Proof.

$$[L^2, L_+] = 0.$$

L.H.S.

$$= [L^2, L_+] = L^2 L_+ - L_+ L^2 \quad \text{— (1)}$$

By Putting the value of $L_+ = L_x + iL_y$ in (1).
we will get.

$$= L^2(L_x + iL_y) - (L_x + iL_y)L^2$$

$$= L^2 L_x + i L^2 L_y - L_x L^2 - i L_y L^2$$

$$= L^2 L_x - L_x L^2 + i L^2 L_y - i L_y L^2$$

$$= (L^2 L_x - L_x L^2) + i(L^2 L_y - L_y L^2) \quad \text{— (2)}$$

as we know L^2, commute with $\hat{L}$ Component

$$[L^2, L_x] = L^2 L_x - L_x L^2 = 0$$

So. equ (2) becomes.

$$= 0 + i(0) = 0,$$ hence proved.

$\boxed{[L^2, L_+] = 0}$, Similarly $\boxed{[L^2, L_-] = 0}$

Proof.

$$[L_+, L_-] = 2\hbar L_z$$

L.H.S.

$$= [L_+, L_-] = L_+L_- - L_-L_+$$

$$= (L_x + iL_y)(L_x - iL_y) - (L_x - iL_y)(L_x + iL_y)$$

$$= L_x^2 - iL_xL_y + iL_yL_x - i^2L_y - [L_x^2 + iL_xL_y - iL_yL_x - i^2L_y^2]$$

$$= \cancel{L_x^2} - iL_xL_y + iL_yL_x - i^2L_y - \cancel{L_x^2} - iL_xL_y + iL_yL_x + \cancel{i^2L_y}$$

$$= -2iL_xL_y + 2iL_yL_x$$

$$= -2i(L_xL_y - L_yL_x)$$

$$= -2i(i\hbar L_z)$$

$$= -2i^2\hbar L_z = 2\hbar L_z$$

Hence,

$$\boxed{[L_+, L_-] = 2\hbar L_z}$$

Proof.

$$[L_z, L_+] = \hbar L_+$$

L.H.S.

$$= [L_z, L_+] = L_z L_+ - L_+ L_z$$

$$= L_z(L_x + iL_y) - (L_x + iL_y) L_z$$

$$= L_z L_x + i L_z L_y - L_x L_z - i L_y L_z$$

$$= (L_z L_x - L_x L_z) - i (L_y L_z - L_z L_y)$$

$$= i\hbar L_y - i(i\hbar L_x)$$

$$= i\hbar L_y - i^2 \hbar L_x \qquad \because [L_z, L_x] = i\hbar L_y$$

$$[L_y, L_z] = i\hbar L_x$$

$$= i\hbar L_y + \hbar L_x$$

$$= \hbar(L_x + iL_y)$$

$$= \hbar L_+$$

Hence prove

$$\boxed{[L_z, L_+] = \hbar L_+}$$

Similarly

$$\boxed{[L_z, L_-] = \hbar L_-}$$

Proof.

$\hat{L}_x = \frac{1}{2}(L_+ + L_-)$ and $\hat{L}_y = \frac{1}{2i}(L_+ - L_-)$

i). $\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$

$= \frac{1}{2}(\hat{L}_+ + \hat{L}_-) = \frac{1}{2}(L_x + \cancel{iL_y} + L_x - \cancel{iL_y})$

$= \frac{1}{2} 2L_x = \hat{L}_x$

(ii) $\hat{L}_y = \frac{1}{2i}(L_+ - L_-)$

$= \frac{1}{2i}(L_x + iL_y - L_x + iL_y)$

$= \frac{1}{2i} 2i\hat{L}_y = \hat{L}_y$

Proof.

$\hat{L}^2 = \frac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+) + \hat{L}_z^2$

$= \frac{1}{2}(L_+L_- + \hat{L}_-\hat{L}_+) + \hat{L}_z^2$ —(1)

we know. $L_+L_- = \hat{L}^2 - L_z^2 + \hbar L_z$.

and $L_-L_+ = L^2 - L_z^2 - \hbar L_z$

Then equ (1) becomes

$= \frac{1}{2}(L^2 - L_z^2 + \cancel{\hbar L_z} + L^2 - L_z^2 - \cancel{\hbar L_z}) + L_z^2$

$$= \frac{1}{2}(2L^2 - 2L_z^2) + L_z^2$$

$$= L^2 - L_z^2 + L_z^2 = L^2 = L.H.S.$$

Now consider

$$L^2(L_+\psi) = (L^2 L_+)\psi$$

so,

As $[L^2, L_+] = 0$

$$= L_+ L^2 \psi$$

$$= L_+ (L^2\psi)$$

$\because L^2\psi = \lambda\psi$

$$= L_+(\lambda\psi)$$

$$= \lambda(L_+\psi)$$

If $L_+\psi \neq 0$, then $L_+\psi$ is an eigenfunction of L^2 with eigenvalue λ.

Similarly

$$L_z(L_+\psi) = (L_z L_+)\psi, \quad \text{As } [L_+, L_z] = -\hbar L_+$$

$$= (L_+L_z + \hbar L_+)\psi$$

$L_+L_z - L_zL_+ = -\hbar L_+$

$$= L_+(L_z\psi) + \hbar(L_+\psi)$$

$L_+L_z + \hbar L_+ = L_zL_+$

$$= L_+(\mu\psi) + \hbar(L_+\psi), \quad \because L_z\psi = \mu\psi$$

$$= \mu(L_+\psi) + \hbar(L_+\psi)$$

$$= (\mu+\hbar)(L_+\psi)$$

If $L_+\psi \neq 0$, then $L_+\psi$ is an eigenfn. of L_z with eigenvalue $\mu+\hbar$. $L_+\psi$ is a simultaneous eigenfunction of L^2 and L_z for which the eigenvalue of L^2 is left unchanged but for which the eigenvalue of L_z is increased by $\hbar$. Thus L_+ can be considered as step up operator.

Similary for $L_-\psi \neq 0$,

$$L^2(L_-\psi) = \lambda(L_-\psi)$$

and $L_z(L_-\psi) = (\mu-\hbar)(L_-\psi)$

We, therefore, get

$$\cdots (L_-)^2\psi, \; L_-\psi, \; \psi, \; L_+\psi, \; (L_+)^2\psi, \cdots$$

as a set of eigenfunction of L^2 and L_z.

The corresponding eigenvalues of L^2

$$\cdots, \lambda, \lambda, \lambda, \lambda, \lambda, \cdots$$

and L_z are

$$\mu-2\hbar, \; \mu-\hbar, \; \mu, \; \mu+\hbar, \; \mu+2\hbar, \cdots$$

Let us consider the inner product

$(L_-\psi, L_-\psi) \geq 0$ for any $\psi \in L^2$

$(\psi, L_+L_-\psi) \geq 0$, $\because L_-^\dagger = L_+$

$(\psi, (L^2 - L_z^2 + \hbar L_z)\psi) \geq 0$ also $L_+L_- = L^2 - L_z^2 + \hbar L_z$

$(\psi, L^2\psi - L_z^2\psi + \hbar L_z\psi) \geq 0.$

$(\psi, \lambda\psi - \mu^2\psi + \hbar\mu\psi) \geq 0.$

or $(\lambda - \mu^2 + \hbar\mu)(\psi, \psi) \geq 0$

$\Rightarrow$

$\lambda - \mu^2 + \hbar\mu \geq 0$

As:

$$\left(L_+ \psi_{max}, L_+ \psi_{max}\right) \geq 0$$

$$\left(\psi_{max}, L_- L_+ \psi_{max}\right) \geq 0$$

$$\Rightarrow \left(\psi_{max}, \left(L^2 - L_z^2 - \hbar L_z\right) \psi_{max}\right) \geq 0$$

$$\left(\psi_{max}, \left(\lambda - \mu_{max}^2 - \hbar \mu_{max}\right) \psi_{max}\right) \geq 0$$

$\Rightarrow$

$$\lambda - \mu_{max}^2 - \hbar \mu_{max} \geq 0$$

$$\mu_{max}^2 + \hbar \mu_{max} - \lambda = 0$$

$$\mu_{max} = \frac{-\hbar \pm \sqrt{\hbar^2 + 4\lambda}}{2}$$

$$\boxed{\mu_{max} = -\frac{\hbar}{2} \pm \sqrt{\lambda + \frac{\hbar^2}{4}}}$$

$$\lambda - (u^2 - t_s u) \geq 0$$

$$\lambda - \left[u^2 - t_s u + \left(\frac{t_s}{2}\right)^2 - \left(\frac{t_s}{2}\right)^2\right] \geq 0$$

$$\lambda - \left[\left(u - \frac{t_s}{2}\right)^2 - \frac{t_s^2}{4}\right] \geq 0$$

$$\lambda + \frac{t_s^2}{4} \geq \left(u - \frac{t_s}{2}\right)^2$$

either

$$\sqrt{\lambda + \frac{t_s^2}{4}} \geq +\left(u - \frac{t_s}{2}\right)$$

or

$$\sqrt{\lambda + \frac{t_s^2}{4}} \geq -\left(u - \frac{t_s}{2}\right)$$

$\Rightarrow$ either

$$\frac{t_s}{2} + \sqrt{\lambda + \frac{t_s^2}{4}} \geq u$$

or

$$-\frac{t_s}{2} + \sqrt{\lambda + \frac{t_s^2}{4}} \geq -u$$

$$\Rightarrow u > \frac{t_s}{2} - \sqrt{\lambda + \frac{t_s^2}{4}}$$

This means that μ lies b/w two values i.e

$$\boxed{\frac{\hbar}{2} - \sqrt{\lambda + \frac{\hbar^2}{4}} \leq \mu \leq \frac{\hbar}{2} + \sqrt{\lambda + \frac{\hbar^2}{4}}}$$ in

$\Rightarrow$ Let suppose μ_{min} and μ_{max} be the minimum and maximum eigenvalues of L_z, respectively, then

$$L_z \psi_{min} = \mu_{min} \psi_{min}$$

$$L_z \psi_{max} = \mu_{max} \psi_{max}$$

and $\quad L_- \psi_{min} = 0, \quad L_+ \psi_{max} = 0$

Again take the inner product of $L_- \psi_{mi}$

$$(L_- \psi_{min}, L_- \psi_{min}) = 0$$

$$(\psi_{min}, L_+ L_- \psi_{min}) = 0$$

$$(\psi_{min}, (L^2 - L_z^2 + \hbar L_z) \psi_{min}) = 0$$

$$\left(\psi_{min}, \left(\lambda - \mu_{min}^2 + k_1 \mu_{min}\right) \psi_{min}\right) = 0$$

$$\left(\lambda - \mu_{min}^2 + k_1 \mu_{min}\right) \left(\psi_{min}, \psi_{min}\right) = 0$$

$$\Rightarrow \lambda - \mu_{min}^2 + k_1 \mu_{min} = 0.$$

$$\Rightarrow \mu_{min} = \frac{k_1 \pm \sqrt{k_1^2 + 4\lambda}}{2}$$

$$\boxed{\mu_{min} = \frac{k_1}{2} \pm \sqrt{\lambda + \frac{k_1^2}{4}}}$$

Similarly

$$\boxed{\mu_{max} = -\frac{k_1}{2} \pm \sqrt{\lambda + \frac{k_1^2}{4}}}$$

This is true for all values of λ.

we choose sign such that

$$\mu_{min} = \frac{k_1}{2} - \sqrt{\lambda + \frac{k_1^2}{4}}$$

and

$$\mu_{max} = -\frac{k_1}{2} + \sqrt{\lambda + \frac{k_1^2}{4}}$$

Hence, the eigenvalues of L_z are therefore

$\mu_{min}, \mu_{min}+\hbar, \ldots, \mu_{max}-\hbar, \mu_{max}$

This shows that the difference $\mu_{max}-\mu_{min}$, is an integral multiple of $\hbar$

Let us write

$$\mu_{max}-\mu_{min} = 2l\hbar, \quad 2l = 0, 1, 2, \ldots$$

$$2\sqrt{\lambda+\frac{\hbar^2}{4}} = 2l\hbar+\hbar$$

$$\boxed{\lambda = l(l+1)\hbar^2}, \text{ where } l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2,$$

from this we get the value of μ_{min} and μ_{max}

$$\mu_{min} = \frac{\hbar}{2}-\sqrt{\lambda+\frac{\hbar^2}{4}}$$

$$= \frac{\hbar}{2}-\sqrt{l(l+1)\hbar^2+\frac{\hbar^2}{4}}$$

$$\boxed{\mu_{min} = -l\hbar}$$

Similarly $\boxed{M_{max} = +l\hbar}$,

Hence, we conclude that the eigenvalues of L^2 are

$$l(l+1)\hbar^2 \quad , \quad l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

and

L_z are $\pm l\hbar$.

For orbital angular Momentum, l takes only integral values:

$$l = 0, 1, 2, 3, \ldots$$

For spin angular momentum of a particle l takes half integral values.

$\Rightarrow$ If ψ_{lm} are the eigen functions of L^2 and L_z corresponding to eigenvalue $l(l+1)\hbar^2$ of L^2 and $m\hbar$ of L_z, we can write.

$$L^2 \psi_{lm} = l(l+1)\hbar^2 \psi_{lm}$$

$$l = 0, 1, 2, 3, \ldots$$

and

$$L_z \psi_{lm} = m\hbar \psi_{lm} \quad , m = \pm l$$

$$m = -l, l+1, \ldots, l-1, l$$

where "m" is azimuthal or magnetic quantum number, which is an for atomic orbitals, Also describe the unique quantum state of an electron and also the shape of the orbitals.

Problem: Eigenvalues of L_+ and L_-!

Let ψ_{lm} be the simultaneous eigen functions of L_z and L^2, so that

$$L_z \psi_{lm} = m\hbar \psi_{lm} \quad \text{--- (1)}$$

$$L^2 \psi_{lm} = l(l+1)\hbar^2 \psi_{lm} \quad \text{--- (2)}$$

Now operating equ (1)' with L_+ both side

$$L_+ L_z \psi_{lm} = m\hbar L_+ \psi_{lm}.$$

$[L_z, L_+] = \hbar L_+ \Rightarrow L_z L_+ - L_+ L_z = \hbar L_+$

$L_+ L_z = L_z L_+ - \hbar L_+$

$$(L_z L_+ - \hbar L_+)\psi_{lm} = m\hbar L_+ \psi_{lm}$$

$$L_z L_+ \psi_{lm} = m\hbar L_+ \psi_{lm} + \hbar L_+ \psi_{lm}$$

$$L_z L_+ \psi_{lm} = (m+1)\hbar L_+ \psi_{lm}.$$

or.

$$L_z (L_+ \psi_{lm}) = (m+1)\hbar (L_+ \psi_{lm}) \quad (3)$$

i.e $(m+1)\hbar$ is the eigen value

Corresponding to eigenfunction $L_+ \psi_{lm}$.

Now apply L_+ on equ (2) both side

$$L_+ L^2 \psi_{lm} = l(l+1)\hbar^2 L_+ \psi_{lm}$$

$$L^2 (L_+ \psi_{lm}) = l(l+1)\hbar^2 (L_+ \psi_{lm}) - (4)$$

From equ. (3) and (4), one can observe that l remains the same but "m" increases by one.

Now. Let us try to find the values λ corresponding to L_+ for the case

$$L_+ \psi_{l,m} = \lambda \psi_{l,m+1}$$

complex conjugation yields

$$L_+^* \psi_{lm}^* = \lambda^* \psi_{l,m+1}^*$$

this gives us

$$\int L_+^* \psi_{lm}^* L_+ \psi_{lm} \, d\underline{x} = \lambda \lambda^* \int \psi_{l,m+1}^* \psi_{l,m+1} \, d\underline{x}$$

$$\int L_+^* \psi_{lm}^* L_+ \psi_{lm} \, d\underline{x} = \lambda \lambda^*$$

or

$$\int \psi_{lm}^* L_+^* L_+ \psi_{lm} \, d\underline{x} = \lambda \lambda^*$$