

$$\frac{1}{ne} \left[\frac{dJ_x}{dt} \right] = \frac{e}{m} E_x$$

$$\frac{dJ_x}{dt} = \text{constant } E_x$$

above eq shows that for constant E_x current increases linearly with time. against ohm's law

So above law can only be satisfied if $\frac{dJ_x}{dt} = 0$ or $J_x = \text{constant}$

$$\text{if } \frac{d}{dt} \langle v_x \rangle = 0$$

we are omitting to include decelerating term, when added to $\frac{d}{dt} \langle v_x \rangle$ gives a net

$$\frac{d}{dt} \langle v_x \rangle = 0$$

so we introduce electron - lattice scattering decelerating term

$$\left[\frac{d}{dt} \langle v_x \rangle \right] + \left[\frac{d}{dt} \langle v_x \rangle \right]_{\text{el-la}} = 0$$

classical free electron theory

mean free path = λ

mean collision time = τ

average speed = \bar{v}

according to drude electron gas model

for unit volume of metal

$$P = \frac{1}{3} \rho \bar{v}^2 = \frac{1}{3} m n \bar{v}^2$$

here ρ is the density of electron gas

For molar volume of metal

$$P = \frac{1}{3} \rho \bar{v}^2 = \frac{1}{3} \frac{m N_A}{V_m} \bar{v}^2$$

V_m = molar volume of metal

$$P V_m = \frac{1}{3} m N_A \bar{v}^2 = R T$$

$$m \bar{v}^2 = \left[\frac{3 R T}{N_A} \right] = 3 K_B T$$

properties of metals

1- at low temp $\rho \propto T^5$

2- at room temp and above
 $\rho \propto T$

3- $\rho \propto \frac{1}{P}$ $P = \text{pressure}$

4- resistivity of an impure specimen is given by Mathiessen's rule

$$\rho = \rho_0 + \rho(T)$$

5- Magneto resistance (magnetic field dependent resistivity)

K.E of electron

$$\frac{1}{2} m \bar{v}^2 = \frac{3}{2} kT$$

$$\bar{v} = \sqrt{\frac{3kT}{m}}$$

$$\bar{v} \propto \sqrt{T}$$

T is the temp of metal

$$\tau = \frac{\lambda}{\bar{v}} = \lambda \left[\sqrt{\frac{m}{3kT}} \right]$$

at room temp $v_d \ll \bar{v}$

Microscopic form of ohm's law

$$J = \frac{I}{A}, \quad E = -\frac{dV}{dx}$$

$$\frac{V}{R} = I = AJ$$

~~R = \rho \frac{l}{A}~~

$$\frac{1}{J} = \frac{AR}{V} = \frac{A \rho l}{A E l} = \frac{\rho}{E}$$

$$R = \rho \frac{l}{A}$$

$$J \rho = E$$

$$\frac{1}{\rho} = \sigma$$

$$J = \frac{1}{\rho} E$$

$$\vec{J} = \sigma \vec{E}$$

when E field is applied in x -direction then all electron accelerate in x -direction,

a_{ix} = acceleration of i th electron in x -direction

$$m a_{ix} = -e E_x$$

$$a_{ix} = - \left[\frac{e E_x}{m} \right]$$

$$\frac{d v_{ix}}{dt} = - \left[\frac{e E_x}{m} \right]$$

$$\frac{d}{dt} \langle v_x \rangle = - \left[\frac{e E_x}{m} \right]$$

$\langle v_x \rangle$ is the average velocity of all n electrons

$$\langle v_x \rangle = \frac{1}{n} \left[\sum_{i=1}^n v_{ix} \right]$$

current density along x -direction

$$J_x = n(-e) \langle v_x \rangle$$

$$\frac{d J_x}{dt} = -ne \frac{d \langle v_x \rangle}{dt}$$

$$-\frac{1}{ne} \left[\frac{d J_x}{dt} \right] = \frac{d \langle v_x \rangle}{dt}$$

It is extremely difficult to solve scattering term. We therefore obtain an expression for $\left[\frac{d \langle v_x \rangle}{dt} \right]_{t=t_0}$ by a method which by-passes these difficulties

at $t=0$, $\langle v_x \rangle_0$ just switch the field off so $\langle v_x \rangle$ tends to zero. Let assume that this process follows simplest law of decay

$$\langle v_x \rangle = \langle v_x \rangle_0 e^{-t/\tau}$$

τ = time constant or relaxation time

in fact it is the time for the drift velocity to decay $1/e$ of its initial value when $E_x = 0$

Now we can obtain $t=t_0$ term without ~~considering~~ considering details of collision process

$$\left[\frac{d \langle v_x \rangle}{dt} \right]_{t=t_0} = - \frac{\langle v_x \rangle_0}{\tau} e^{-t/\tau} = - \frac{\langle v_x \rangle}{\tau}$$

$$\left[\frac{d}{dt} \langle v_x \rangle \right] = - \left[\frac{e E_x}{m} \right] \quad \text{--- 1}$$

$$\left[\frac{d}{dt} \langle v_x \rangle \right] + \left[\frac{d}{dt} \langle v_x \rangle \right]_{d=0} = 0 \quad \text{--- 2}$$

$$\left[\frac{d}{dt} \langle v_x \rangle \right]_{d=0} = - \frac{\langle v_x \rangle}{\tau} \quad \text{--- 3}$$

$$- \left[\frac{e E_x}{m} \right] - \frac{\langle v_x \rangle}{\tau} = 0$$

$$\langle v_x \rangle = - \left[\frac{e \tau E_x}{m} \right]$$

So the steady average velocity is proportional to the applied field E_x , $\frac{e \tau}{m}$ is constant of proportionality known as mobility ' μ '

$$\langle v_x \rangle = - \mu E_x$$

$$\mu = \frac{\langle v_x \rangle}{E_x} = \frac{e \tau}{m}$$

1. The three integers n_x , n_y , and n_z which are called quantum numbers, are required to specify completely each energy state. No quantum number can be zero, because if any one of them is taken as zero, then $\psi(x, y, z) = 0$, which would mean that the particle does not exist in the box.
2. It should be noted, however, that the energy E depends only upon the sum of the squares of the quantum numbers n_x , n_y , and n_z and not on their individual values.
3. Several combinations of the three integers may give different stationary states or different wave functions, but of the same energy value. Such states and energy levels are said to be *degenerate*.

Example:

1. $n_x = 1, n_y = 1, n_z = 2$
2. $n_x = 2, n_y = 1, n_z = 1$
3. $n_x = 1, n_y = 2, n_z = 1$

The corresponding wave functions are then

$$\Psi_{112} = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \sin \frac{2\pi z}{a} \quad \text{where } A = \sqrt{\frac{8}{a^3}}$$

$$\Psi_{211} = A \sin \frac{2\pi x}{a} \sin \frac{\pi y}{a} \sin \frac{\pi z}{a}$$

and

$$\Psi_{121} = A \sin \frac{\pi x}{a} \sin \frac{2\pi y}{a} \sin \frac{\pi z}{a}$$

but in each case the energy value is exactly the same; that is

$$E_{n_x, n_y, n_z} = \frac{6h^2}{8ma^2}$$

We know that if several states have the same energy, they are said to be *degenerate*. Thus for the level for which the energy is $\frac{6h^2}{8ma^2}$, there are three independent states having quantum numbers (1, 1, 2), (2, 1, 1) and (1, 2, 1). The level is, therefore, said to be three fold degenerate or *triple degenerate*. It will be noted that the groundstate (1, 1, 1), as also several other states like (2, 2, 2), are *non-degenerate*.

It can be shown with an example that the degeneracy breaks on introducing a small modification to the system. Consider, for example, the triply degenerate level, having $E = \frac{6h^2}{8ma^2}$. For this energy value, there are three independent energy states having quantum numbers (2, 1, 1), (1, 2, 1) and (1, 1, 2) for n_x, n_y, n_z . Let the length of the cubical box be increased by a small amount, da along the x -axis keeping breadth and thickness unaltered. The corresponding change in energy of the first state (2, 1, 1), which has an energy $\frac{4h^2}{8ma^2}$ associated with the x -direction, will be different from that for the other two states (1, 2, 1) and (1, 1, 2), which have an energy $\frac{h^2}{8ma^2}$ associated with the x -direction. The decrease in energy of the first state (2, 1, 1) is

energy $\frac{3}{2}k_B T$ [refer equation (6.1)]. Thus one kmol of a metal which has N_A atoms will therefore have N_A free electrons, assuming that each atom contributes one valence electron to the electron gas. Now the energy associated with one kmol of such a metal is

$$U = \frac{3}{2} N_A k_B T \quad (6.18)$$

If heat is supplied to the metal, these free electrons also absorb part of the heat, and the *molar electronic specific heat* is obtained as follows:

$$[C_{v,el}] = \left[\frac{dU}{dT} \right] = \frac{3}{2} k_B N_A = 1.5 R_u \quad (6.19)$$

where R_u is the universal gas constant
i.e.,

$$[C_{v,el}] = 1.5 \times 1.38 \times 10^{-23} \times 6.02 \times 10^{26} = 12.5 \times 10^3 \text{ J/kmol/K}$$

That is, *molar electronic specific heat* = 12.5 kJ/kmol/K. The above value of $1.5 R_u$ due to free electrons is about hundred times greater than the experimentally predicted value. Since the heat capacity of a solid due to atomic vibrations in $3 R_u$, free electrons should make a significant contribution to the total specific heat of a metal. It is, however, that at least at high temperatures, Dulong and Petit law (lattice specific heat) holds good and the total specific heat of a solid is $3 R_u$; this means that the free electrons do not contribute significantly to the heat capacity of a metal. It is, therefore, concluded that the law of equipartition and hence classical Maxwell-Boltzmann statistics must not be applied to evaluate the electronic specific heat in metals.

(ii) Computation of mean free path

The microscopic expression for the resistivity of a metal is given by the equation (6.12),

$$\rho = \frac{m}{n e^2 \tau}$$

The resistivity of the most useful metal copper at 20°C is 1.69×10^{-8} ohm-m and the concentration of free electrons in copper, $n = 8.5 \times 10^{28}/\text{m}^3$. Thus

$$\tau = \frac{m}{n e^2 \rho}$$

$$= \frac{9.11 \times 10^{-31}}{8.5 \times 10^{28} \times (1.6 \times 10^{-19})^2 \times 1.69 \times 10^{-8}}$$

i.e.,

$$\tau = 2.47 \times 10^{-14} \text{ sec}$$

But

$$\tau = \frac{\lambda}{c} \quad (6.20)$$

Referring equation (6.20)

$$\lambda = \tau \bar{c} = 2.47 \times 10^{-14} \times 1.154 \times 10^5$$

$$\lambda = 2.85 \text{ nm}$$

The experimentally found value for λ is about ten times above this value. Classical theory could not explain the large variation in λ values.

(iii) Relation between electrical conductivity and thermal conductivity (Wiedemann-Franz law)

Heat conduction in solids may take place either through the mechanism of the atoms or through that of the free electrons. In metals, electrons are the principal carriers of heat energy. Free electrons do this by being excited by energetic scattering centres and carrying the extra energy to another scattering centre in a cooler part of the metal.

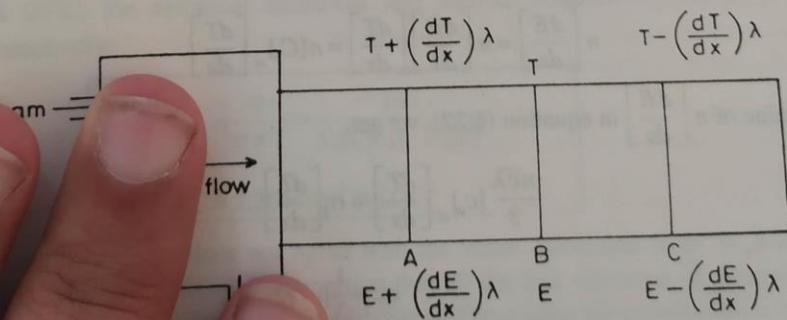


Fig. 6.10 Heat conduction through a conductor.

Consider a rod of uniform cross-section (say 1 sq. m). Let T be the temperature at section B. Consider three layers at A, B and C normal to the direction of flow. At A, the temperature is $T + \left(\frac{dT}{dx}\right)\lambda$ and the energy is $E + \left(\frac{dE}{dx}\right)\lambda$. At C, the temperature is $T - \left(\frac{dT}{dx}\right)\lambda$ and the energy is $E - \left(\frac{dE}{dx}\right)\lambda$. The distance between A and B is λ , the mean free path of the electron gas in the metal.

Similarly, we can solve equation (6.91) to obtain

$$Z = \left[\sqrt{\frac{2}{c}} \right] \sin \frac{n_z \pi z}{c} \quad (6.96)$$

The complete wave function

$$\Psi(x, y, z) = XYZ$$

i.e.,

$$\Psi = \left[\sqrt{\frac{2}{a}} \right] \sin \frac{n_x \pi x}{a} \left[\sqrt{\frac{2}{b}} \right] \sin \frac{n_y \pi y}{b} \left[\sqrt{\frac{2}{c}} \right] \sin \frac{n_z \pi z}{c}$$

$$\Psi_{n_x, n_y, n_z} = \frac{2\sqrt{2}}{\sqrt{abc}} \sin \frac{n_x \pi x}{a} \sin \frac{n_y \pi y}{b} \sin \frac{n_z \pi z}{c} \quad (6.97)$$

Equation (6.88) is

i.e.,

$$E = \frac{\hbar^2}{2m} (k_x^2 + k_y^2 + k_z^2)$$

$$E = \frac{\hbar^2}{2m} \left[\frac{n_x^2 \pi^2}{a^2} + \frac{n_y^2 \pi^2}{b^2} + \frac{n_z^2 \pi^2}{c^2} \right]$$

$$E = \frac{\hbar^2 \pi^2}{2m} \left[\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right]$$

$$E_{n_x, n_y, n_z} = \left(\frac{h^2}{8m} \right) \left[\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right] \quad (6.98)$$

where n_x , n_y and n_z denote any set of three positive integers. Equation (6.97) gives the total normalized wave functions inside the rectangular box for the stationary states. Similarly equation (6.98) gives the eigen values of the energy of the particle. These values are called the energy-levels of the particle.

If the particle is confined in a cubical box in which $a = b = c$, the eigen values of energy are given by

$$E_{n_x, n_y, n_z} = \frac{h^2}{8ma^2} [n_x^2 + n_y^2 + n_z^2] \quad (6.99)$$

$$= \frac{h^2 n^2}{8ma^2}, \text{ where } n^2 = n_x^2 + n_y^2 + n_z^2$$

and the normalized wave-functions are

$$\Psi_{n_x, n_y, n_z} = \left[\sqrt{\frac{8}{a^3}} \right] \sin \frac{n_x \pi x}{a} \sin \frac{n_y \pi y}{a} \sin \frac{n_z \pi z}{a} \quad (6.100)$$

From equations (6.99) and (6.100), we derive the following conclusions:

$$\begin{aligned} \frac{4h^2}{8ma^2} - \frac{4h^2}{8m(a+da)^2} &= \frac{4h^2}{8m} \left[\frac{1}{a^2} - \frac{1}{(a+da)^2} \right] \\ \frac{4h^2}{8m} \left[\frac{a^2 + 2da(a) + (da)^2 - a^2}{a^4} \right] &\equiv \left(\frac{4h^2}{8m} \right) \left[\frac{2da}{a^3} \right] \\ &= \frac{h^2}{8ma^2} \left(\frac{8da}{a} \right) \end{aligned} \tag{6.101}$$

But for the other two states (1, 2, 1) and (1, 1, 2), the energy will decrease only by $\left(\frac{h^2}{8ma^2} \right) \left(\frac{2da}{a} \right)$, which is shown below:

$$\begin{aligned} &= \frac{h^2}{8m} \left[\frac{1}{a^2} - \frac{1}{(a+da)^2} \right] \\ &= \frac{h^2}{8m} \left[\frac{a^2 + (da)^2 + 2a(da) - a^2}{a^4} \right] \\ &= \left(\frac{h^2}{8m} \right) \left(\frac{2da}{a^3} \right) \\ &= \left(\frac{h^2}{8ma^2} \right) \left(\frac{2da}{a} \right) \end{aligned} \tag{6.102}$$

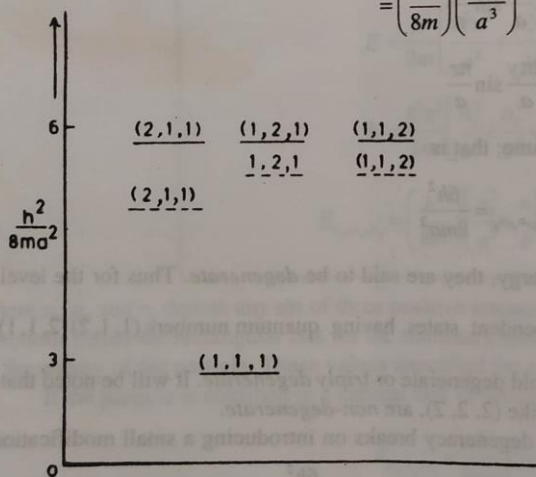


Fig. 6.15 Breakdown of degeneracy of the level (2, 1, 1).

This breakdown of degeneracy is shown in Fig. 6.15 by comparing equation (6.102) with (6.101).

Breakdown of degeneracy, which thus occurs due to some modification applied to the system, has relevance to what is often observed in practice, such as the splitting of spectral lines in a magnetic field or electric field. As a result of the application of a magnetic or electric field the degenerate level breaks up into separate levels, and consequently, spectral lines which were single become multiples.

XIV. FERMI-DIRAC STATISTICS AND ELECTRONIC DISTRIBUTION IN SOLIDS

The most important assumption of classical statistics is that any number of particles may have identical energies. Though the classical approach gives an understanding of the conductivity of metals, it fails to give a proper explanation for the contribution by electrons to the specific heat and magnetic susceptibility. As the electrons are free, the total molar specific heat of a metal at room temperature and above should be $\left[3R_u + \frac{3R_u}{2} \right], 3R_u$

being the contribution from the lattices and $\frac{3R_u}{2}$ from the electrons. But experimentally the specific heat of a metal near room temperature is only $3R_u$. Thus the specific heat associated with the electron must be very small and not $\frac{3R_u}{2}$.

In order to account for the unexpectedly low specific heat of metals, we must look more closely into the statistical behaviour of the electron gas. Such an electron gas behaves like a system of *Fermi particles* and hence obeys *Fermi-Dirac statistics*. Fermi-Dirac statistics is applied to indistinguishable particles (called *Fermions*) which are governed by Pauli's exclusion principle. Sommerfeld investigated the behaviour of a free electron gas taking into account the quantum theory and the Pauli's exclusion principle. He considered the distribution of a large number of electrons of the order of $10^{28}/\text{m}^3$ in thermal equilibrium among the various states in a three dimensional box.

Assuming that the metal is in the form of a cube of side a , we have immediately the wave function of the stationary states as

$$\Psi_{n_x, n_y, n_z} = \left[\sqrt{\frac{8}{a^3}} \right] \sin \frac{n_x \pi x}{a} \sin \frac{n_y \pi y}{a} \sin \frac{n_z \pi z}{a}$$

within the metal and zero outside, and the energy levels are

$$E_{n_x, n_y, n_z} = \frac{h^2}{8ma^2} (n_x^2 + n_y^2 + n_z^2)$$

where n_x, n_y and n_z are integers.

$$E_n = \frac{h^2 n^2}{8ma^2} \text{ with } n^2 = n_x^2 + n_y^2 + n_z^2$$

According to the Pauli's principle, not more than two electrons may occupy any orbital state, so that, at the absolute zero of temperature, two electrons will go into the ground state, two into each state of next higher energy, and so on, until all the electrons are allocated to states of lowest possible energy. Since the number of electrons is very large, it is thus understandable that, even at the absolute zero of temperature, some electrons have kinetic energies of several electron volt. For a piece of metal of macroscopic dimensions, say centimetre cube, the energy of the ground state ($n_x = n_y = n_z = 1$) is of the order of 10^{-15} eV and hence may be taken to be zero for all practical purposes. Also the maximum spacing between consecutive energy level is less than 10^{-6} eV, so that the distribution of energy may be regarded as almost continuous or sometimes '*quasi-continuous*'.

Now we shall consider how the many electrons of a real material distribute among these available energy states. Since we are talking about an almost continuous range of energies, we can represent the probability of occupying a given state by a continuous distribution function. The probability $F(E)$ of an electron occupying a given energy level is given by

$$F(E) = \frac{1}{1 + e^{\frac{E - E_F}{k_B T}}} \quad (6.103)$$

Here $F(E)$ is called the Fermi function, E is the energy of the level whose occupancy is being considered, E_F is the Fermi level and is a constant for the particular system. At absolute zero $F(E) = 0$ for $E > E_F$ and $F(E) = 1$ for $E < E_F$. Thus at absolute zero, the Fermi level divides the occupied states from the unoccupied states; i.e., it is the highest energy state for the electrons to occupy at absolute zero.

near the Fermi level take part in thermal and electrical conductions.

IV. RELAXATION TIME, COLLISION TIME AND MEAN FREE PATH

(i) *Relaxation time*

Suppose at a given time $t = 0$, the average velocity of the electron is $\langle v_x \rangle_0$. Just at that instant we switch the field off, so that $\langle v_x \rangle$ will subsequently tend to zero. In other words, the velocity will be randomized by electron-lattice collisions. Now let us assume that this process follows the simplest law of decay.

$$\langle v_x \rangle = \langle v_x \rangle_0 e^{-t/\tau} \quad (6.27)$$

The factor τ is a constant called *relaxation time*, because it gives a measure of the time that the system takes to relax when a constraint (the electric field) is removed. When $t = \tau$, the equation (6.27) becomes

$$\langle v_x \rangle = \frac{\langle v_x \rangle_0}{e}$$

Thus relaxation time may be stated as the time taken for the drift velocity to decay to $\frac{1}{e}$ of its initial value.

Differentiating equation (6.27)

ssp
page 1

$$\left[\frac{d}{dt} \langle v_x \rangle \right]_{elec-latt} = - \frac{\langle v_x \rangle_0 e^{-t/\tau}}{\tau}$$

$$\left[\frac{d}{dt} \langle v_x \rangle \right]_{elec-latt} = - \frac{\langle v_x \rangle}{\tau} \tag{6.28}$$

(ii) Collision time

In the last section we introduced the concept of relaxation time in a highly mathematical way. However, we shall now show that it also has a very simple physical meaning. Suppose that the probability of an electron making a collision in time dt is $\frac{dt}{\tau_c}$, so that on an average there are $\frac{1}{\tau_c}$ collisions per second. It is now our

intention in what follows is to show that $\tau_c = \tau$.

Let us assume that all electron collisions are elastic; that is, energy is conserved, so that the speed of the electron is the same before and after collision. This assumption is not strictly correct, because a conductor warms up when current is passing; it implies that energy is transferred to the lattice during electron-lattice collisions. However, the actual energy change at each collision is minute, and the fractional change in velocity in a metal is typically 1 in 10^6 , so that the assumption is entirely reasonable. If we assume that after collision the velocities are completely random, we have

$$\langle v_x \rangle_{after} = 0. \text{ Since } \langle v_x \rangle_{before} = \langle v_x \rangle,$$

it follows that the change of $\langle v_x \rangle$ on collision = $-\langle v_x \rangle$.

There are $\frac{1}{\tau_c}$ collisions per second, so that

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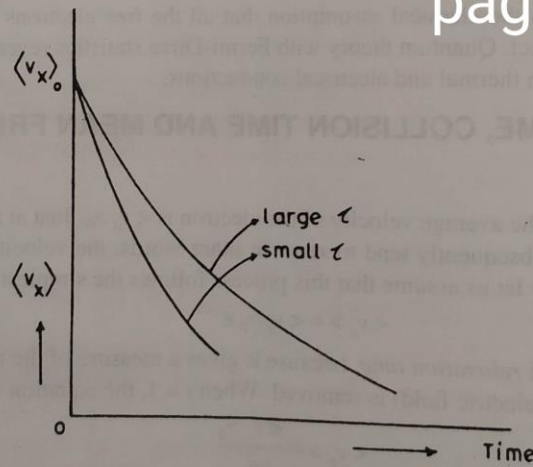


Fig. 6.3 Relaxation of a system of electrons to equilibrium after removing the electric field as given by equation (6.27).

the rate of change of

$$\frac{d}{dt} \langle v_x \rangle = - \frac{\langle v_x \rangle}{\tau_c} \tag{6.29}$$

However, we could equally write the left-hand side of this equation as

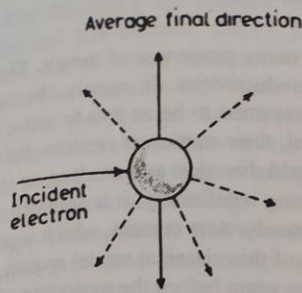


Fig. 6.4(a) Scattering equally probable in all directions, so that the average angle is 90°.

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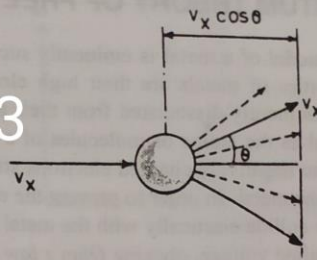


Fig. 6.4(b) Scattering predominantly in the forward direction at average angle θ . Change in any v_x is therefore, on an average, $v_x (1 - \cos \theta)$.

$$\left[\frac{d \langle v_x \rangle}{dt} \right]_{\text{elect-latt}}$$

so that on comparing equations (6.29) and (6.28) we see that $\tau = \tau_c$ and the relaxation time, therefore, equals the mean time between collisions.

In deriving this result, we assumed that the velocities took up random directions after each collision [Fig. 6.4(a)]; this is equivalent to stating that after each collision the electron had no memory of what went before. However, it is possible that the scattering is very weak, in which case such an assumption is unrealistic. If, on an average, the velocity direction changes by θ on collision [see Fig. 6.4 (b)], then the change of $\langle v_x \rangle$ is on an average, $-\langle v_x \rangle (1 - \cos \theta)$. Consequently

$$\tau = \frac{\tau_c}{[1 - \langle \cos \theta \rangle]} \tag{6.30}$$

where τ is the relaxation time and τ_c is the mean time between collisions.

In other words, the relaxation time is then much longer than the mean time between collisions; these electrons have a memory.

(iii) Mean free path

The mean free path λ of an electron is defined as below:

$$\lambda = \bar{c} \tau_c$$

In case, the obstacles are hard spheres, the mean free path λ is determined by the concentration of these obstacles. Then for a given concentration of obstacles, the collision time τ_c becomes inversely proportional to the concentration of obstacles, independent of the velocity of the electrons, in accor-

Temperature dependence of electrical resistivity

In the absence of an electric field, the free electrons in a metal will be moving about at random in all directions and will be in temperature equilibrium with it. The kinetic energy associated with the electron is

$$\frac{1}{2} m(\bar{c})^2 = \frac{3}{2} k_B T$$

When an electric field is applied, the electron will acquire a drift velocity and the resulting acceleration is, $a = \frac{eE}{m}$. The drift velocity is small compared to the random velocity \bar{c} . Further the drift velocity is not retained after a collision with an atom because of the relatively large mass of the atom. Hence just after a collision the drift velocity is zero. If the mean free path is λ , then the time that elapses before the next collision takes place is $\frac{\lambda}{\bar{c}}$. Hence the drift velocity acquired before the next collision takes place is

$$u = \text{acceleration} \times \text{time interval}$$

$$= \left(\frac{eE}{m} \right) \left(\frac{\lambda}{\bar{c}} \right)$$

Thus the average drift velocity is

$$\frac{u}{2} = \frac{eE\lambda}{2m\bar{c}}$$

If n is the number of electrons per unit volume, then the current flowing through unit area for unit time is

$$J_x = \frac{neu}{2} = \frac{ne^2 E \lambda}{2m\bar{c}}$$

or

$$\sigma = \frac{J_x}{E} = \frac{ne^2 \lambda}{2m\bar{c}}, \text{ or } \rho = \frac{2m\bar{c}}{ne^2 \lambda} \quad (6.16)$$

i.e.

$$\rho = 2 \times \sqrt{\frac{3mk_B T}{ne^2 \lambda}}; \sigma = \frac{ne^2 \lambda}{\sqrt{12mk_B T}} \quad (6.17)$$

It was assumed by Drude and Lorentz that λ is independent of temperature and that is of the order of interatomic distance. Hence $\rho \propto \sqrt{T}$. This means that the specific resistance of an electric conductor is directly proportional to the square root of the absolute temperature. This is not in agreement with the experimental observation that $\rho \propto T$. Apart from this discrepancy, it is also not correct to assume that the mean free path is independent of temperature and hence this classical theory is almost an unacceptable one. However, the Ohm's law is derived, since the conductivity in equation (6.16) is independent of the field.

III. DRAWBACKS OF CLASSICAL THEORY

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Debye's model of Lattice heat capacity

Einstein model assumed that atoms vibrate totally independent, and with same frequency ω_0 natural frequency of vib of a single atom

but in real these atoms are coupled together which cannot vibrate independently Debye consider this fact in his model

A crystal can have a number of modes of vib, number of vibrational mode per unit length frequency range is called density of states of modes $Z(f)$

1.1 Density of modes

one dimensional wave equation

$$\frac{\partial^2 Y}{\partial x^2} = \frac{1}{v_s^2} \frac{\partial^2 Y}{\partial t^2} \quad \text{--- (i)}$$

v_s = velocity of propagation of wave

solution of above eq is

$$Y(x,t) = A \sin\left(\frac{n\pi}{L}x\right) \cos 2\pi ft$$

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put this value of $Y(x,t)$ in above eq (i)

$$A \cos 2\pi f t \frac{\partial}{\partial x} \left[\cos \left(\frac{n\pi x}{L} \right) \right] \cdot \frac{n\pi}{L} = \frac{2\pi f}{v_s^2} \frac{\partial}{\partial t} \left[A \sin \left(\frac{n\pi x}{L} \right) (-\sin 2\pi f t) \right]$$

$$A \cos 2\pi f t \left[-\sin \left(\frac{n\pi x}{L} \right) \cdot \frac{n^2 \pi^2}{L^2} \right] = \frac{4\pi^2 f^2}{v_s^2} \left[A \sin \left(\frac{n\pi x}{L} \right) \cos 2\pi f t \right]$$

$$\frac{n^2}{L^2} = \frac{4f^2}{v_s^2}$$

$$n^2 = \frac{4f^2 L^2}{v_s^2}$$

three dimensional wave eq.

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 y}{\partial z^2} = \frac{1}{v_s^2} \frac{\partial^2 y}{\partial t^2}$$

Solution

$$Y(x,y,z,t) = A \sin \left(\frac{n_x \pi}{L} x \right) \sin \left(\frac{n_y \pi}{L} y \right) \sin \left(\frac{n_z \pi}{L} z \right) \cos 2\pi f t$$

put in above

$$n_x^2 + n_y^2 + n_z^2 = \frac{4f^2 L^2}{v_s^2}$$

$$n_x^2 + n_y^2 + n_z^2 = R^2 = \frac{4f^2 L^2}{v_s^2} \quad \text{--- (ii)}$$

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$$x^2 + y^2 + z^2 = R^2$$

this is the eq of sphere of radius
 R

volume of the sphere

$$V = \frac{4}{3} \pi R^3$$

$$\frac{dV}{dR} = \frac{4}{3} \pi R^2$$

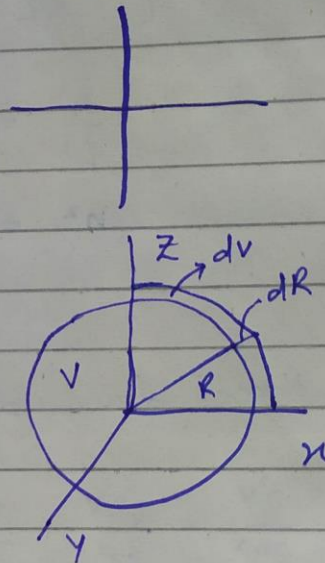
$$dV = 4\pi R^2 dR$$

number of modes present
in the frequency range
 f and $f + df$ should be
the same as number of
points in volume V and $V + dV$
or in the range R and $R + dR$ of
radius vector

number of points is directly proportional
to the volume of sphere

$$n = \frac{4}{3} \pi R^3$$

$$\frac{dn}{dR} = 4\pi R^2$$



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$$dn = 4\pi R^2 dR$$

we consider only positive values of n_x , n_y and n_z lies in $\frac{1}{8}$ portion of sphere
so number of possible modes of vib

$$\sum (f) df = \frac{1}{8} (4\pi R^2 dR)$$

put value of R^2 in above from eq (ii)

$$= \frac{1}{8} \left(4\pi \frac{4f^2 L^2}{v_s^2} dR \right) \quad \text{--- (iii)}$$

eq (ii)

$$R^2 = \frac{4f^2 L^2}{v_s^2}$$

$$2R \frac{dR}{df} = \frac{4 \times 2f L^2}{v_s^2}$$

$$dR = \frac{4f L^2}{v_s^2 R} df = \frac{4f L^2 df}{v_s^2 \sqrt{\frac{4f^2 L^2}{v_s^2}}}$$

$$dR = \frac{2L}{v_s} df$$

put dR in eq (iii)

$$\sum (f) df = \frac{1}{8} \left[8 \times 2 \pi \frac{f^2 L^2}{v_s^2} \left(\frac{2L}{v_s} df \right) \right]$$

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$$Z(f) df = \frac{4f^2 L^3 \pi df}{V_s^3}$$

as L^3 is volume of solid cube

$$L^3 = V$$

$$Z(f) df = \frac{4L^3 f^2 \pi df}{V_s^3}$$

$$= 4\pi V \left[\frac{1}{V_s^3} \right] f^2 df$$

elastic wave propagating in solid are of two types transverse & longitudinal

V_l = longitudinal wave velocity

V_t = transverse wave velocity

two vib modes perpendicular to direction of propagation in transverse and one mode of vib in longitudinal

So the total vib modes

$$Z(f) df = 4\pi V \left[\frac{2}{V_t^3} + \frac{1}{V_l^3} \right] f^2 df$$

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Debye approximation

- if wavelength is large then interatomic distance \rightarrow medium continuum

- All modes of vib are possible in continuum medium.

- crystal consists of finite number of atoms say N

so the maximum number of mode of vib is $3N$

so there is a limit on the highest frequency prop through the crystal this maximum frequency is called debye frequency ' f_D '.

total number of vib modes from frequency $0 \rightarrow f_D$ are

$$\int_0^{f_D} Z(f) df = 3N$$

$$\int_0^{f_D} 4\pi V \left[\frac{2}{V_t^3} + \frac{1}{V_l^3} \right] f^2 df = 3N$$

$$4\pi V \left[\frac{2}{V_t^3} + \frac{1}{V_l^3} \right] \frac{f^3}{3} \Big|_0^{f_D} = 3N$$

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$$f_D^3 = \frac{9N}{4\pi V} \left[\frac{2}{V_t^3} + \frac{1}{V_l^3} \right]^{-1}$$

by putting the values of V_t and V_l

$$f_D \approx 10^{13} \text{ s}^{-1} \longrightarrow \lambda \approx 1 \text{ \AA}$$

this wavelength is same as interatomic distance this violates the basic assumption of debye

To calculate the vib energy of the crystal we use plank's law as average energy of oscillator at T

$$\bar{\epsilon} = \frac{hf}{e^{hf/KT} - 1}$$

Thus vib energy of crystal

$$E = \int_0^{f_D} \bar{\epsilon} z(f) df$$

$$= 4\pi hV \left[\frac{2}{V_t^3} + \frac{1}{V_l^3} \right] \int_0^{f_D} \frac{f^3 df}{e^{hf/KT} - 1}$$

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$$\text{as } 4\pi V \left[\frac{2}{V_1^3} + \frac{1}{V_2^3} \right] = \frac{9N}{f_0^3}$$

$$E = \frac{9Nh}{f_0^3} \int_0^{f_0} \frac{f^3 df}{e^{hf/KT} - 1}$$

$$\frac{hf}{KT} = x \quad \text{and} \quad \frac{hf_0}{KT} = x_m$$

$$\text{we get } f = \frac{xKT}{h}$$

$$df = \frac{KT}{h} dx$$

$$E = \frac{9Nh}{f_0^3} \left(\frac{KT}{h} \right)^4 \int_0^{x_m} \frac{x^3 dx}{e^x - 1}$$

$$= 9N \left(\frac{KT}{hf_0} \right)^3 KT$$

$$\text{Debye temperature } \theta_D = \frac{hf_0}{K}$$

$$x_m = \frac{\theta_D}{T}$$

$$E = 9NKT \left(\frac{T}{\theta_D} \right)^3 \int_0^{\theta_D/T} \frac{x^3 dx}{e^x - 1}$$

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heat capacity

$$C_v = \left(\frac{\partial E}{\partial T} \right)_v$$

$$C_v = 9NK \left(\frac{T}{\theta_D} \right)^3 \int_0^{x_m} \frac{e^x x^4}{(e^x - 1)^2} dx$$

$$= 3R \left(\frac{\theta_D}{T} \right) F_D \quad \because N = N_A$$

$$F_D = 3 \left(\frac{T}{\theta_D} \right)^4 \int_0^{\theta_D/T} \frac{e^x x^4}{(e^x - 1)^2} dx$$

(i) high temp case $T \gg \theta_D$

$$e^x - 1 \approx x$$

$$E = 9NKT \left(\frac{T}{\theta_D} \right)^3 \int_0^{\theta_D/T} x^2 dx$$

$$E = 3NKT$$

$$C_v = 3R \quad (\text{Dulong and Petit's law})$$

ii) low temp $T \ll \theta_D$

$$\frac{\theta_D}{T} = x_m \rightarrow \infty$$

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$$E = 9NKT \left(\frac{T}{\theta_D}\right)^3 \int_0^{\infty} \frac{x^3 dx}{e^x - 1}$$

$$E = 9NKT \left[\frac{T}{\theta_D}\right]^3 \left[\frac{\pi^4}{15}\right]$$

$$= \frac{3}{5} \pi^4 NK \frac{T^4}{\theta_D^3}$$

vib energy proportional T^4 which is analogous stefen's law of Blackbody

so photons and phonons obey the same statistics

photon obey T^4 at all temp

phonon obey T^4 at low temp

$$C_v = \frac{\partial E}{\partial T} = \frac{12}{5} \pi^4 NK \left(\frac{T}{\theta_D}\right)^3$$

$$= \frac{12}{5} \pi^4 R \left(\frac{T}{\theta_D}\right)^3$$

at very low temp $C_v \propto T^3$

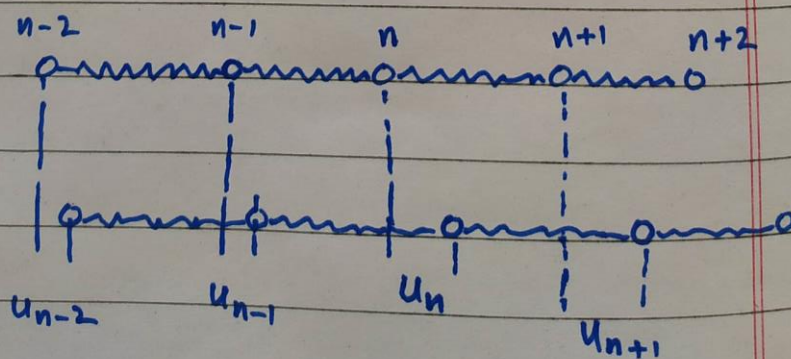
this is called Debye T^3 law

Lattice vibrations

- motion of atom is coupled
- Lattice may vib freely due to internal energy
- Lattice also vib under the effect of dynamical external forces which may be mechanical or electromagnetic in nature

these vib describes the thermal properties such as specific heat and thermal conductivity.

one dimensional chain of atoms



The equilibrium state of atom is $n, n-1, n-2, n+1, n+2$ if interatomic distance is 'a' then x-coordinates of the atoms are $na, (n-1)a, (n+1)a, \dots$ when atoms vibrates the distance from mean position be $u_n, u_{n-1}, u_{n+1}, \dots$

according to Hook's law

$$F = k u$$

Spring constant \swarrow \searrow displacement

the net force on nth atom is

$$F = k(u_{n+1} - u_n) - k(u_n - u_{n-1})$$

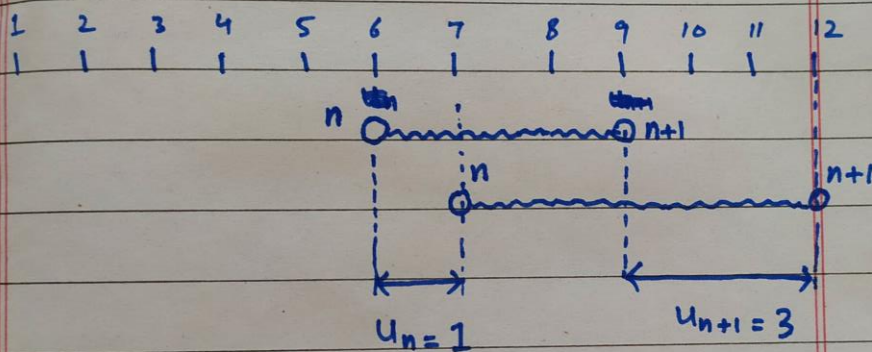
$$F = k [u_{n+1} + u_{n-1} - 2u_n]$$

as $F = m \frac{d^2 u_n}{dt^2}$

Hint

How this equation comes

$$F = k(u_{n+1} - u_n) - k(u_n - u_{n-1})$$



$$F = -k \Delta x$$

$$x_f = 12 - 7 = 5$$

$$F = -k(2)$$

$$x_i = 9 - 6 = 3$$

$$\Delta x = x_f - x_i$$

$$F = k(u_{n+1} - u_n)$$

$$= 5 - 3$$

$$= 2$$

$$F = k(3 - 1)$$

$$\text{as } u_{n+1} = 3$$

$$F = k(2)$$

$$u_n = 1$$

$$m \frac{d^2 u_n}{dt^2} = k [u_{n+1} + u_{n-1} - 2u_n] \quad \text{--- (i)}$$

Solution of above differential eq is

$$u_n = u_0 e^{i(\omega t - kna)} \quad \text{--- (iv)}$$

displacement of n th atom $x = na$

$$k = 2\pi/\lambda$$

Similarly for $(n+1)$ th and $(n-1)$ th atoms

$$(ii) \quad u_{n+1} = u_0 e^{i[\omega t - k(n+1)a]}$$

$$(iii) \quad u_{n-1} = u_0 e^{i[\omega t - k(n-1)a]}$$

put value of u_n , u_{n+1} and u_{n-1} in eq (i)

$$m \omega^2 u_0 e^{i(\omega t) - ikna} = k \left[u_0 e^{i\omega t - ikna - ika} + u_0 e^{i\omega t - ikna + ika} - 2u_0 e^{i\omega t - ikna} \right]$$

$$-m\omega^2 u_0 e^{i\omega t - ikna} = k u_0 e^{i\omega t - ikna} \left[e^{-ika} + e^{ika} - 2 \right]$$

$$-m\omega^2 = k \left[e^{ika} + e^{-ika} - 2 \right]$$

$$= k \left[e^{ika/2} - e^{-ika/2} \right]^2$$

$$\text{as } \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\sin^2 x = -\frac{1}{4} \left(e^{ix} - e^{-ix} \right)^2$$

$$-m\omega^2 = -4k \sin^2 \left(\frac{ka}{2} \right)$$

$$\omega = \pm \sqrt{\frac{4k}{m}} \sin \left(\frac{ka}{2} \right)$$

longitudinal stiffness $c = ka$

mass per unit length $\rho = m/a$

$$\omega = \pm 2 \sqrt{\frac{c/a}{\rho a}} \sin \left(\frac{ka}{2} \right)$$

$$\omega = \pm \frac{2}{a} \sqrt{\frac{c}{\rho}} \sin \left(\frac{ka}{2} \right)$$

V_s = velocity of sound wave
in solids

$$V_s = \sqrt{\frac{c}{\rho}}$$

$$\omega = \pm \frac{2}{a} V_s \sin\left(\frac{Ka}{2}\right)$$

for 1st Brillion Zone $K = \pm \pi/a$

frequency should always positive
irrespective of sign of K
thus we have

$$\omega = \sqrt{\frac{4\beta k}{m}} \left| \sin\left(\frac{Ka}{2}\right) \right|$$

$$\omega = \frac{2}{a} V_s \left| \sin\left(\frac{Ka}{2}\right) \right|$$

this relation is called dispersion
relation

Case (i)

low frequency \rightarrow large λ

$$K = \frac{2\pi}{\lambda} = \frac{2\pi}{\omega} \approx 0$$

So

$$K \rightarrow 0$$

As for small angle $\theta \approx 0$

$$\sin \theta \approx \theta$$

$$\sin\left(\frac{Ka}{2}\right) \rightarrow \frac{Ka}{2}$$

$$\omega = \frac{z}{a} v_s \frac{Ka}{2}$$

$$(i) \text{ --- } \omega = v_s K$$

$$v_s = \frac{\omega}{K}$$

we know that

$$\text{phase velocity} = v_p = \frac{\omega}{K}$$

$$\text{group velocity} = v_g = \frac{d\omega}{dK}$$

As

$$v_s = \frac{\omega}{K} = v_p$$

Now to calculate V_g diff
eq (i) w.r.t 'K'

$$\omega = V_s K$$

$$\frac{d\omega}{dK} = V_s$$

So $V_g = V_s$

it is concluded that at low
frequency

$$V_s = V_p = V_g$$

So, at low frequency or greater
(larger) wavelength the atomic
nature of solids is of little
importance and it behaves like
a homogenous and continuous medium

Case (ii)

high frequencies \rightarrow shorter λ

$$V_p = \frac{\omega}{K} = \frac{\frac{2}{a} V_s \sin\left(\frac{Ka}{2}\right)}{K}$$

$$V_p = \frac{2V_s}{ka} \sin\left(\frac{ka}{2}\right)$$

$$V_g = \frac{d\omega}{dk} = \frac{2V_s}{ka} \cos\left(\frac{ka}{2}\right) \frac{a}{2}$$

$$V_g = V_s \cos\left(\frac{ka}{2}\right)$$

both V_p and V_g depends on k and k is related with frequency $\omega = \frac{2}{a} V_s \left| \sin \frac{ka}{2} \right|$

So V_p and V_g are functions of frequency. This is referred to as phenomenon of dispersion and medium is called dispersive medium

Case (iii)

at frequency $\omega = \sqrt{\frac{4k}{m}}$

which is called maximum angular frequency

$$\omega = \sqrt{\frac{4R}{m}} \left| \sin \frac{Ka}{2} \right|$$

$$\sqrt{\frac{4R}{m}} = \sqrt{\frac{4R}{m}} \left| \sin \frac{Ka}{2} \right|$$

$$1 = \sin \frac{Ka}{2}$$

for ~~when~~ $K = \frac{\pi}{a}$ above condition satisfied

$$\sin \frac{\pi a}{a 2} = \sin \frac{\pi}{2} = 1$$

$$\text{As } K = \frac{2\pi}{\lambda} = \frac{\pi}{a}$$

$$\boxed{2a = \lambda}$$

$$V_p = \frac{\omega}{K} = \frac{2V_s}{Ka} \left| \sin \frac{Ka}{2} \right|$$

$$\boxed{K = \frac{\pi}{a}}$$

$$\boxed{Ka = \pi}$$

$$V_p = \frac{2V_s}{\pi} \left| \sin \frac{\pi}{2} \right|$$

$$V_p = \frac{2V_s}{\pi}$$

group velocity $V_g = \frac{d\omega}{dk}$

$$V_g = V_s \cos \frac{ka}{2}$$

as $ka = \pi$

$$V_g = V_s \cos \left(\frac{\pi}{2} \right)$$

$$V_g = 0$$

it follows that there is no transfer of signal at this frequency ' $\omega = \sqrt{\frac{4F}{m}}$ '

It is concluded from the above description that only the vib of $\omega < \sqrt{\frac{4F}{m}}$ can propagate

through the Lattice
hence lattice behaves as low pass filter



SOLID STATE PHYSICS

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Note: This lecture is a condensed version extracted from several full-semester lectures posted by Prof. Beşire Gönül , Turkey. <http://www1.gantep.edu.tr/~bgonul/dersnotlari/ss/>