In principle one can obtained energy correction and wavefn correction to any order. However, the above the 2nd order calculation are not significantly effective, since the first and two orders are generally sufficiently accurate.

If the unperturbed energy levels are $E_n^{(0)}$ and $E_m^{(0)}$ are equ. (i.e degenerate) the finite conditions of E_n and ψ_n will break down. So the degenerate energy levels require an approach that is different from the non-degenerate treatment.

Degenerate Perturbation theory:

Now we apply perturbation theory to determine the energy spectrum and the state of a system whose unperturbed Hamiltonian H_0 is degenerate.

Let $\psi_1^{(0)}$ and $\psi_2^{(0)}$ be degenerate (9).

$$H_0 \psi_1^{(0)} = E^{(0)} \psi_1^{(0)}$$

$$H_0 \psi_2^{(0)} = E^{(0)} \psi_2^{(0)}$$ with $\langle \psi_1 | \psi_2 \rangle = 0$.

Any linear combination of these states

$$\psi^{(0)} = a\,\psi_1^{(0)} + b\,\psi_2^{(0)}$$

is also an eigenstate of H_0 with the same eigenvalue.

$$H_0 \psi^{(0)} = E^{(0)} \psi^{(0)}$$

when the system is perturbed, the Hamiltonian is expressed as

$$H = H_0 + \lambda H_1$$

obeying the Schrodinger equation.

$$H\psi = E\psi$$

with.

$$E = E^{(0)} + \lambda E^{(1)} + \lambda^2 E^{(2)} + \cdots$$

$$\psi = \psi^{(0)} + \lambda \psi^{(1)} + \lambda^2 \psi^{(2)} + \cdots$$

With this expansion the Schrodinger equ above is written as.

1:

$$H_0\psi^{(0)} + \lambda(H_1\psi^{(0)} + H_0\psi^{(1)}) + \cdots$$
$$= E^0\psi^{(0)} + \lambda(E^{(1)}\psi^{(0)} + E^{(0)}\psi^{(1)}) + \cdots$$

Now take the first order perturbation:

$O(\lambda)$

$$H_0\psi^{(1)} + H_1\psi^{(0)} = E^{(0)}\psi^{(1)} + E^{(1)}\psi^{(0)}$$

Taking inner product with $\psi_1^{(0)}$

$$\langle\psi_1^{(0)}|H_0\psi^{(1)}\rangle + \langle\psi_1^{(0)}|H_1\psi^{(0)}\rangle$$
$$= E^{(0)}\langle\psi_1^{(0)}|\psi^{(1)}\rangle + E^{(1)}\langle\psi_1^{(0)}|\psi^{(0)}\rangle$$

Since H_0 is Hermitian, then

$$\langle H_0\psi_1^{(0)}|\psi^{(1)}\rangle + \langle\psi_1^{(0)}|H_1\psi^{(0)}\rangle$$
$$= 0 + E^{(1)}\langle\psi_1^{(0)}|\psi^{(0)}\rangle$$

$$0 + \langle\psi_1^{(0)}|H_1\psi^{(0)}\rangle = E^{(1)}\langle\psi_1^{(0)}|\psi^{(0)}\rangle$$

or

$$\langle\psi_1^{(0)}|H_1(a\psi_1^{(0)} + b\psi_2^{(0)})\rangle$$
$$= E^1\langle\psi_1^{(0)}|(a\psi_1^{(0)} + b\psi_2^{(0)})\rangle$$

$$a\langle \psi_1^{(0)} | H_1 | \psi_1^{(0)} \rangle + b \langle \psi_1^{(0)} | H_1 | \psi_2^{(0)} \rangle = a E^{(1)}$$

Can be written as

$$a\,(H_1)_{11} + b\,(H_1)_{12} = a E^{(1)} \quad \text{---} (*)$$

Similarly,

$$a\,(H_1)_{21} + b\,(H_1)_{22} = b E^{(1)} \quad \text{---} (**)$$

From equ. $(*)$

$$a = \frac{-(H_1)_{12}}{(H_1)_{11} - E^{(1)}}\, b$$

$$a = \frac{(H_1)_{12}}{E^{(1)} - (H_1)_{11}}\, b$$

Put this value of "a" in $(**)$, on arranged

$$(H_1)_{12}(H_1)_{21} + \left[(H_1)_{22} - E^{(1)}\right]\left[E^{(1)} - (H_1)_{11}\right] = 0 \longrightarrow (A)$$

or for $E^{(1)}$, we have

$$\left(E^{(1)}\right)^2 - E^{(1)}\left[(H_1)_{11} + (H_1)_{22}\right] + \left[(H_1)_{11}(H_1)_{22} - (H_1)_{12}(H_1)_{21}\right] = 0$$

By solving this quadratic equ, we get two roots.

$$E_{\pm}^{(1)} = \frac{1}{2}\left[(H_1)_{11} + (H_1)_{22} \pm \sqrt{\left((H_1)_{11} - (H_1)_{22}\right)^2 + 4\,|(H_1)_{12}|^2}\right]$$

since $(H_1)_{12}^{*} = (H_1)_{21}$

The two roots correspond to two energies. The states are no longer degenerate.

For two-fold degeneracy, equ (A) can be expressed as

$$\begin{vmatrix} (H_1)_{11} - E^{(1)} & (H_1)_{12} \\ H_{21} & (H_1)_{22} - E^{(1)} \end{vmatrix} = 0$$

For n-fold degeneracy, one can have

$$\begin{vmatrix} (H_1)_{11} - E^{(1)} & (H_{12}) & \cdots & (H_1)_{nn} \\ (H_1)_{21} & (H_1)_{22} - E^{(1)} & \cdots & (H_1)_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (H_1)_{n1} & \cdots & \cdots & (H_1)_{nn} - E^{(1)} \end{vmatrix} = 0$$

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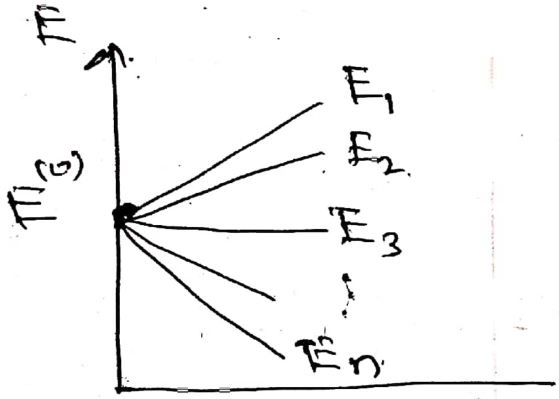
7. The n-fold degeneracy in H_0 has been removed, the n-fold degenerate level is separated into n-distinct levels. The separation of these distinct levels is proportional to the strength of the perturbation.



Time Dependent Perturbation Theory: (1)

So far we have discussed the time-independent Hamiltonian, Now we would like to discuss problems for which the Hamiltonian contains a part which has a time-dependence. This will leads to a kind of quantum dynamics which deals with the probability that a system under goes a transition from one state to another

Let us start with a free particle Schrodinger wave equation

$$i\hbar \frac{\partial}{\partial t} |\psi_0\rangle = H_0 |\psi_0\rangle$$

where $|\psi_0\rangle$ is the state of the free particle and H_0 is the free particle Hamiltonian.

In the presence of an interactioning potential $V(\underline{x},t)$, the Hamiltonian is

$$H = H_0 + V(\underline{x},t),$$

where all time-dependence enters through the potential $V(t)$. $V(t)$ characterizes the interaction of the system with external source.

Q. Perturbation.

Now the Schrodinger equation is

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = (H_0 + V(x,t))|\psi(t)\rangle$$

The main task of time-dependent perturbation theory is that How does $V(t)$ affect the system.

→ May be the system is initially in an unperturbed eigenstate $|\psi_i\rangle$ of H_0 what is the probability that the system will be found at a later time in an other unperturbed eigen state $|\psi_f\rangle$.

The state $|\psi(t)\rangle$ can be expanded as

$$|\psi(t)\rangle = \sum_n a_n(t)|\phi_n\rangle e^{-\frac{i}{\hbar}E_n t}$$

called interaction representation.

with

$$H_0|\phi_n\rangle = E_n|\phi_n\rangle$$

wheree

$$\langle\phi_m|\phi_n\rangle = \delta_{mn}$$

$$\sum_n |\phi_n\rangle\langle\phi_n| = 1$$

The time-dependence of the co-efficients a_n is a consequence of interaction $\underline{V}(\underline{x},t)$. Now the schrodinger equation is expressed as

$$i\hbar \frac{\partial}{\partial t}\left(\sum_n a_n(t)|\phi_n\rangle e^{-iE_nt/\hbar}\right)$$

$$= \left(H_0 + V(t)\right)\sum_n a_n(t)|\phi_n\rangle e^{-iE_nt/\hbar}$$

$$i\hbar \sum_n \frac{\partial}{\partial t} a_n(t)|\phi_n\rangle e^{-iE_nt/\hbar} + i\hbar \sum_n a_n(t)|\phi_n\rangle \times\left(-i\frac{E_n}{\hbar}\right) e^{-iE_nt/\hbar}$$

$$= \sum_n a_n(t)\, \overset{E_n|\phi_n\rangle}{H_0|\phi_n\rangle}\, e^{-iE_nt/\hbar}$$

$$+ \sum_n V(t)\, a_n(t)|\phi_n\rangle e^{-iE_nt/\hbar}$$

$$i\hbar \sum_n \dot{a}_n(t)|\phi_n\rangle e^{-iE_nt/\hbar} + \cancel{\sum_n a_n(t) E_n|\phi_n\rangle e^{-iE_nt/\hbar}}$$

$$= \cancel{\sum_n a_n(t) E_n|\phi_n\rangle e^{-iE_nt/\hbar}}$$

$$+ \sum_n V(t)\, a_n(t)|\phi_n\rangle e^{-iE_nt/\hbar}$$

or

$$i\hbar \sum_n \dot{a}_n(t)|\phi_n\rangle e^{-iE_nt/\hbar}$$

$$= \sum_n V(t)\, a_n(t)|\phi_n\rangle e^{-iE_nt/\hbar}$$

Taking an inner Product with $|\phi_f\rangle$ on both sides

$$i\hbar \sum_n \dot{a}_n(t) \langle \phi_f | \phi_n \rangle e^{-iE_n t/\hbar}$$

$$= \sum_n a_n(t) \langle \phi_f | V(t) | \phi_n \rangle e^{-iE_n t/\hbar}$$

$\Rightarrow$

$$i\hbar \sum_n \dot{a}_n(t)\, \delta_{fn}\, e^{-iE_n t/\hbar}$$

$$= \sum_n a_n(t)\, V_{fn}(t)\, e^{-iE_n t/\hbar}$$

where

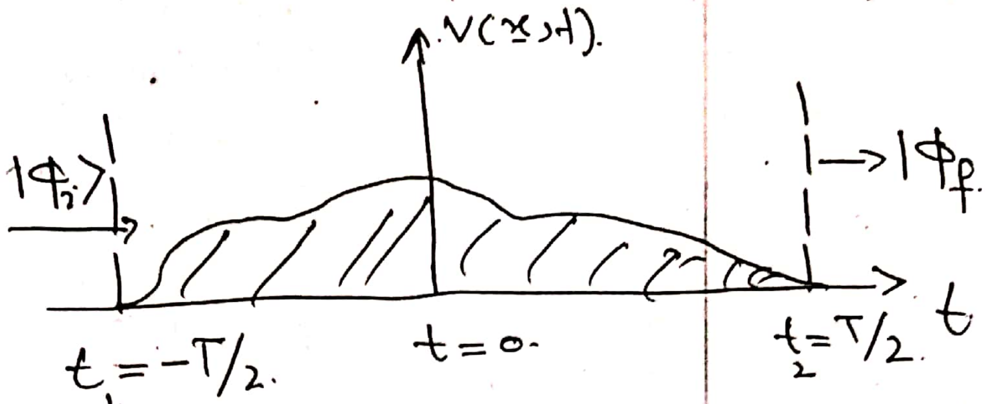
$V_{fn}(t) = \langle \phi_f | V(t) | \phi_n \rangle$ is the matrix element of $V(t)$

$\Rightarrow$ for $f = n$, above equ will become

$$i\hbar \sum_n \dot{a}_f(t)\, e^{-iE_f t/\hbar} = \sum_n a_n(t) V_{fn}(t) e^{\frac{-iE_n t}{\hbar}}$$

∴ Hence

$$\dot{a}_f = \frac{-i}{\hbar} \sum_n a_n(t) \cdot V_{fn}(t)\, e^{i(E_f - E_n)t/\hbar} \quad (1)$$



Before the action of V, the Particle is in an eigenstate ϕ_i of H_0 i.e at $t_1 = -T/2$ we take

$$a_n(-T/2) = \delta_{ni}$$

i.e put $n = i$

$$a_i(-T/2) = 1$$

Hence Equ. 1 can be expressed as

$$\dot{a}_f = \frac{-i}{\hbar} V_{fi}(t)\, e^{i(E_f - E_i)t/\hbar} \rightarrow (2)$$

Transition Probability:

The transition probability corresponding to a transition from an initial unperturbed state $|\phi_i\rangle$ to another unperturbed final state $|\phi_f\rangle$ is obtained by integrating of equ. 2

$$\Rightarrow a_f(t) = \frac{-i}{\hbar} \int_{t_1}^{t_2} V_{fi}(t)\, e^{i(E_f - E_i)t/\hbar}\, dt$$

$$t_1 = -T/2, \quad t_2 = T/2$$

or

$$T_{fi} = a_f = \frac{-i}{\hbar} \int_{-T/2}^{T/2} V_{fi}(t)\, e^{i(E_f - E_i)t/\hbar}\, dt$$

$$T_{fi} = \frac{-i}{\hbar} \int_{-T/2}^{T/2} dt \langle \phi_f | V(t) | \phi_i \rangle e^{i(E_f - E_i)t/\hbar} \quad \rightarrow 3$$

The quantity $|T_{fi}|^2$ is the transition probability of a particle from an initial state $|\phi_i\rangle$ to a final state $|\phi_f\rangle$.

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