y* (Output)	*MPP*	*APP*
0	0		——
		——	
10	50		——
		——	
25	75		——
		——	
40	80		——
		——	
50	85		——

2. For the following production functions, does the law of diminishing returns hold?

a. $y = x^{0.2}$
b. $y = 3x$
c. $y = x^3$
d. $y = 6x - 0.10x^2$

3. Find the corresponding *MPP* and *APP* functions for the production functions given in problem number 2.

4. Assume a general multiplicative production function of the form

$$y = 2x^b$$

Derive the corresponding *MPP* and *APP* functions, and draw on a sheet of graph paper *TPP*, *APP* and *MPP* when the value of *b* is

a. 5	f. 0.7
b. 3	g. 0.3
c. 2	h. 0
d. 1.5	i. -0.5
e. 1.0	j. -1.0

Be sure to show the sign, slope and curvature of *MPP* and *APP*. What is the value for the elasticity of production in each case? Notice that the curves remain at fixed proportion from each other.

5. Graph the production function

$$y = 0.4x + 0.09x^2 - 0.003x^3$$

for values of x between 0 and 20. Derive and graph the corresponding *MPP* and *APP*. What is the algebraic expression for the elasticity of production in this case? Is the elasticity of production constant or variable for this function? Explain.

6. Suppose that the coefficients or parameters of a production function of the polynomial form are to be found. The production function is

$$y = ax + bx^2 + cx^3$$

where y = corn yield in bushels per acre

x = nitrogen application in pounds per acre

a, b and c are coefficients or unknown parameters

The production function should produce a corn yield of 150 bushels per acre when 200 pounds of nitrogen is applied to an acre. This should be the maximum corn yield (*MPP* = 0). The maximum *APP* should occur at a nitrogen application rate of 125 pounds per acre. Find the parameters a, b and c for a production function meeting these restrictions. *Hint:* First find the equation for *APP* and *MPP*, and the equations representing maximum *APP* and zero *MPP*. Then insert the correct nitrogen application levels in the three equations representing *TPP*, maximum *APP* and zero *MPP*. There are three equations in three unknowns (a, b, and c). Solve this system for a, b, and c.