2

Production With One Variable Input

This chapter introduces the concept of a production function and uses the concept as a basis for the development of the factor-product model. An agricultural production function in presented using graphical and tabular approaches. Algebraic examples of simple production functions with one input and one output are developed. Key features of the neoclassical production function are outlined. The concept of marginal and average physical product is introduced. The use of the first, second, and third derivatives in determining the shape of the underlying total, marginal, and average product is illustrated, and the concept of the elasticity of production is presented.

Key terms and definitions:

Production Function
Domain
Range
Continuous Production Function
Discrete Production Function
Fixed Input
Variable Input
Short Run
Long Run
Intermediate Run
Sunk Costs
Law of Diminishing (Marginal) Returns
Total Physical Product (*TPP*)
Marginal Physical Product (*MPP*)
Average Physical Product (*APP*)
$\Delta y/\Delta x$
Sign
Slope
Curvature
First Derivative
Second Derivative
Third Derivative
Elasticity of Production

2.1 What Is a Production Function?

A production function describes the technical relationship that transforms inputs (resources) into outputs (commodities). A mathematician defines a *function* as a rule for assigning to each value in one set of variables (the domain of the function) a single value in another set of variables (the range of the function).

A general way of writing a production function is

(2.1) $$y = f(x)$$

where y is an output and x is an input. All values of x greater than or equal to zero constitute the domain of this function. The range of the function consists of each output level (y) that results from each level of input (x) being used. Equation (2.1) is a very general form for a production function. All that is known about the function $f(x)$ so far is that it meets the mathematician's definition of a function. Given this general form, it is not possible to determine exactly how much output (y) would result from a given level of input (x). The specific form of the function $f(x)$ would be needed, and $f(x)$ could take on many specific forms.

Suppose the simple function

(2.2) $$y = 2x.$$

For each value of x, a unique and single value of y is assigned. For example if $x = 2$, then $y = 4$; if $x = 6$ then $y = 12$ and so on. The domain of the function is all possible values for x, and the range is the set of y values corresponding to each x. In equation (2.2), each unit of input (x) produces 2 units of output (y).

Now consider the function

(23) $$y = \sqrt{x}$$

It is not possible to take the square root of a negative number and get a real number. Hence the domain (x) and range (y) of equation (2.3) includes only those numbers greater than or equal to zero. Here again the function meets the basic definition that a single value in the range be assigned to each value in the domain of the function. This restriction would be all right for a production function, since it is unlikely that a farmer would ever use a negative quantity of input. It is not clear what a negative quantity of an input might be.

Functions might be expressed in other ways. The following is an example:

If $x = 10$, then $y = 25$.
If $x = 20$, then $y = 50$.
If $x = 30$, then $y = 60$.
If $x = 40$, then $y = 65$.
If $x = 50$, then $y = 60$.

Notice again that a single value for y is assigned to each x. Notice also that there are two values for x (30 and 50) that get assigned the same value for y (60). The mathematician's definition of a function allows for this. But one value for y must be assigned to each x. It does not matter if two different x values are assigned the same y value.

The converse, however, is not true. Suppose that the example were modified only slightly:

If $x = 25$, then $y = 10$.
If $x = 50$, then $y = 20$.
If $x = 60$, then $y = 30$.
If $x = 65$, then $y = 40$.
If $x = 60$, then $y = 50$.

This is an example that violates the definition of a function. Notice that for the value $x = 60$, two values of y are assigned, 30 and 50. This cannot be. The definition of a function stated that a single value for y must be assigned to each x. The relationship described here represents what is known as a *correspondence*, but not a function. A correspondence describes the relationship between two variables. All functions are correspondences, but not all correspondences are functions.

Some of these ideas can be applied to hypothetical data describing the production of corn in response to the use of nitrogen fertilizer. Table 2.1 represents the relationship and provides specific values for the general production function $y = f(x)$. For each nitrogen application level, a single yield is defined. The yield level is sometimes referred to as the total physical product (*TPP*) resulting from the nitrogen that is applied.

Table 2.1 Corn Yield Response to Nitrogen Fertilizer

Quantity of Nitrogen (Pounds/Acre)	Yield in Bushels/Acre
0	50
40	75
80	105
120	115
160	123
200	128
240	124

From Table 2.1, 160 pounds of nitrogen per acre will result in a corn yield or *TPP* of 123 bushels per acre. The concept of a function has a good deal of impact on the basic assumptions underlying the economics of agricultural production.

Another possible problem exists with the interpretation of the data contained in Table 2.1. The exact amount of corn (*TPP*) that will be produced if a farmer decides to apply 120 pounds of nitrogen per acre can be determined from Table 2.1, but what happens if the farmer decides to apply 140 pounds of nitrogen per acre? A yield has not been assigned to this nitrogen application level. A mathematician might say that our production function $y = f(x)$ is discontinuous at any nitrogen application level other than those specifically listed in Table 2.1.

A simple solution might be to interpolate between the known values. If 120 pounds per acre produces 115 bushels of corn, and 160 pounds of nitrogen produces 123 bushels of corn, the yield at 140 pounds might be $(115 + 123)/2$ or 119 bushels per acre. However, incremental increases in nitrogen application do not provide equal incremental increases in corn production throughout the domain of the function. There is no doubt that some nitrogen is available in the soil from decaying organic material and nitrogen applied in previous seasons, and nitrogen need not be applied in order to get back the first 50 bushels of corn.

The first 40 pounds of nitrogen applied produces 25 additional bushels, for a total of 75 bushels, the next 40 pounds produces 30 bushels of corn, for a total of 105 bushels, but the productivity of the remaining 40 pound increments in terms of corn production declines. The next 40 pounds increases yield by only 10 bushels per acre, the 40 pounds after that by only 8 bushels per acre, and the final 40 pounds by only 5 bushels per acre.

Following this rationale, it seems unlikely that 140 pounds of nitrogen would produce a yield of 119 bushels, and a more likely guess might be 120 or 121 bushels. These are only guesses. In reality no information about the behavior of the function is available at nitrogen application levels other than those listed in Table 2.1. A yield of 160 bushels per acre at a nitrogen application level of 140 pounds per acre could result- or, for that matter, any other yield.

Suppose instead that the relationship between the amount of nitrogen that is applied and corn yield is described as

(2.4) $$y = 0.75x + 0.0042x^2 - 0.000023x^3$$

where y = corn yield (total physical product) in bushels per acre

x = nitrogen applied in pounds per acre

Equation (2.4) has some advantages over the tabular function presented in Table 2.1. The major advantage is that it is possible to calculate the resultant corn yield at any fertilizer application level. For example, the corn yield when 200 pounds of fertilizer is applied is $0.75(200) + 0.0042(200^2) - 0.000023(200^3) =$ 134 bushels per acre.

Moreover, a function such as this is continuous. There are no nitrogen levels where a corn yield cannot be calculated. The yield at a nitrogen application level of 186.5 pounds per acre can be calculated exactly. Such a function has other advantages, particularly if the additional output resulting from an extra pound of nitrogen is to be calculated. The yields of corn at the nitrogen application rates shown in Table 2.1 can be calculated and are presented in Table 2.2.

Table 2.2 Corn Yields at Alternative Nitrogen Application Rates for the Production Function $y = 0.75x + 0.0042x^2 - 0.000023x^3$

Quantity of Nitrogen, x (lb/acre)	Corn Yield, y or *TPP* (bu/Acre)
0	0.0
20	16.496
40	35.248
60	55.152
80	75.104
100	94.000
120	110.736
140	124.208
160	133.312
180	136.944
200	134.000
220	123.376
240	103.968

The corn yields (*TPP*) generated by the production function in Table 2.2 are not the same as those presented in Table 2.1. There is no reason for both functions to generate the same yields. A continuous function that would generate exactly the same yields as those presented in Table 2.1 would be very complicated algebraically. Economists like to work with continuous functions, rather than discrete production functions from tabular data, in that the yield for any level of input use can be readily obtained without any need for interpolation. However, a tabular presentation would probably make more sense to farmers.

The yields generated in Table 2.2 also differ from those in Table 2.1 in another important way. Table 2.1 states that if a farmer applied no nitrogen to corn, a yield of 50 bushels per acre is obtained. Of course, nitrogen is absolutely essential for corn to grow. As indicated earlier, the data contained in Table 2.1 assume that there is some residual nitrogen in the soil on which the corn is grown. The nitrogen is in the soil because of decaying organic material and leftover nitrogen from fertilizers applied in years past. As a result, the data in Table 2.1 reveal higher yields at low nitrogen application levels than do the data contained in Table 2.2.

The mathematical function used as the basis for Table 2.2 could be modified to take this residual nitrogen into account by adding a constant such as 50. The remaining coefficients of the function (the 0.75, the 0.0042, and the − 0.000023) would also need to be altered as well. Otherwise, the production function would produce a possible but perhaps unrealistic corn yield of $50 + 136.944 = 186.944$ bushels per acre when 180 pounds of fertilizer were applied. For many production processes in agriculture, no input produces no output. Consider the case of the production of beef using feed as an input. No feed would indeed produce no beef. In the case of crop production, some yield will normally result without chemical fertilizers.

A production function thus represents the relationship that exists between inputs and outputs. For each level of input use, the function assigns a unique output level. When a zero level of input is used, output might be zero, or, in some instances, output might be produced without the input.

2.2 Fixed Versus Variable Inputs and the Length of Run

So far, examples have included only one input or factor of production. The general form of the production function was

(2.5) $$y = f(x)$$

where y = an output

x = an input

Equation (2.5) is an ultrasimplistic production function for agricultural commodities. Such a function assumes that the production process can be accurately described by a function in which only one input or factor of production is used to produce an output. Few, if any, agricultural commodities are produced in this manner. Most agricultural commodities require several, if not a dozen or more, inputs. As an alternative, suppose a production function where there are several inputs and all but one are assumed to be held fixed at some constant level. The production function would thus become

(2.6) $$y = f(x_1, \mid x_2, x_3, x_4, x_5, x_6, x_7).$$

For example, y might be the yield of corn in bushels per acre, and x_1 might represent the amount of nitrogen fertilizer applied per acre. Variables $x_2, ..., x_7$ might represent each of the other inputs used in the production of corn, such as land, labor, and machinery.

Thus, in this example, the input x_1 is treated as the "variable" input, while the remaining inputs (x_2, ..., x_7) are assumed to be held constant at some fixed level. The "$|$" can be read as the word "given". As the use of x_1 is "varied" or increased, units of the variable input x_1 are added to units of the fixed inputs x_2, ..., x_7.

How can it be determined if an input should be treated as fixed or variable? A *variable input* is often thought of as an input that the farm manager can control or for which he or she can alter the level of use. This implies that the farmer has sufficient time to adjust the amount of input being used. Nitrogen in corn production has often been cited as an example of a variable input, in that the farmer can control the amount to be applied to the field.

A *fixed input* is usually defined as an input which for some reason the farmer has no control over the amount available. The amount of land a farmer has might be treated as a fixed input.

However, these distinctions become muddy and confused. Given sufficient time, a farmer might be able to find additional land to rent or purchase, or the farmer might sell some of the land owned. If the length of time were sufficient to do this, the land input might be treated as a variable input.

The categorization of inputs as either fixed or variable is closely intertwined with the concept of time. Economists sometimes define the *long run* as time of sufficient length such that all inputs to the production function can be treated as variable. The *very short run* can be defined as a period of time so short that none of the inputs are variable. Other lengths of time can also be defined. For example, the *short run* is a period of time long enough such that a few of the inputs can be treated as variable, but most are fixed. The *intermediate run* is long enough so that many, but not all inputs are treated as variable.

These categories again are somewhat arbitrary. If an economist were asked "How long is the short run?", the answer would probably be that the short run is a period of time sufficiently long that some inputs can be treated as variable, but sufficiently short such that some inputs can be treated as fixed. Does this imply a length of time of a day, a week, a month, or a crop production season? The length of time involved could be any of these.

Once fertilizer has been applied, a farmer no longer has control over application levels. The input that was previously classified as variable becomes fixed. Seed before planting is classified as a variable input. Once it is planted in the ground, seed can no longer be treated as a variable input.

Some production economists have argued that inputs should not be arbitrarily categorized as either fixed or variable. These arbitrary categories can be highly misleading. Production economists argue that in the case of crop production, prior to planting, nearly all inputs are variable. Farmers might rent additional land, buy or sell machinery, or adjust acreages of crops. Here is where real decision making can take place. Once planting begins, more and more of the inputs previously treated as variable become fixed. Tractor time and labor for tillage operations cannot be recovered once used. Acreages of crops once planted largely cannot be altered. Insecticides and herbicides are variable inputs before application, but must be treated as fixed or "sunk" once they have been applied. At the start of harvest, the only variable input is the labor, fuel, and repairs to run the harvesting equipment and to move the grain to market.

This view treats the input categories as a continuum rather than as a dichotomy. As inputs are used, costs are treated as sunk. Inputs, once used, can no longer be sold, or used on the farm for a different enterprise, such as another crop.

2.3 The Law of Diminishing Returns

The law of diminishing returns is fundamental to all of production economics. The law is misnamed. It should be called the law of diminishing *MARGINAL* returns, for the law deals with what happens to the incremental or marginal product as units of input or resource are added. The *law of diminishing marginal returns* states that as units of an variable input are added to units of one or more fixed inputs, after a point, each incremental unit of the variable input produces less and less additional output. As units of the variable input are added to units of the fixed inputs, the proportions change between fixed and variable inputs. The law of diminishing returns has sometimes been referred to as the law of variable proportions.

For example, if incremental units of nitrogen fertilizer were applied to corn, after a point, each incremental unit of nitrogen fertilizer would produce less and less additional corn. Were it not for the law of diminishing returns, a single farmer could produce all the corn required in the world, merely by acquiring all of the available nitrogen fertilizer and applying it to his or her farm.

The key word in the law of diminishing returns is *additional*. The law of diminishing returns does not state that as units of a variable input are added, each incremental unit of input produces less output in total. If it did, a production function would need to have a negative slope in order for the law of diminishing returns to hold. Rather, the law of diminishing returns refers to the rate of change in the slope of the production function. This is sometimes referred to as the *curvature* of the production function.

Figure 2.1 illustrates three production functions. The production function labeled A has no curvature at all. The law of diminishing returns does not hold here. Each incremental unit of input use produces the exact same incremental output, regardless of where one is at on the function. An example of a function such as this is

$$y = 2x. \tag{2.7}$$

Each incremental unit of x produces 2 units of y, regardless of the initial value for x, whether it be 0, 24, 100 or 5000.

A slightly more general form of this function is

$$y = bx. \tag{2.8}$$

where b is some positive number. If b is a positive number, the function is said to exhibit *constant marginal returns* to the variable input x, and the law of diminishing returns does not hold. Each incremental unit of x produces bx units of y.

The production function labeled B represents another kind of relationship. Here each incremental unit of x produces more and more additional y. Hence the law of diminishing returns does not hold here either. Notice that as the use of input x is increased, x becomes more productive, producing more and more additional y. An example of a function that would represent this kind of a relationship is

$$y = x^2. \tag{2.9}$$

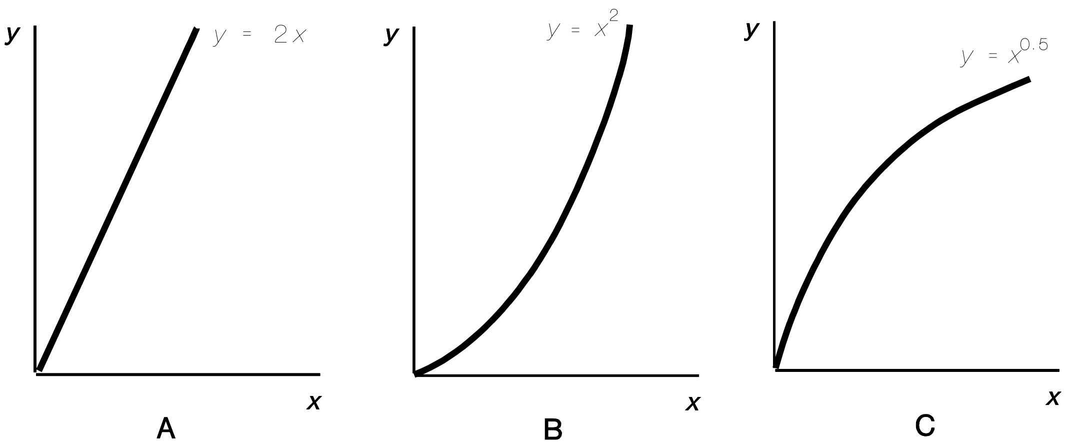

Figure 2.1 Three Production Functions

A slightly more general form of the function might be

(2.10) $y = ax^b$,

where both *a* and *b* are positive numbers, and *b* is greater than 1. Notice that if $b = 1$, the function is the same as the one depicted in diagram A of figure 2.1. The value of a must be positive if the input is to produce a positive quantity of output.

The production function labeled C represents the law of diminishing returns throughout its range. Here each incremental unit of *x* produces less and less additional *y*. Thus each unit of *x* becomes less and less productive. An example of a function that represents this kind of relationship is

(2.11) $y = \sqrt{x}$.

Another way of writing equation (2.11) is

(2.12) $y = x^{0.5}$.

Both are exactly the same thing. For this production function, total product (*TPP* or *y*) will never decline.

A slightly more general form of the function is

(2.13) $y = ax^b$,

where *a* and *b* are positive numbers. However, here *b* must be less than 1 but greater than zero, if diminishing (marginal) returns are to hold. This function will forever increase, but at a decreasing rate.

2.4 Marginal and Average Physical Product

The *marginal physical product* (*MPP*) refers to the change in output associated with an incremental change in the use of an input. The incremental increase in input use is usually taken to be 1 unit. Thus *MPP* is the change in output associated with a 1 unit increase in the input. The *MPP* of input x_i might be referred to as MPP_{x_i}. Notice that *MPP*, representing the incremental change in *TPP*, can be either positive or negative.

Average physical product (*APP*) is defined as the ratio of output to input. That is, *APP* = *y*/*x*. For any level of input use (*x*), *APP* represents the average amount of output per unit of *x* being used.

Suppose that the production function is

(2.14) $y = f(x)$.

One way of expressing *MPP* is by the expression $\Delta y/\Delta x$, where the Δ denotes change. The expression $\Delta y/\Delta x$ can be read as "the change in *y* (Δy) with respect to a change in *x* (Δx)." For the same function *APP* is expressed either as *y*/*x* or as *f*(*x*)/*x*.

For the production function

(2.15) $y = 2x$,

MPP is equal to 2. The change in *y* with respect to a 1 unit change in *x* is 2 units. That is, each additional or incremental unit of *x* produces 2 additional or incremental units of *y*. For each additional unit of *x* that is used, *TPP* increases by 2 units. In this example *APP* equals *y*/*x*, or *APP* equals 2*x*/*x*, or *APP* equals 2. For this simple production function *MPP* = *APP* = 2 for all positive values for *x*.

For the production function

(2.16) $y = bx$,

MPP is equal to the constant coefficient *b*. The change in *y* with respect to a change in *x* is *b*. Each incremental or additional unit of *x* produces *b* incremental or additional units of *y*. That is, the change in *TPP* resulting from a 1 unit change in *x* is *b*. Moreover, *APP* = *bx*/*x*. Thus, *MPP* = *APP* = *b* everywhere.

Marginal and average physical products for the tabular data presented in Table 2.1 may be calculated based on the definition that *MPP* is the change in output (Δy) arising from an incremental change in the use of the input (Δx) and that *APP* is simply output (*y*) divided by input (*x*). These data are presented in Table 2.3. *MPP* is calculated by first making up a column representing the rate of change in corn yield. This rate of change might be referred to as Δy or perhaps ΔTPP. Then the rate of change in nitrogen use is calculated. This might be referred to as Δx. Since 40 pound units were used in this example, the rate of change in each case for *x* is 40. The corresponding *MPP* over the increment is $\Delta y/\Delta x$. *MPP* might also be thought of as $\Delta TPP/\Delta x$. The corresponding calculations are shown under the column labeled *MPP* in Table 2.3. For example, if nitrogen use increases from 120 to 160 pounds per acre, or 40 pounds, the corresponding increase in corn yield will be from 123 to 128 bushels per acre, or 5 bushels. The *MPP* over this range is approximately 5/40 or 0.125.

The *MPP*'s are positioned at the midpoint between each fertilizer increment. The *MPP*'s calculated here are averages that apply only approximately at the midpoints between each increment, that is at nitrogen application levels of approximately 20, 60, 100, 140 and 180 pounds per acre. Since no information is available with respect to what corn might have