(i.e)

$(k_1 a)$ is a zero of the lth-order spherical Bessel function. Since $J_l(x)$ are oscillatory, each having infinite number of zeros.

Let β_{nl} be the nth zero of the lth spherical Bessel function, then

$$k_1 a = \beta_{nl}$$

$$k_1 = \frac{\beta_{nl}}{a}$$

$$\frac{2mE_{nl}}{\hbar^2} = \frac{\beta_{nl}^2}{a^2}$$

$$\boxed{E_{nl} = \frac{\hbar^2}{2ma^2}\beta_{nl}^2}$$

This the required eigenvalue formula of the problem.

Since the Neumann fns $N_l(x)$ diverge at the origin, and ~~since~~ the wavefn ψ_{nlm} are required to be finite everywhere in space, the functions

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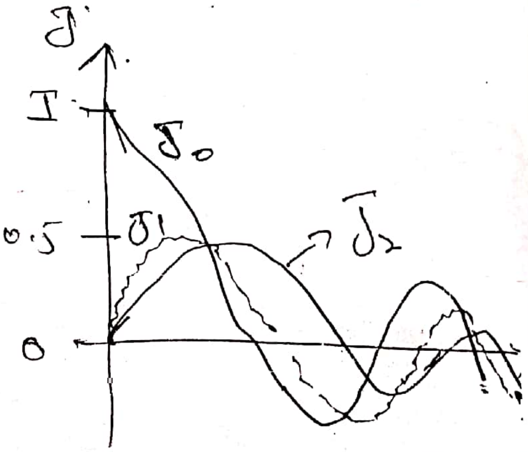
$N_\ell(x)$ are unacceptable solutions to the problem. Hence only the spherical Bessel functions $J_\ell(x)$ contribute to the eigenfns of the free particle.

The first few spherical Bessel and Neumann functions are and also their shapes are displayed in fig below.

$$J_0(x) = \frac{\sin x}{x}$$

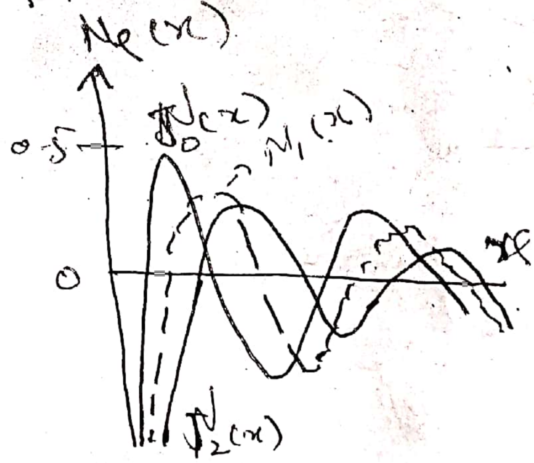


$$J_1(x) = \frac{\sin x}{x} - \frac{\cos x}{x}$$

$$J_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3\cos x}{x^2}$$

⟹ Similarly

$$N_0(x) = -\frac{\cos x}{x}$$



$$N_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$N_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$$

One can observe that the amplitude of the fns becomes smaller and smaller as "x" increases.

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The corresponding wavefns. are

$$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}^{(r)}\, Y_{\ell m}(\theta,\phi).$$

where

$$R_{n\ell}(r) = A_{n\ell}\, J_{\ell}\left(\frac{\beta_{n\ell}\, r}{a}\right), \quad \because K_1 = \frac{\beta_{n\ell}}{a}.$$

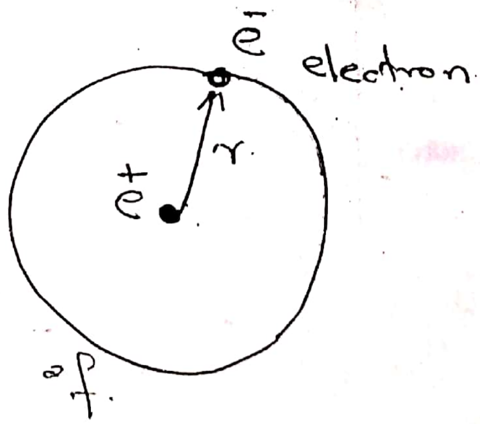
Therefore

$$\psi_{n\ell m}(r,\theta,\phi) = A_{n\ell}\, J_{\ell}\left(\frac{\beta_{n\ell}\, r}{a}\right) Y_{\ell m}(\theta,\phi)$$

The constant $A_{n\ell}$ can be determined using the normalization condition.

The HYDROGEN ATOM :-

The hydrogen atom consists of an electron revolving around a motionless proton.



The charges of the two particles are equal but opposite in sign. The motion of the electron can be considered as the motion of a particle moving in the field produced by the proton. From the ~~Coto~~ Coulomb's law the potential energy is

$$V(r) = -\frac{e^2}{r}$$

Since the potential is centrally symmetric. The wavefn. of the revolving electron is of the form

$$\psi_{n\ell m} = A_{n\ell} R_{n\ell}(r) \cdot Y_{\ell m}(\theta, \phi)$$

$$= A_{n\ell} \frac{U_{n\ell}(r)}{r} C_{\ell m} P_{\ell m}(\cos\theta) e^{im\phi}$$

where 'u' is the solution of the differential equation → $\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}(E-V)u - \frac{\ell(\ell+1)}{r^2}u = 0$

$$\frac{d^2u}{dr^2} + \frac{2m_e}{\hbar^2}\left[E + \frac{e^2}{r} - \frac{\hbar^2 \ell(\ell+1)}{2m_e r^2}\right]u = 0.$$

$$\ell = 0, 1, 2, \ldots$$

In order to solve this equ, Let us introduce new variables ρ and Z such that

$$E = -\frac{\rho^2 m_e e^4}{2\hbar^2}$$

and

$$r = \frac{\hbar^2 Z}{m e^2}$$

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Now.

$$\frac{du}{dr} = \frac{du}{dz} \cdot \frac{dz}{dr}$$

Putting the value of $\frac{dz}{dr} = \frac{m_e e^2}{\hbar^2}$

$$\frac{du}{dr} = \frac{m_e e^2}{\hbar^2} \frac{du}{dz}$$

$$\frac{d^2u}{dr^2} = \left(\frac{m_e e^2}{\hbar^2}\right)^2 \frac{d^2u}{dz^2}$$

In terms of the new variables, equ. (2) can be expressed as.

$$\left(\frac{m_e e^2}{\hbar^2}\right)^2 \frac{d^2u}{dz^2} + \frac{2m_e}{\hbar^2}\left[-\frac{p^2 m_e e^4}{2\hbar^2} + \frac{m_e e^2 e^2}{\hbar^2 z}\right.$$

$$\left. - \frac{\hbar^2 l(l+1) m_e^2 e^4}{2m_e \hbar^4 z^2}\right] u = 0$$

$$\frac{d^2u}{dz^2} + \left[-p^2 + \frac{2}{z} - \frac{l(l+1)}{z^2}\right] u = 0 \quad \text{--- (3)}$$

For large z, equ (3) can be written as.

$$\frac{d^2u}{dz^2} - p^2 u = 0 \quad \text{--- (4)}$$

for $|z| \to \infty$

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The Solutions are.

$$e^{-\rho z} \quad \text{and} \quad e^{\rho z}$$

So the general sol of equ. (3) can now be written as.

$$U(z) = \omega(z)\, e^{-\rho z} + \chi(z)\, e^{\rho z}.$$

Since $e^{\rho z} \longrightarrow \infty$, as $z \longrightarrow \infty$.

therefor the acceptable asymptotic Solution of the problem is.

$$U(z) = \omega(z)\, e^{-\rho z}$$

Now

$$\frac{du}{dz} = \frac{dw}{dz} e^{-\rho z} - \rho w e^{-\rho z}$$

$$\frac{d^2u}{dz^2} = \frac{d^2w}{dz^2} e^{-\rho z} - \rho \frac{dw}{dz} e^{-\rho z}$$
$$- \rho \frac{dw}{dz} e^{-\rho z} + \rho^2 w\, e^{-\rho z}$$

Put these expressions in ~~equ.~~ (3), we get

$$\frac{d^2w}{dz^2} - 2\rho \frac{dw}{dz} + \left[\frac{2}{z} - \frac{\ell(\ell+1)}{z^2}\right] w = 0$$

$\longrightarrow$ (5)

Power Series Solution :-

Let us expand $w(z)$ as a power series in z

$$w(z) = \sum_{i=0}^{\infty} a_i z^{i+\alpha}$$

where α can be determined as follows

$$\frac{dw}{dz} = \sum_{i=0}^{\infty} a_i (i+\alpha) z^{i+\alpha-1}$$

$$\frac{d^2w}{dz^2} = \sum_{i=0}^{\infty} a_i (i+\alpha)(i+\alpha-1) z^{i+\alpha-2}$$

Put these values in equ (5), we get

$$\boxed{\sum_{i=0}^{\infty} a_i \left[(i+\alpha)(i+\alpha-1) z^{i+\alpha-2} - 2P(i+\alpha) z^{i+\alpha-1} + \left(\frac{2}{z} - \frac{l(l+1)}{z^2}\right) z^{i+\alpha} \right] = 0}$$

Now compare the co-efficients of $z^{\alpha-2}$, mean $i=0$

$$a_0(\alpha)(\alpha-1) - a_0 l(l+1) = 0$$

$$a_0 \left[\alpha(\alpha-1) - l(l+1)\right] = 0$$

Since $a_0 \neq 0$, therefore

$$\alpha(\alpha-1) - l(l+1) = 0$$

$$\alpha^2 - \alpha - l^2 - l = 0$$

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$$(\alpha^2 - \ell^2) - (\alpha + \ell) = 0.$$

$$(\alpha+\ell)(\alpha-\ell) - (\alpha+\ell) = 0.$$

$$\Rightarrow \boxed{\alpha = -\ell,} \quad \text{or} \quad \boxed{\alpha = \ell+1}$$

If $\alpha = -\ell$, then.

$w(z) \longrightarrow \infty$, as $z \longrightarrow 0$

~~Ag.~~ Therefore, we take $\alpha = \ell+1$

Again Compare the Co-efficients of

$z^{k+\alpha-2}$ $\boxed{i = k}$

$$a_k\left[(k+\alpha)(k+\alpha-1) - \ell(\ell+1)\right] z^{k+\alpha-2}$$

$$= a_k\left[2p(k+\alpha) - 2\right] z^{k+\alpha-1}$$

Put $k = k-1$, in right hand side

we will get

$$a_k\left[(k+\alpha)(k+\alpha-1) - \ell(\ell+1)\right]$$

$$= a_{k-1}\left[2p(k+\alpha-1) - 2\right] z^{k+\alpha-2}$$

So.

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$$\Rightarrow \quad a_K = \frac{2p(k+\alpha-1)-2}{(k+\alpha)(k+\alpha-1)-\ell(\ell+1)} a_{k-1}$$

$K = 1, 2, 3, \ldots$

Since $\alpha = \ell+1$.

therefore

$$a_K = \frac{2P(K+\ell)-2}{(K+\ell+1)(K+\ell)-\ell(\ell+1)} a_{k-1}$$

$$a_K = \frac{2[P(K+\ell)-1]}{K^2+2K\ell+K} a_{K-1}$$

which is required recurrence relation.

From this relation, one can see that

$$\lim_{K\to\infty} \frac{a_K}{a_{K-1}} = \lim_{K\to\infty} \frac{2[P(K+\ell)-1]}{(K^2+2K\ell+K)}$$

$$= \lim_{K\to\infty} \frac{2K[P(1+\frac{\ell}{K})-\frac{1}{K}]}{K^2[1+\frac{2\ell}{K}+\frac{1}{K}]}$$

$$\simeq \frac{2P}{K} \longrightarrow 0 \qquad \lim_{n\to\infty} \frac{1}{n} = 0$$

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Let us look at the ratio test of the series

$$e^{2\rho z} = \sum_{k=0}^{\infty} a_k$$

$$\lim_{k\to\infty} \frac{a_k}{a_{k-1}} = \lim_{k\to\infty} \frac{2\rho}{k!} \cdot \frac{(k-1)!}{(2\rho)^{k-1}}$$

$$\cong \frac{2\rho}{k} \to 0$$

Note that $w(z)$ has the same convergence behaviour as that of $e^{2\rho z}$

So we can write

$$u(z) = w(z)\, e^{-\rho z}$$
$$\cong e^{2\rho z}\, e^{-\rho z}$$
$$u(z) = e^{\rho z}$$

where

$$e^{\rho z} \to \infty \quad \text{as } z \to \infty$$

$e^{\rho z}$ must be terminated after a finite number of terms.

Let us terminate the series at

$$k = k_r$$

i.e

$$a_{k_r} = 0 \Rightarrow a_{k_r+p} = 0$$
$$p = 1, 2, 3, \ldots$$

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a_{kr} will be zero when.

$$\rho(k_r+\ell)-1=0$$

$$\rho=\frac{1}{k_r+\ell}=\frac{1}{n}$$

where
$n = k_r+\ell$
is the principle quantum number.

$\Rightarrow$

$$E_n=-\frac{\rho^2 m e^4}{2\hbar^2}=\frac{-me^4}{2\hbar^2}\frac{1}{n^2}$$

$$E_n=-\frac{1}{2}\frac{e^2}{an^2}$$

where $a=\frac{\hbar^2}{me^2}=0.53\,A^\circ$

is called Bohr radius.

So

$$E_n=\frac{E_1}{n^2}, \quad n=2,3,\ldots$$

This is the famous Bohr formula.

The wavefns for hydrogen are labeled by three quantum numbers. (n,ℓ,m).

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$$\psi_{nlm}(r,\theta,\phi) = A_{nl} \frac{U_{nl}(r)}{r} Y_{lm}(\theta, \phi)$$

$$= \frac{A_{nl}}{r} e^{-\ell z} \omega_{nl}(z) P_{lm}(\cos\theta) e^{im\phi}$$

where

ω_{nl} is the polynomial of degree

$$K_r - 1 + \alpha = K_r - 1 + \ell + 1 \qquad (\alpha = \ell + 1)$$

$$= K_r + \ell$$

$$= n.$$

when $n = 1, \quad \ell = 0, \quad m = 0$

$$K_r = 1.$$

$$a_1 = 0 \Rightarrow a_2 = a_3 = a_4 \cdots = 0$$

$$\boxed{E_1 = -\frac{m_e e^4}{2\hbar^2} = -13.6 \text{ eV}}$$

This is the ground state energy of the hydrogen atom.

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Eigenfunctions of Hydrogen Atom: (41)

Let us start with the equ.

$$\frac{d^2w}{dz^2} - 2\rho\frac{dw}{dz} + \left[\frac{2}{z} - \frac{l(l+1)}{z^2}\right]w = 0.$$

Now write

$$w(z) = t^{l+1}\, v(z) \qquad z = r/a$$

where $t = \frac{2z}{n} = \frac{2r}{na}$.

and $\boxed{\rho = \frac{1}{n}}$

Then

$$\frac{dw}{dt} = (l+1)t^{l}v + t^{l+1}\frac{dv}{dt}$$

$$\frac{d^2w}{dt^2} = l(l+1)t^{l-1}v + (l+1)t^{l}\frac{dv}{dt} + (l+1)t^{l}\frac{dv}{dt} + t^{l+1}\frac{d^2v}{dt^2}$$

Therefore the equation above now can be expressed as

$$t\frac{d^2v}{dt^2} + [2(l+1) - t]\frac{dv}{dt} + (n - (l+1))v = 0 \qquad (1)$$

Let us consider the Laguerre Polynomial L_q, which are the solutions of