4 → Particles with half integral spins called fermions — quarks, electrons, protons, neutrons etc.

Particles with integer spins — called bosons — pions, photons, gravitons, etc.

Conclusion :-

1) A beam splits into two distinct set of components rather than a continuous band, it provides an additional confirmation of quantum hypothesis on the discrete character of microphysical world.

2) This experiment can also be used to determine the total angular momentum $\bar{J}$ of an atom, which in case of $l \neq 0$, is given by the sum of orbital and spin angular momentum.

$$\bar{J} = \underline{L} + \underline{S}$$

vector space

$H =$ square integrable wav

+ inner

scalar sp

$(\psi, \psi) = \delta$

$$\boxed{\int \psi^* \psi \, dx < \infty}$$

$L^2(\mathbb{R}^3)$

$$\sum_n |\phi_n\rangle\langle\phi_n| = 1$$

(B) On Sc...

•). It can also be used to determine the total angular momentum of an atom. $\bar{J}$,

$$\boxed{\bar{J} = \bar{L} + \bar{S}}$$

General theory of spin :-

Likewise orbital angular momentum $\underline{L}$, the operator $\underline{S}$ obeys the relation.

$$\underline{S} \times \underline{S} = i\hbar \underline{S}.$$

In components

$$[S_x, S_y] = i\hbar S_z$$

$$[S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

we introduce the eigenstate χ

such that

$$\hat{S}_z \chi = m_s \hbar \chi \quad , m_s = \pm S$$

$$\hat{S}^2 \chi = S(S+1)\hbar^2 \chi$$

for $S = 1/2$, $m_s = \pm 1/2$.

In this case the vector space representin spin states has only two dimensions.

<spin 1/2>

Let χ_+ and χ_- be the basis. Corresponding to states $m_s = \pm 1/2$. The spin state χ is called a spinor which is not a function. The general state of a spin (1/2) particle can be expressed a two component spinor

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

with.

$$\chi^{t} = \begin{pmatrix} \chi_+^{*} & \chi_-^{*} \end{pmatrix}$$

The spin states with $s = 1/2$ are

$$|+\rangle = \chi_+ = \chi_{\underset{s,\ m_s}{1/2,\, 1/2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

spin up state

$$|-\rangle = \chi_- = \chi_{1/2,\, -1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle = \left|\tfrac{1}{2}, \tfrac{-1}{2}\right\rangle$$

spin down state

Similarly we can calculate for $\underline{\underline{S = 1/2}}$, $m_s = \pm\frac{1}{2}$

where.

$$S^2 |\uparrow\rangle = \frac{3}{4}\hbar^2 |\uparrow\rangle$$

$$\langle\uparrow|\uparrow\rangle = 1$$

$$S^2 |\downarrow\rangle = \frac{3}{4}\hbar^2 |\downarrow\rangle$$

$$\langle\uparrow|\downarrow\rangle = 0$$

$$\langle\downarrow|\uparrow\rangle = 0$$

$$S_z |\uparrow\rangle = 1/2\,\hbar |\uparrow\rangle$$

$$\langle\downarrow|\downarrow\rangle = 1$$

$$S_z |\downarrow\rangle = -1/2\,\hbar |\downarrow\rangle$$

The ladder operators of $\underline{S}$ are defined as

$$S_\pm = S_x \pm i S_y$$

with.

$$S_\pm \chi_{s,m} = \sqrt{s(s+1) - m(m\pm 1)}\,\hbar\,\chi_{s,m\pm 1}$$

$$S_\pm |s,m\rangle = \sqrt{s(s+1) - m(m\pm 1)}\,\hbar\,|s, m\pm 1\rangle$$

with.

$$S_+ |\downarrow\rangle = \hbar |\uparrow\rangle$$

$$S_- |\uparrow\rangle = \hbar |\downarrow\rangle$$

$$S_+ |\uparrow\rangle = S_- |\downarrow\rangle = 0$$

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Now.

$$S_x = \frac{1}{2}(S_+ + S_-)$$

$$S_y = \frac{1}{2i}(S_+ - S_-)$$

it follows that

$$S_x|\uparrow\rangle = \frac{1}{2}(S_+ + S_-)|\uparrow\rangle = \frac{1}{2} S_-|\uparrow\rangle = \frac{1}{2}\hbar|\downarrow$$

$$S_x|\downarrow\rangle = \frac{1}{2}(S_+ + S_-)|\downarrow\rangle = \frac{1}{2} S_+|\downarrow\rangle = \frac{1}{2}\hbar|\uparrow\rangle$$

$$S_y|\uparrow\rangle = \frac{1}{2i}(S_+ - S_-)|\uparrow\rangle = -\frac{1}{2i} S_-|\uparrow\rangle = \frac{-1}{2i}\hbar|\downarrow$$

$$S_y|\downarrow\rangle = \frac{1}{2i}(S_+ - S_-)|\downarrow\rangle = \frac{1}{2i} S_+|\downarrow\rangle = \frac{1}{2i}\hbar|$$

Since the matrix elements of S_+ are

$$(S_+)_{sm,s'm'} = (\chi_{sm}, S_+ \chi_{s'm'}) = \langle sm|S_+|s'm'$$

$$= \sqrt{s'(s'+1) - m'(m'+1)}\,\hbar\,(\chi_{sm}, \chi_{s,m}$$

$$= \sqrt{s'(s'+1) - m'(m'+1)}\,\hbar\,\delta_{ss'}, \delta_{m,m}$$

for $s = s' = 1/2$

$$(S_+)_{mm'} = \sqrt{\frac{3}{4} - m'(m'+1)}\,\hbar\,\delta_{m,m'+1}$$

The matrix elements are:

$m \backslash m'$	$1/2$	$-1/2$
$1/2$	0	$\hbar$
$-1/2$	0	0

i.e.
$$S_+ = \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Similarly
$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Since,
$$S_x = \frac{1}{2}\left(S_+ + S_-\right)$$
$$= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and
$$S_y = \frac{1}{2i}\left(S_+ - S_-\right)$$
$$= \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\left(S_z\right)_{sm,\, s'm'} = \left(\chi_{sm},\, S_z\, \chi_{s'm'}\right)$$
$$= \langle sm | S_z | s'm' \rangle$$
$$= m'\hbar\, \delta_{ss'}\, \delta_{mm'}$$

then.

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{for } s = s' = 1/2$$

$m = 1/2, -1/2$

$m' = 1/2, -1/2$

Thus

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Pauli Matrices: -

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is convenient to write

$$\underline{S} = \frac{\hbar}{2}\underline{\sigma}, \quad \text{when } s = 1/2.$$

or.

$$(S_x, S_y, S_z) = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z).$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

are the famous Pauli spin matrices. Noted S_x, S_y, S_z and S^2 are all Hermitian while S_+ and S_- are not Hermitian, therefore ~~don't~~ do not represent observables.

Properties of the Paulie Spin Matrices:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1} \text{ (unit matrix)}$$

$$[\sigma_x, \sigma_y] = \sigma_x \sigma_y - \sigma_y \sigma_x = 2i\,\sigma_z$$

↳ and cyclic permutations

In classical poission Brackets:

$$\{\sigma_x, \sigma_y\} = \sigma_x \sigma_y + \sigma_y \sigma_x = 0$$

↳ and cyclic permutations

$$\sigma_x \sigma_y = -\sigma_y \sigma_x = i\,\sigma_x$$

↳ and cyclic permutations

$$\sigma_x \sigma_y \sigma_z = i \cdot \mathbb{1} \text{ (unit matrix)}$$

$$\mathrm{Tr}\,\sigma_x = \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_z = 0$$

$$\det \sigma_x = \det \sigma_y = \det \sigma_z = -1$$

$$\{\sigma_x, \sigma_x\} = 2\,\mathbb{1}$$

$$[\sigma_x, \sigma_x] = 0$$

→ Traces: sum of diagonal entries.

The Eigenspinors:

In ~~order~~ to find the eigenspinors of S_z we ~~have~~ solve the eigenvalue equ

$$S_z \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

put the value of S_z.

$$\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

or

$$\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

For Plus sign, $\beta = 0$.

and for minus sign, $\alpha = 0$.

then.

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \left(\text{eigenvalue } + \frac{\hbar}{2}\right)$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \left(\text{eigenvalue } -\hbar/2\right).$$

are called the eigenspinors of S_z.

spinors σ_x.

Eigen~~values~~ of σ_x

The eigenvalues of $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is obtained by the characteristic equation $\rightarrow$ $\det(A - \lambda I) = 0$

also called secular equ.

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0.$$

$$\lambda = \pm 1$$

The eigenspinors corresponding to $\lambda = +1$ is obtained by the eigenvalue equ. as

$$\sigma_x \chi_+^{(x)} = +1 \, \chi_+^{(x)}$$

so $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = +1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$$

$$\begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}.$$

$$\Rightarrow \quad \alpha_1 = \beta_1.$$

then $$\chi_+^{(x)} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_1 \end{pmatrix}.$$

$$\chi_+^{(x)} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The normalization Condition gives.

$$\left(\chi_+^{(x)}, \chi_+^{(x)}\right) = |\alpha_1|^2 + |\beta_1|^2 = 1.$$
$$= 2|\alpha_1|^2$$

$$\alpha_1 = \frac{1}{\sqrt{2}}.$$

So the normalized eigenspinors of σ_x Corresponding to eigenvalue +1 is.

$$\boxed{\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}.$$

Similarly, the normalized eigenspinor of σ_x Corresponding to eigenvalue −1. is

$$\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Now it is Simple to find the eigen~~spinors~~ values. of σ_y and σ_z

$$\sigma_y : \begin{cases} \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \\ \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}. \end{cases} \qquad \sigma_z = \begin{cases} \chi_+^{(z)} = \begin{pmatrix}1\\0\end{pmatrix} \\ \chi_-^{(z)} = \begin{pmatrix}0\\1\end{pmatrix}. \end{cases}$$

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Eigen spinor for σ_z

In order to find the eigen spinor of σ_z, we have to solve the eigen value equ. $\sigma_z \chi = \lambda \chi$ $\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\lambda = \pm 1$

$$\sigma_z \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

for Plus sign (χ_+^z)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Hence $\alpha = \alpha$

$-\beta = \beta \Rightarrow \beta = 0.$

Therefore

$$\chi_{+}^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For -ve sign (χ_{-}^{z})

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}$$

$$\alpha = -\alpha$$

$$-\beta = -\beta$$

it means $\alpha = 0$

So.

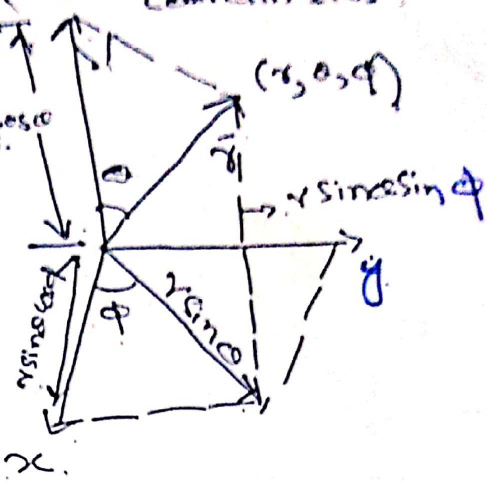
$$\chi_{-}^{(z)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

①

Angular Momentum in Spherical Polar Coordinates:

Let us make a transformation of Cartesian coordinates to spherical polar coordinates.

We know that



$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Since

$$L_x = -i\hbar\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]$$

$$L_y = -i\hbar\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right]$$

$$L_z = -i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$$

Let us operate "$\frac{\partial}{\partial x}$" on any arbitrary function $f(r, \theta, \phi)$, so we will get

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\cdot\frac{\partial \phi}{\partial x}.$$

then we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \phi}\cdot\frac{\partial \phi}{\partial x}.$$

Similarly

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \phi}\cdot\frac{\partial \phi}{\partial y}.$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\cdot\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial \phi}\cdot\frac{\partial \phi}{\partial z}.$$

Now we need to calculate

$\left(\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial \phi}{\partial x}\right), \left(\frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial y}, \frac{\partial \phi}{\partial y}\right)$, and

$\left(\frac{\partial r}{\partial z}, \frac{\partial \theta}{\partial z}, \frac{\partial \phi}{\partial z}\right)$.

Hence.

$$r^2 = x^2 + y^2 + z^2.$$

$$2r \frac{\partial r}{\partial x} = 2x.$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta \cos\phi}$$

Similary $$\boxed{\frac{\partial r}{\partial y} = \sin\theta \sin\phi}$$

$$\boxed{\frac{\partial r}{\partial z} = \cos\theta}$$

Also $$\tan\phi = \frac{y}{x}.$$

→ differentiate w.r.t.

$$\sec^2\phi \frac{\partial \phi}{\partial x} = -\frac{y}{x^2}$$

$$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2}\cos^2\phi = \frac{-r\sin\theta\sin\phi \cos^2\phi}{r^2\sin^2\theta\cos^2\phi}$$

$$\boxed{\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}}$$

∴ Similarly $\boxed{\dfrac{\partial\phi}{\partial y} = \dfrac{\cos\phi}{r\sin\theta}}$

$$\sec^2\phi \cdot \frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{x}\right) = 0$$

$$\boxed{\frac{\partial\phi}{\partial z} = 0}$$

Now.

$$z = r\cos\theta$$

$$\frac{\partial z}{\partial x} = \frac{\partial r}{\partial x}\cos\theta - r\sin\theta\,\frac{\partial\theta}{\partial x}$$

$$\frac{\partial\theta}{\partial x} = \cancel{\frac{\partial r}{\partial x}\cos\theta \cdot \frac{1}{r\sin\theta}}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial r}{\partial x}\cos\theta \cdot \frac{1}{r\sin\theta}$$

$$= \frac{1}{r}\cdot\frac{x}{r}\,\frac{\cos\theta}{\sin\theta}$$

$$= \frac{r\sin\theta\cos\phi}{r^2}\cdot\frac{\cos\theta}{\sin\theta}$$

$$\boxed{\frac{\partial\theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}}$$

$$\boxed{\frac{\partial\theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}}$$

$$\boxed{\frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r}}$$