or

$$\int \psi^*_{lm}\, L_- L_+ \psi_{lm}\, d\underline{x} = \lambda\lambda^* \quad \text{since } L_+^\dagger = L_-$$

Let us first calculate

$$L_- L_+ \psi_{lm} = \left(L^2 - L_z^2 - \hbar L_z\right)\psi_{l,m}$$

$$= \left(l(l+1)\hbar^2 - m^2\hbar^2 - m\hbar^2\right)\psi_{l,m}$$

Now

$$\lambda^*\lambda = |\lambda^2| = \int \psi^*_{lm}\left\{l(l+1) - m^2 - m\right\}\hbar^2\,\psi_{lm}\,d$$

$$|\lambda^2| = \left(l(l+1) - m(m+1)\right)\hbar^2 \int \psi^*_{lm}\,\psi_{lm}\,d\underline{x}$$

$$\lambda = \sqrt{l(l+1) - m(m+1)}\;\hbar$$

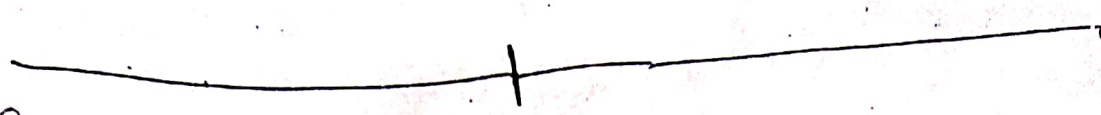
Therefore,

$$L_+\,\psi_{l,m} = \sqrt{l(l+1) - m(m+1)}\;\hbar\,\psi_{l,m+1}$$

Similarly

$$L_-\,\psi_{l,m} = \sqrt{l(l+1) - m(m-1)}\;\hbar\,\psi_{l,m-1}$$

L^2 L_z

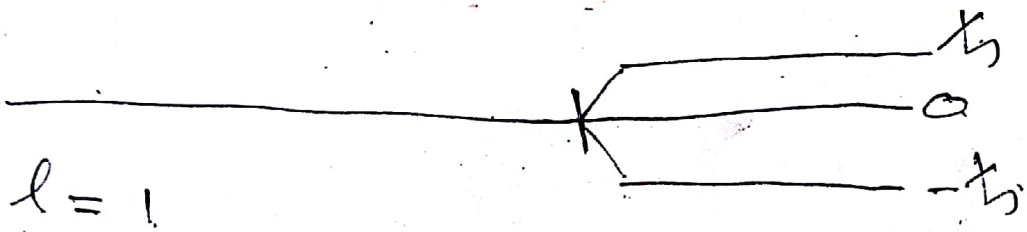


1-fold

$l = 0$

$\lambda = l(l+1)\hbar$

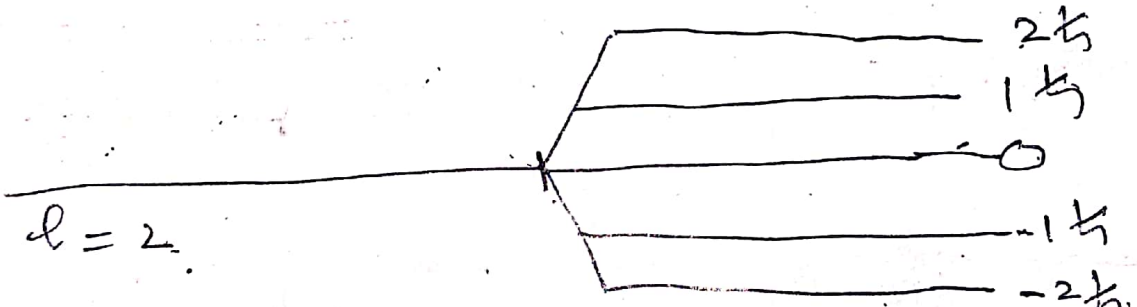
$m = 0$



3-fold

$l = 1$

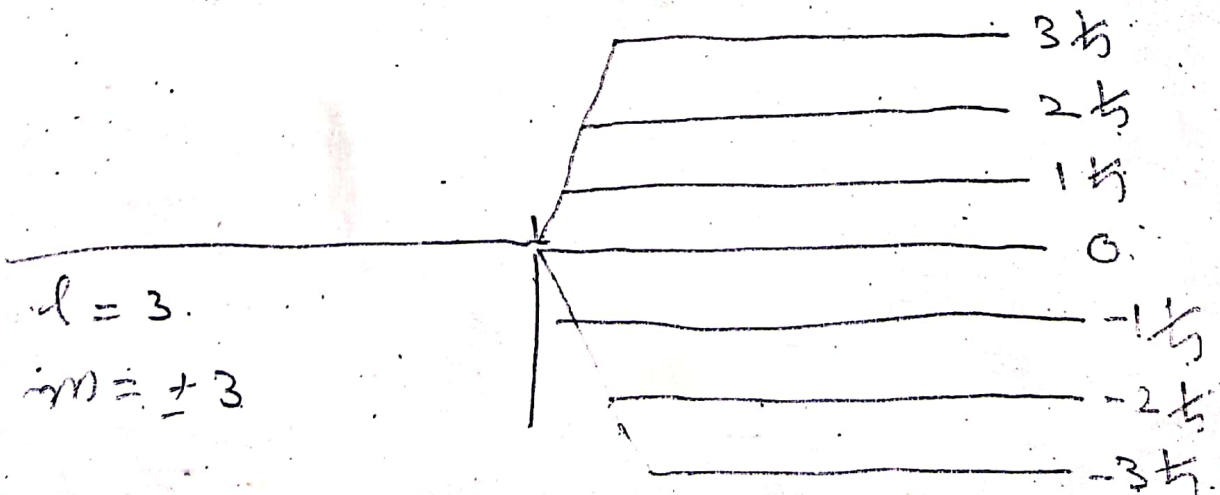
$m = \pm 1$



5-fold

$l = 2$

$m = \pm 2$



7-fold

$l = 3$

$m = \pm 3$

The eigenvalues of L^2 is $(2l+1)$-fold degenerate

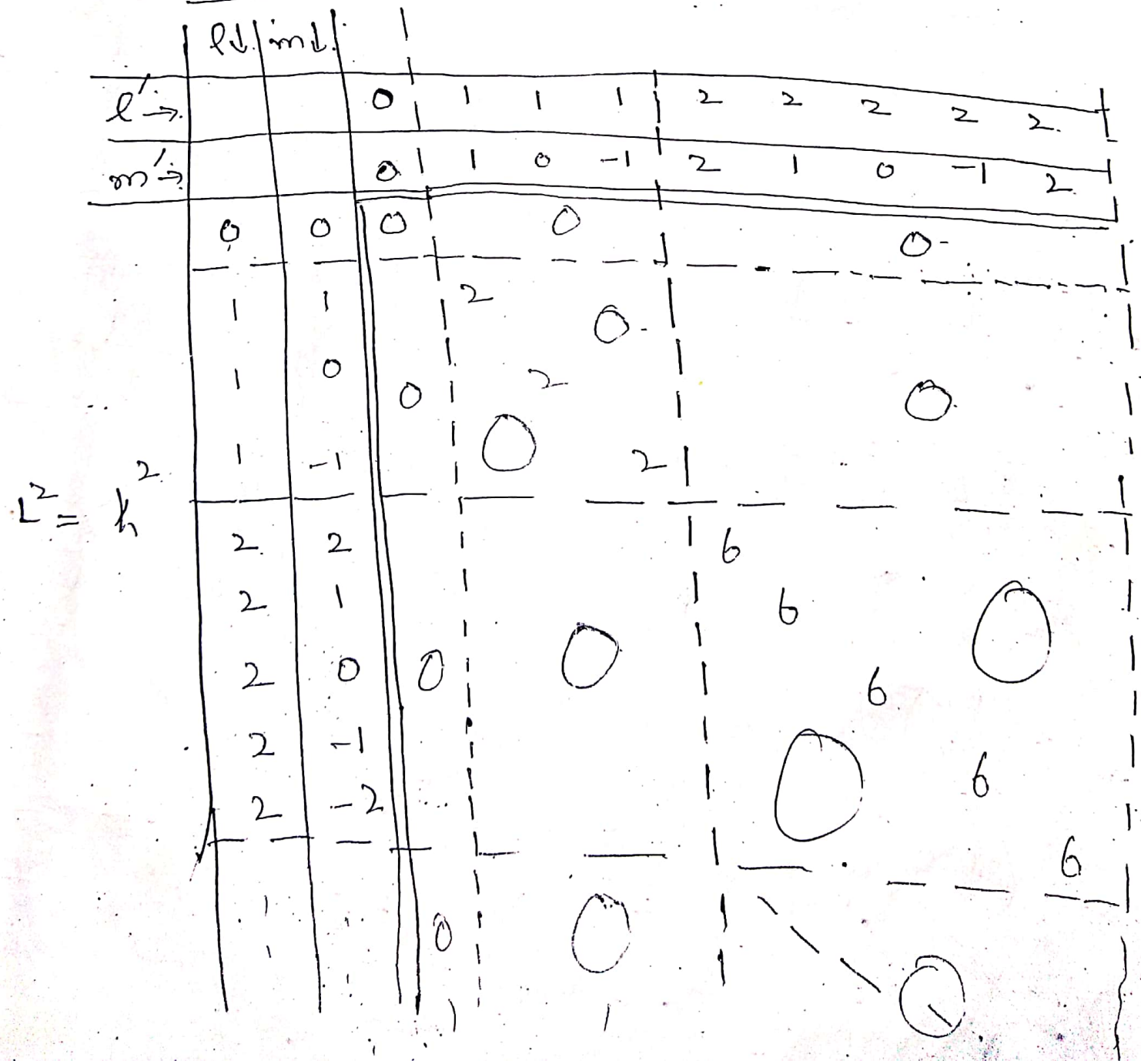
Matrix Elements of $\hat{L}^2$ and $\hat{L}_z$.

The matrix elements of L^2 and L_z are

$$L^2_{lm,l'm'} = (Y_{lm}, L^2 Y_{l'm'}) = \langle lm|L^2|l'm'\rangle$$

$$= l'(l'+1)\hbar^2 \delta_{ll'} \delta_{mm'}$$

For L^2:



So

$$L^2 = \hbar^2 \begin{pmatrix} 0 & & & & \\ & 2 & & 0 & \\ & & 2 & & \\ & & & 6 & \\ & 0 & & & 6 \\ & & & & & \ddots \end{pmatrix}$$

Similarly the matrix elements of L_z are

$$L_z = \hbar \begin{pmatrix} 0 & & & & \\ & 0 & & 0 & \\ & & -1 & & \\ & & & 2 & \\ & 0 & & & 0 \\ & & & & & \ddots \end{pmatrix}$$

$$(L_z)_{00,00} = 0$$

$$(L_z)_{11,11} = 1\hbar$$

$$(L_z)_{12,12} = 2\hbar \quad \text{and so on}$$

The matrix elements of $L_\pm$ are

$$(L_\pm)_{lm,l'm'} = (Y_{lm}, L_\pm Y_{l'm'}) = \langle lm|L_\pm|l'm'\rangle$$

$$= \sqrt{l'(l'+1) - m'(m'\pm 1)}\,\hbar\,(Y_{lm}, Y_{l',m'\pm 1})$$

$$= \sqrt{l'(l'+1) - m'(m'\pm 1)}\,\hbar\,\delta_{ll'}\,\delta_{m,m'\pm 1}$$

So.

$l\downarrow$	$m\downarrow$	$l'\to$: 0	1	1	1	2	2	2	2	2
		$m'\to$: 0	1	0	−1	2	1	0	−1	−2
0	0	0	0					0		
1	1		0	0	0					
1	0		$\sqrt{2}$	0	0			0		
1	−1		0	$\sqrt{2}$	0					
2	2					0	0	0	0	0
2	1					2	0	0	0	0
2	0	0	0			0	$\sqrt{6}$	0	0	0
2	−1					0	0	$\sqrt{6}$	0	0
2	−2					0	0	0	2	0

$L_- = \hbar$

Similarly

$$L_+ = \hbar \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix elements of L_x and L_y are, as.

$$L_x = \frac{1}{2}(L_+ + L_-)$$

and $$L_y = \frac{-i}{2}(L_+ - L_-)$$

So

$$L_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6} & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$L_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

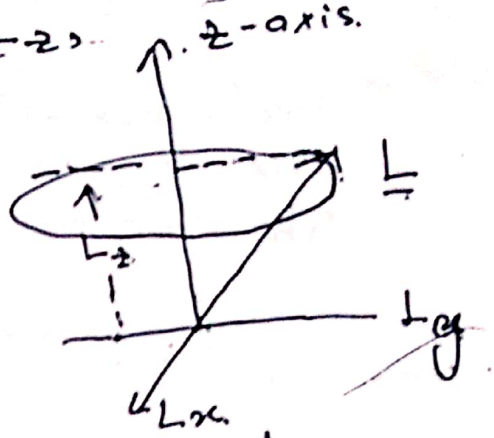
The matrix elements of L^2, L_z, L_x, L_y can also be expressed in terms of submatrices corresponding to a given value of l.

For $l = 1$

$$L^2 = \hbar^2\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad L_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Geometrical Representation of Angular Momentum:

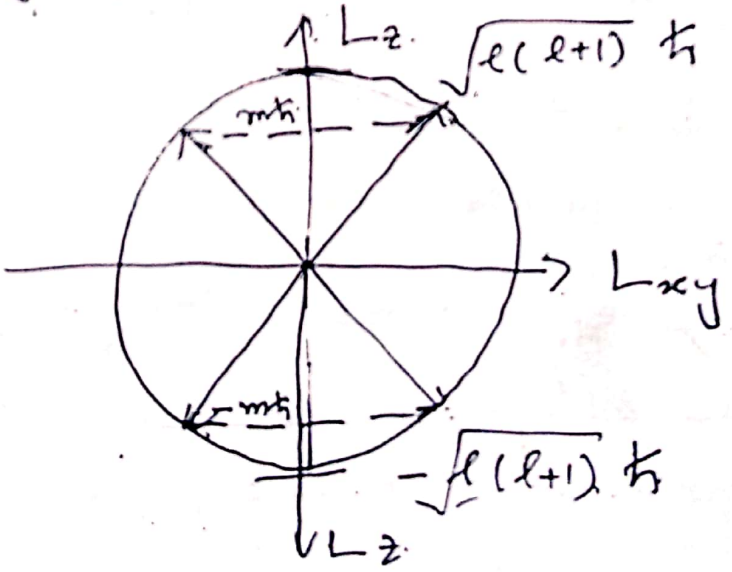
The projection of vector $\underline{L}$ on z-axis gives its z-component L_z, as shown in fig.



The eigenvalue of $\underline{L}$ is $\sqrt{l(l+1)}\,\hbar$

and z-component L_z is $m\hbar$.

The vector $\underline{L}$ lies within the $L_z L_{xy}$ plane, on a circle of radius $\sqrt{l(l+1)}\,\hbar$ - see below.



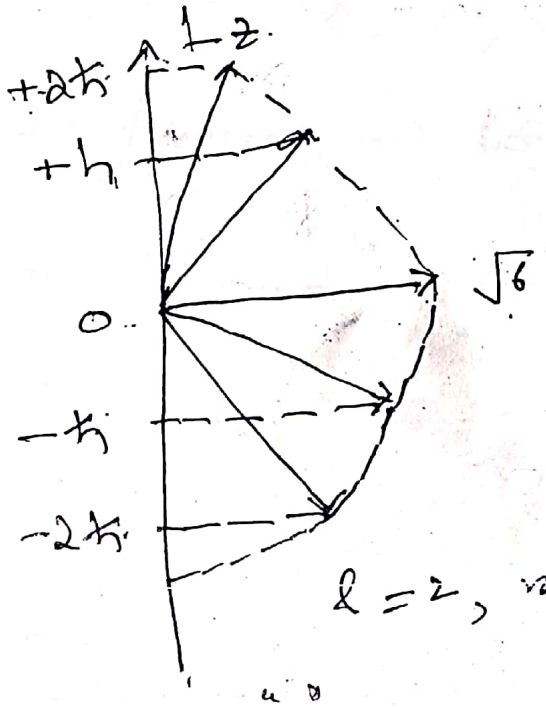
Suppose that we measure the eigenvalue of $\underline{L}$ for $l = 2$, then the eigenvalue is $\sqrt{6}\,\hbar$

and the number of different eigenvalues of L_z are ± 2. $\therefore m = \pm l$

as $L_z = m\hbar$

The possible values of m are $-2, -1, 0, 1, 2$

The vector diagram is below for $l = 2$ and $m = \pm 2$



The quantum number "l" is refferred to as the orbital quantum number and "m" is refferred to as the azimuthal or magnetic quantum number.

In atomic physics the state "s" stands for $l = 0$ and "p" corresponds to $l = 1$ and so on.

$l =$	0	1	2	3	4	5	6 ...
	s	p	d	f	g	h	i ...

The states are also specified by Principal quantum number 'n', which contain all the information of other remaining quantum numbers, as

$n = 1, 2, 3, \ldots$

$l = 0, 1, 2, \ldots, n-1$

$m = 0, \pm 1, \pm 2, \ldots, = \pm l$

for example, the atomic state in which $n=2$, $l=0$ is a 2s state and one in which $n=4$, $l=2$ is a 4d atomic state. Atomic states in Hydrogen atom are

	$l=0$	$l=1$	$l=2$	$l=3$
$n=1$	1s			
$n=2$	2s	2p		
$n=3$	3s	3p	3d	
$n=4$	4s	4p	4d	4f

→ number should have only +ve integer values.

As "n" increases the orbitals become larger and electrons lies at higher potential energy, therefore less tightly bound to the nucleus.

The principal quantum number is the only quantum number introduced by Bohr model

Spin Angular Momentum :

Experimental Observation :

In 1922, Stern and Gerlach confirmed the existence of spin using silver (Ag) atom.

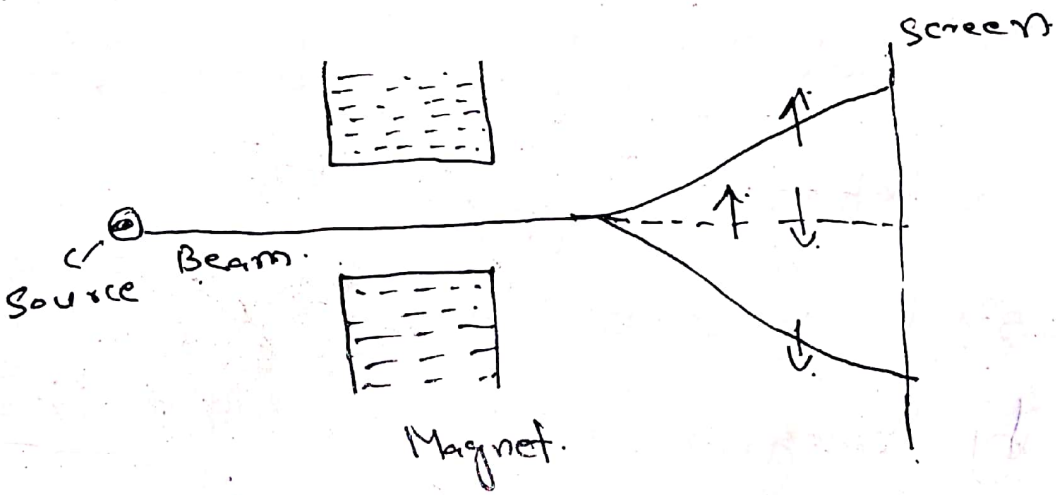
→ Silver – atomic number 47.

and electronic configuration is.

	$\ell = 0$	$\ell = 1$	$\ell = 2$
$n = 1$	$1s^2$		
$n = 2$	$2s^2$	$2p^6$	
$n = 3$	$3s^2$	$3p^6$	$3d^{10}$
$n = 4$	$4s^2$	$4p^6$	$4d^{10}$
$n = 5$	$5s^1$		

→. Silver atom was in ground state. then its total orbital angular momentum should be zero i.e $l = 0$, means 5s' shell contain one electron.

→. A beam of silver atoms passes through an inhomogeneous (non-uniform) magnetic field B in z-direction.



→. Classically expectation was that one would ~~expect~~ get a continuous band on the screen.

→ But wave theory of Schroedinger tells us that, if a particle have angular momentum, one would expect that beam should split into discrete number of $2l+1$ components.

→ for example, if beam atom in their ground state i.e. $l = 0$, in s-state. electron shoul be, then one would expect one spot on the screen.

→ for $l = 1$, 3 – spots should be on the screen, and electrons will be in p-state.

→ However, experimentally observe that the beam splits into two distinct compone as shown in Fig.

→ This results was also confirmed for by Hydrog atoms in their ground state. i.e $l = 0$, where no splitting is expectec

→ To solve this problem, in 1925, Goudsmith and Uhlenbeck postulated that electron possesses an intrinsic angular momentum, which is totally inde independent of its orbital angular momen

→. Analogous to the motion of earth, which consists of an orbital motion around the Sun as well as the internal rotational or spinning motion about it axis.

→. Therefore all the microscopic particles including $\bar{e}$s keep orbital motion and the intrinsic or spinning motion.

→. The spin angular momentum of a particle does not depend on it's spatial degrees of freedom. Hence, the spin, an intrinsic degree of freedom, is purely a quantum mechanical concept with no classical analogy.

→ According to classical theory of electromagnetism, an orbital magnetic dipole moment with the orbital motion of a particle of charge q is

$$\vec{\mu}_L = \frac{q}{2mc} \vec{L}$$

where.

$\underline{L}$ ⇒ orbital angular momentum.

m — mass, c is speed of light.

→ q is +ve, $\bar{\mu}_L$ and $\underline{L}$ has same direction.

→ If q is -ve, both $\bar{\mu}_L$ and $\underline{L}$ will be in opposite direction.

Similary, spin magnetic dipole moment in classical theory.

$$\bar{\mu}_s = \frac{-e}{2m_e c} \bar{S}$$

→ electron's spin magnetic moment is twice its classical expression

Since $$\bar{\mu}_s = -g_s \frac{e}{2m_e c} \bar{S}$$

where g_s is the Lande factor or gyromagnetic ration of the electron.

i.e. $g_s \simeq 2$ — derived using Dirac's relativistic theory of electron

⇒ when electron is placed in a magnetic field $\bar{B}$, a force will be exerted on its intrinsic dipole moment $\bar{\mu}_s$.

→ The direction and magnitude of the force depends on the relative orientation of the field and the dipole.

→ For example, if $\mu_s \parallel B$ – electron will move in the direction in which field increases. ~~On the~~ and vice versa. i.e.

→ μ_s ~~is~~ is anti-parallel to $\bar{B}$, then the $\bar{e}$ will move in the direction in which the field decreases.

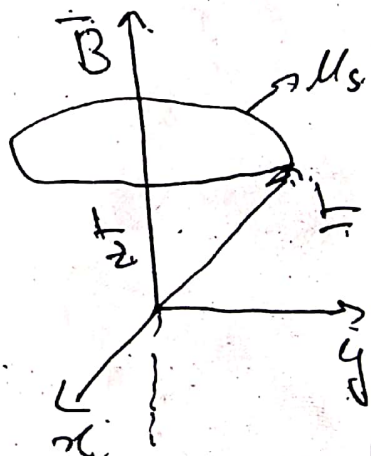


Fig. The force tends to align $\bar{\mu}_s$ along $\bar{B}$, producing a precessional motion of $\bar{\mu}_s$ around $\bar{B}$

~~For~~ The hydrogen or Silver atoms, are in ground state, Hence $l = 0$, therefore, the dipole moment of the atom will be entirely due to spin of the $\bar{e}$.

The atomic beam will therefore deflect according to orientation of the electron's spin. §.

→ Since, experimentally, the beam splits into two components. The electron's spin must have only two orientations relative to the magnetic field, either parallel or anti-parallel.

→ Analogues to orbital quantum number l, spin also characterized by two quantum numbers, the spin "s" and its projection m_s on the z-axis (the direction of the magnetic field. The quantum numbers of the electrons must be given by $s = 1/2$ and $m_s = \pm 1/2$.

→ It is found that by nature some Particles have integer spin i.e. $s = 0, 1, 2, \ldots$ such that π-meson $- s = 0$, photons $- s = 1$.