# Exponential Smoothing Methods

# Chapter Topics

- Introduction to exponential smoothing
- Simple Exponential Smoothing
- Holt's Trend Corrected Exponential Smoothing
- Holt-Winters Methods
  - Multiplicative Holt-Winters method
  - Additive Holt-Winters method

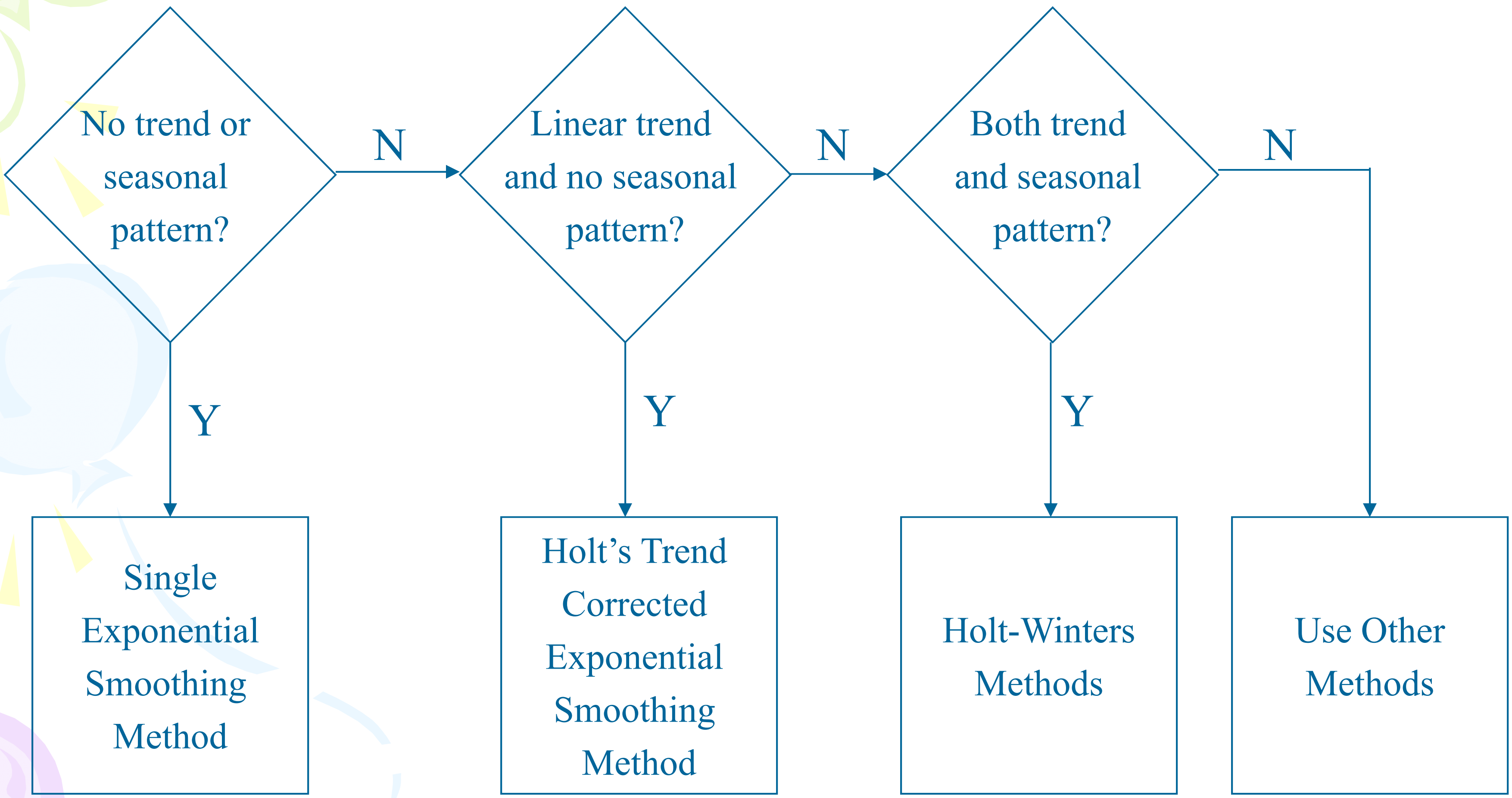
# Motivation of Exponential Smoothing

- Simple moving average method assigns equal weights ($1/k$) to all $k$ data points.
- Arguably, recent observations provide more relevant information than do observations in the past.
- So we want a weighting scheme that assigns decreasing weights to the more distant observations.

# Exponential Smoothing

- Exponential smoothing methods give larger weights to more recent observations, and the weights decrease exponentially as the observations become more distant.
- These methods are most effective when the parameters describing the time series are changing SLOWLY over time.

# Data vs Methods



# Simple Exponential Smoothing

- The Simple Exponential Smoothing method is used for forecasting a time series when there is no trend or seasonal pattern, but the mean (or level) of the time series $y_t$ is slowly changing over time.
- NO TREND model

$$y_t = \beta_o + \varepsilon_t$$

# Procedures of Simple Exponential Smoothing Method

- **Step 1**: Compute the initial estimate of the mean (or level) of the series at time period $t = 0$

$$\ell_0 = \bar{y} = \frac{\sum_{t=1}^{n} y_t}{n}$$

- **Step 2**: Compute the updated estimate by using the smoothing equation

$$\ell_T = \alpha y_T + (1-\alpha)\ell_{T-1}$$

where $\alpha$ is a smoothing constant between 0 and 1.

# Procedures of Simple Exponential Smoothing Method

Note that

$$\ell_T = \alpha y_T + (1-\alpha)\ell_{T-1}$$

$$= \alpha y_T + (1-\alpha)[\alpha y_{T-1} + (1-\alpha)\ell_{T-2}]$$

$$= \alpha y_T + (1-\alpha)\alpha y_{T-1} + (1-\alpha)^2 \ell_{T-2}$$

$$= \alpha y_T + (1-\alpha)\alpha y_{T-1} + (1-\alpha)^2 \alpha y_{T-2} + \ldots + (1-\alpha)^{T-1} \alpha y_1 + (1-\alpha)^T \ell_0$$

The coefficients measuring the contributions of the observations decrease exponentially over time.

# Simple Exponential Smoothing

- Point forecast made at time $T$ for $y_{T+p}$

$$\hat{y}_{T+p}(T) = \ell_T \qquad (p = 1, 2, 3, \ldots)$$

- SSE, MSE, and the standard errors at time $T$

$$SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2$$

$$MSE = \frac{SSE}{T-1}, \qquad s = \sqrt{MSE}$$

Note: There is no theoretical justification for dividing SSE by (T – number of smoothing constants). However, we use this divisor because it agrees to the computation of $s$ in Box-Jenkins models introduced later.

# Example: Cod Catch

- The Bay City Seafood Company recorded the monthly cod catch for the previous two years, as given below.

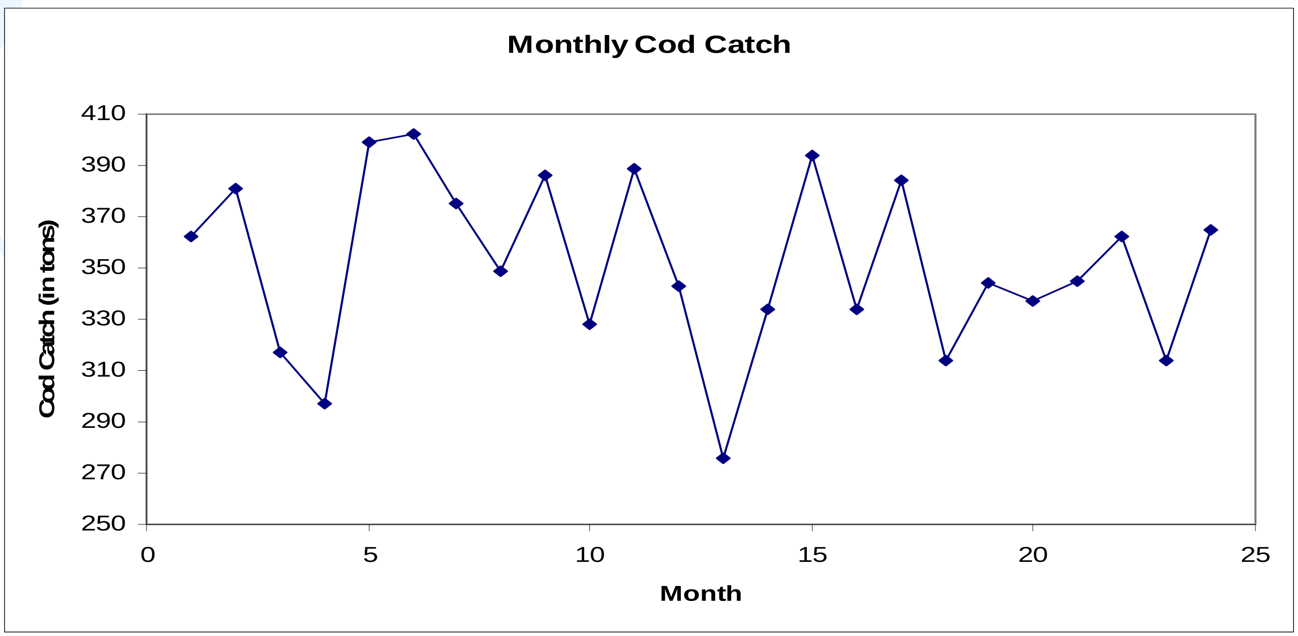
**Cod Catch (In Tons)**

| **Month** | **Year 1** | **Year 2** |
|---|---|---|
| January | 362 | 276 |
| February | 381 | 334 |
| March | 317 | 394 |
| April | 297 | 334 |
| May | 399 | 384 |
| June | 402 | 314 |
| July | 375 | 344 |
| August | 349 | 337 |
| September | 386 | 345 |
| October | 328 | 362 |
| November | 389 | 314 |
| December | 343 | 365 |

# Example: Cod Catch

- The plot of these data suggests that there is no trend or seasonal pattern. Therefore, a NO TREND model is suggested: $y_t = \beta_o + \varepsilon_t$

  It is also possible that the mean (or level) is slowly changing over time.



# Example: Cod Catch

- **Step 1**: Compute $\ell_0$ by averaging the first twelve time series values.

$$\ell_0 = \frac{\sum_{t=1}^{12} y_t}{12} = \frac{362 + 381 + \ldots + 343}{12} = 360.6667$$

Though there is no theoretical justification, it is a common practice to calculate initial estimates of exponential smoothing procedures by using HALF of the historical data.

# Example: Cod Catch

- **Step 2**: Begin with the initial estimate $\ell_0$ = 360.6667 and update it by applying the smoothing equation to the 24 observed cod catches.

  Set α = 0.1 arbitrarily and judge the appropriateness of this choice of α by the model's in-sample fit.

$$\ell_1 = \alpha y_1 + (1-\alpha)\ell_0 = 0.1(362) + 0.9(360.6667) = 360.8000$$

$$\ell_2 = \alpha y_2 + (1-\alpha)\ell_1 = 0.1(381) + 0.9(360.8000) = 362.8200$$

# One-period-ahead Forecasting

| n | alpha | SSE | MSE | s | |
|---|---|---|---|---|---|
| 24 | 0.1 | 28735.1092 | 1249.3526 | 35.3462 | |
| | | | | | |
| Time | | Smoothed Estimate | Forecast Made | Forecast | Squared Forecast |
| Period | y | for Level | Last Period | Error | Error |
| 0 | | 360.6667 | | | |
| 1 | 362 | 360.8000 | 360.6667 | 1.3333 | 1.7777 |
| 2 | 381 | 362.8200 | 360.8000 | 20.2000 | 408.0388 |
| 3 | 317 | 358.2380 | 362.8200 | -45.8200 | 2099.4749 |
| 4 | 297 | 352.1142 | 358.2380 | -61.2380 | 3750.0956 |
| 5 | 399 | 356.8028 | 352.1142 | 46.8858 | 2198.2762 |
| 6 | 402 | 361.3225 | 356.8028 | 45.1972 | 2042.7869 |
| 7 | 375 | 362.6903 | 361.3225 | 13.6775 | 187.0735 |
| 8 | 349 | 361.3212 | 362.6903 | -13.6903 | 187.4234 |
| 9 | 386 | 363.7891 | 361.3212 | 24.6788 | 609.0411 |
| 10 | 328 | 360.2102 | 363.7891 | -35.7891 | 1280.8609 |
| 11 | 389 | 363.0892 | 360.2102 | 28.7898 | 828.8523 |
| 12 | 343 | 361.0803 | 363.0892 | -20.0892 | 403.5753 |
| 13 | 276 | 352.5722 | 361.0803 | -85.0803 | 7238.6517 |
| 14 | 334 | 350.7150 | 352.5722 | -18.5722 | 344.9281 |
| 15 | 394 | 355.0435 | 350.7150 | 43.2850 | 1873.5899 |
| 16 | 334 | 352.9392 | 355.0435 | -21.0435 | 442.8295 |
| 17 | 384 | 356.0452 | 352.9392 | 31.0608 | 964.7756 |
| 18 | 314 | 351.8407 | 356.0452 | -42.0452 | 1767.8027 |
| 19 | 344 | 351.0566 | 351.8407 | -7.8407 | 61.4769 |
| 20 | 337 | 349.6510 | 351.0566 | -14.0566 | 197.5894 |
| 21 | 345 | 349.1859 | 349.6510 | -4.6510 | 21.6317 |
| 22 | 362 | 350.4673 | 349.1859 | 12.8141 | 164.2015 |
| 23 | 314 | 346.8206 | 350.4673 | -36.4673 | 1329.8638 |
| 24 | 365 | 348.6385 | 346.8206 | 18.1794 | 330.4918 |

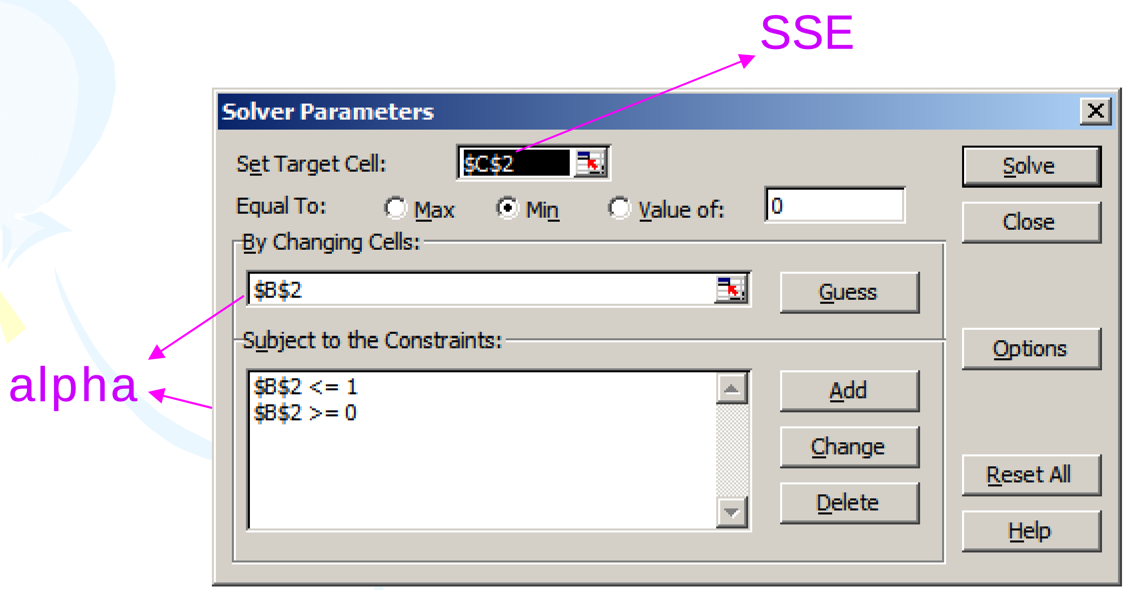
# Example: Cod Catch

- Results associated with different values of $\alpha$

| Smoothing Constant | Sum of Squared Errors |
|---|---|
| 0.1 | 28735.11 |
| 0.2 | 30771.73 |
| 0.3 | 33155.54 |
| 0.4 | 35687.69 |
| 0.5 | 38364.24 |
| 0.6 | 41224.69 |
| 0.7 | 44324.09 |
| 0.8 | 47734.09 |

# Example: Cod Catch

- Step 3: Find a good value of α that provides the minimum value for MSE (or SSE).
  - Use Solver in Excel as an illustration



# Example: Cod Catch

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | n | alpha | SSE | MSE | s | |
| 2 | 24 | 0.03435 | 28089.1479 | 1221.2673 | 34.9466 | |
| 3 | | | | | | |
| 4 | Time | | Smoothed Estimate | Forecast Made | Forecast | Squared Forecast |
| 5 | Period | y | for Level | Last Period | Error | Error |
| 6 | 0 | | 360.6667 | | | |
| 7 | 1 | 362 | 360.7125 | 360.6667 | 1.3333 | 1.7777 |
| 8 | 2 | 381 | 361.4095 | 360.7125 | 20.2875 | 411.5825 |
| 9 | 3 | 317 | 359.8838 | 361.4095 | -44.4095 | 1972.1994 |
| 10 | 4 | 297 | 357.7235 | 359.8838 | -62.8838 | 3954.3759 |
| 11 | 5 | 399 | 359.1415 | 357.7235 | 41.2765 | 1703.7457 |
| 12 | 6 | 402 | 360.6139 | 359.1415 | 42.8585 | 1836.8477 |
| 13 | 7 | 375 | 361.1081 | 360.6139 | 14.3861 | 206.9605 |
| 14 | 8 | 349 | 360.6921 | 361.1081 | -12.1081 | 146.6059 |
| 15 | 9 | 386 | 361.5616 | 360.6921 | 25.3079 | 640.4879 |
| 16 | 10 | 328 | 360.4086 | 361.5616 | -33.5616 | 1126.3778 |
| 17 | 11 | 389 | 361.3908 | 360.4086 | 28.5914 | 817.4685 |
| 18 | 12 | 343 | 360.7590 | 361.3908 | -18.3908 | 338.2219 |
| 19 | 13 | 276 | 357.8472 | 360.7590 | -84.7590 | 7184.0916 |
| 20 | 14 | 334 | 357.0280 | 357.8472 | -23.8472 | 568.6912 |
| 21 | 15 | 394 | 358.2981 | 357.0280 | 36.9720 | 1366.9281 |
| 22 | 16 | 334 | 357.4634 | 358.2981 | -24.2981 | 590.3991 |
| 23 | 17 | 384 | 358.3750 | 357.4634 | 26.5366 | 704.1911 |
| 24 | 18 | 314 | 356.8506 | 358.3750 | -44.3750 | 1969.1431 |
| 25 | 19 | 344 | 356.4091 | 356.8506 | -12.8506 | 165.1376 |
| 26 | 20 | 337 | 355.7424 | 356.4091 | -19.4091 | 376.7141 |
| 27 | 21 | 345 | 355.3733 | 355.7424 | -10.7424 | 115.3981 |
| 28 | 22 | 362 | 355.6010 | 355.3733 | 6.6267 | 43.9130 |
| 29 | 23 | 314 | 354.1718 | 355.6010 | -41.6010 | 1730.6402 |
| 30 | 24 | 365 | 354.5438 | 354.1718 | 10.8282 | 117.2494 |

# Holt's Trend Corrected Exponential Smoothing

- If a time series is increasing or decreasing approximately at a fixed rate, then it may be described by the LINEAR TREND model

  $$y_t = \beta_0 + \beta_1 t + \varepsilon_t$$

  If the values of the parameters $\beta_0$ and $\beta_1$ are slowly changing over time, Holt's trend corrected exponential smoothing method can be applied to the time series observations.

  <u>Note</u>: When neither $\beta_0$ nor $\beta_1$ is changing over time, regression can be used to forecast future values of $y_t$.

- Level (or mean) at time $T$: $\beta_0 + \beta_1 T$

  Growth rate (or trend): $\beta_1$

# Holt's Trend Corrected Exponential Smoothing

- A smoothing approach for forecasting such a time series that employs two smoothing constants, denoted by $\alpha$ and $\gamma$.
- There are two estimates $\ell_{T-1}$ and $b_{T-1}$.
  - $\ell_{T-1}$ is the estimate of the level of the time series constructed in time period $T$–1 (This is usually called the <u>permanent component</u>).
  - $b_{T-1}$ is the estimate of the growth rate of the time series constructed in time period $T$–1 (This is usually called the <u>trend component</u>).

# Holt's Trend Corrected Exponential Smoothing

- Level estimate

$$\ell_T = \alpha y_T + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

- Trend estimate

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

where $\alpha$ = smoothing constant for the level ($0 \le \alpha \le 1$)

$\gamma$ = smoothing constant for the trend ($0 \le \gamma \le 1$)

# Holt's Trend Corrected Exponential Smoothing

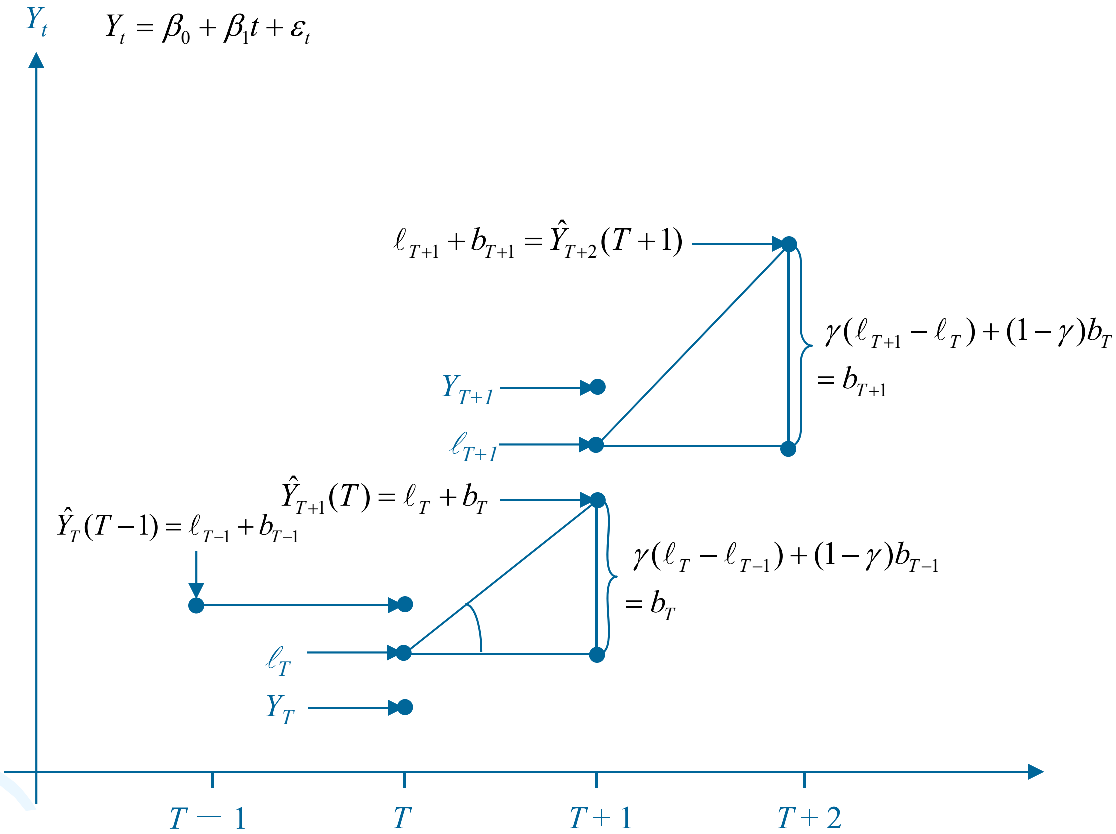
- Point forecast made at time $T$ for $y_{T+p}$

$$\hat{y}_{T+p}(T) = \ell_T + pb_T \qquad (p = 1, 2, 3, \ldots)$$

- MSE and the standard error $s$ at time $T$

$$SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2$$

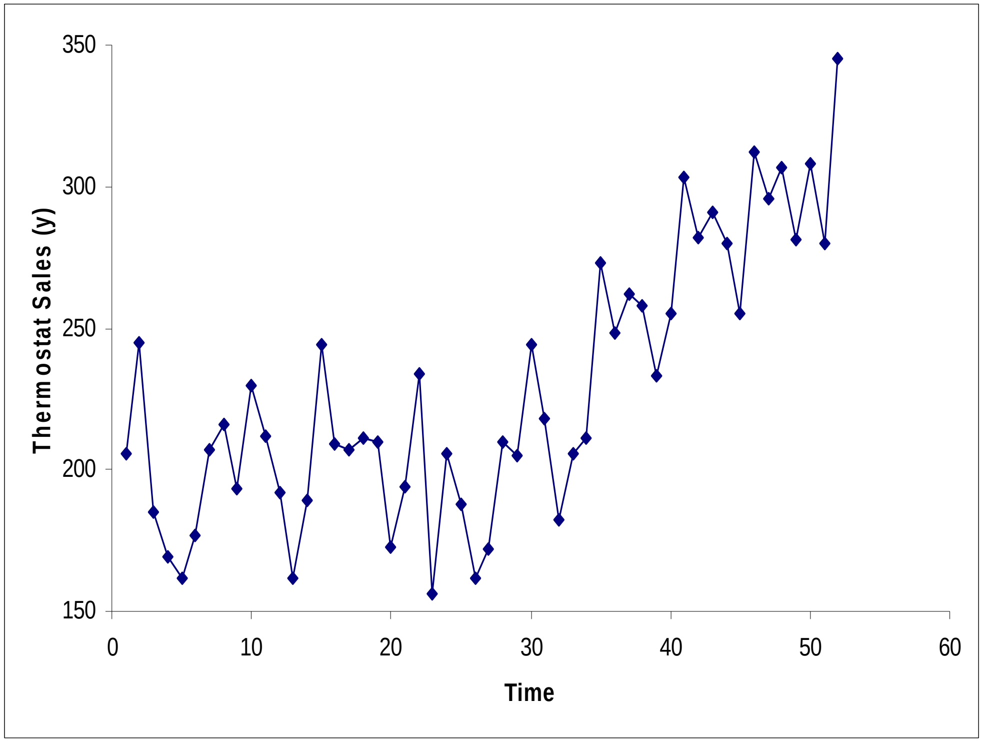
$$MSE = \frac{SSE}{T-2}, \qquad s = \sqrt{MSE}$$


$Y_t$
$Y_t = \beta_0 + \beta_1 t + \varepsilon_t$
$\ell_{T+1} + b_{T+1} = \hat{Y}_{T+2}(T+1)$
$\gamma(\ell_{T+1} - \ell_T) + (1-\gamma)b_T$
$= b_{T+1}$
$Y_{T+1}$
$\ell_{T+1}$
$\hat{Y}_{T+1}(T) = \ell_T + b_T$
$\hat{Y}_T(T-1) = \ell_{T-1} + b_{T-1}$
$\gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$
$= b_T$
$\ell_T$
$Y_T$
$T-1$
$T$
$T+1$
$T+2$

# Procedures of Holt's Trend Corrected Exponential Smoothing

- Use the example of Thermostat Sales as an illustration

| Weekly Thermostat Sales | | | |
|---|---|---|---|
| 206 | 189 | 172 | 255 |
| 245 | 244 | 210 | 303 |
| 185 | 209 | 205 | 282 |
| 169 | 207 | 244 | 291 |
| 162 | 211 | 218 | 280 |
| 177 | 210 | 182 | 255 |
| 207 | 173 | 206 | 312 |
| 216 | 194 | 211 | 296 |
| 193 | 234 | 273 | 307 |
| 230 | 156 | 248 | 281 |
| 212 | 206 | 262 | 308 |
| 192 | 188 | 258 | 280 |
| 162 | 162 | 233 | 345 |



# Procedures of Holt's Trend Corrected Exponential Smoothing

- Findings:
  - Overall an upward trend
  - The growth rate has been changing over the 52-week period
  - There is no seasonal pattern

  $\Rightarrow$ Holt's trend corrected exponential smoothing method can be applied

# Procedures of Holt's Trend Corrected Exponential Smoothing

- **Step 1**: Obtain initial estimates $\ell_0$ and $b_0$ by fitting a least squares trend line to HALF of the historical data.
  - $y$-intercept = $\ell_0$; slope = $b_0$

# Procedures of Holt's Trend Corrected Exponential Smoothing

- Example
  - Fit a least squares trend line to the first 26 observations
  - Trend line

$$\hat{y}_t = 202.6246 - 0.3682t$$

  - $\ell_0 = 202.6246$; $b_0 = -0.3682$

| SUMMARY OUTPUT | |
|---|---|
| | |
| *Regression Statistics* | |
| Multiple R | 0.111769555 |
| R Square | 0.012492433 |
| Adjusted R Square | -0.028653715 |
| Standard Error | 25.5551741 |
| Observations | 26 |
| | |
| ANOVA | |
| | *df* |
| Regression | 1 |
| Residual | 24 |
| Total | 25 |
| | |
| | *Coefficients* |
| Intercept | 202.6246154 |
| X Variable 1 | -0.368205128 |

# Procedures of Holt's Trend Corrected Exponential Smoothing

- **Step 2**: Calculate a point forecast of $y_1$ from time 0

$$\hat{y}_{T+p}(T) = \ell_T + pb_T \qquad T = 0, p = 1$$

- Example

$$\hat{y}_1(0) = \ell_0 + b_0 = 202.6246 - 0.3682 = 202.2564$$

# Procedures of Holt's Trend Corrected Exponential Smoothing

- **Step 3**: Update the estimates $\ell_T$ and $b_T$ by using some predetermined values of smoothing constants.
- Example: let $\alpha = 0.2$ and $\gamma = 0.1$

$$\begin{aligned}\ell_1 &= \alpha y_1 + (1-\alpha)(\ell_0 + b_0) \\ &= 0.2(206) + 0.8(202.6246 - 0.3682) = 203.0051\end{aligned}$$

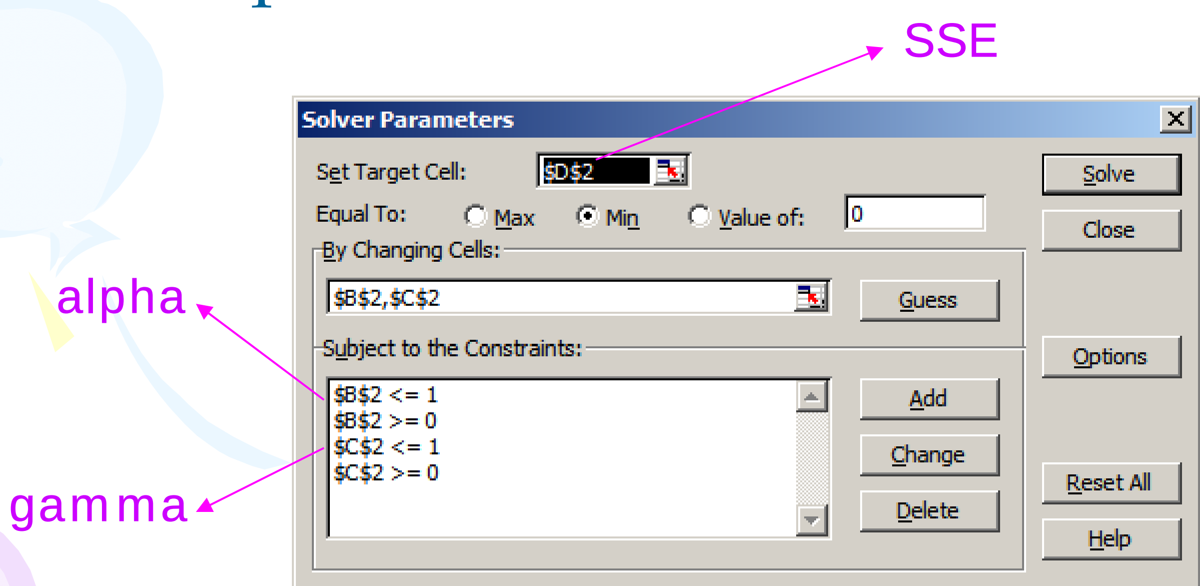
$$\begin{aligned}b_1 &= \gamma(\ell_1 - \ell_0) + (1-\gamma)b_0 \\ &= 0.1(203.0051 - 202.6246) + 0.9(-0.3682) = -0.2933\end{aligned}$$

$$\hat{y}_2(1) = \ell_1 + b_1 = 203.0051 - 0.2933 = 202.7118$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | n | alpha | gamma | SSE | MSE | s | |
| 2 | 52 | 0.2 | 0.1 | 39182.4705 | 783.6494 | 27.993739 | |
| 3 | | | | | | | |
| 4 | | | | | Forecast | | Squared |
| 5 | Time | | | Growth | Made Last | Forecast | Forecast |
| 6 | Period | y | Level | Rate | Period | Error | Error |
| 7 | 0 | | 202.6246 | -0.3682 | | | |
| 8 | 1 | 206 | 203.0051 | -0.2933 | 202.2564 | 3.7436 | 14.0145 |
| 9 | 2 | 245 | 211.1694 | 0.5524 | 202.7118 | 42.2882 | 1788.2923 |
| 10 | 3 | 185 | 206.3775 | 0.0180 | 211.7219 | -26.7219 | 714.0582 |
| 11 | 4 | 169 | 198.9164 | -0.7299 | 206.3955 | -37.3955 | 1398.4224 |
| 12 | 5 | 162 | 190.9492 | -1.4536 | 198.1865 | -36.1865 | 1309.4608 |
| 13 | 6 | 177 | 186.9964 | -1.7036 | 189.4955 | -12.4955 | 156.1383 |
| 14 | 7 | 207 | 189.6343 | -1.2694 | 185.2929 | 21.7071 | 471.1995 |
| 15 | 8 | 216 | 193.8919 | -0.7167 | 188.3649 | 27.6351 | 763.6997 |
| 16 | 9 | 193 | 193.1402 | -0.7202 | 193.1752 | -0.1752 | 0.0307 |
| 17 | 10 | 230 | 199.9360 | 0.0314 | 192.4199 | 37.5801 | 1412.2609 |
| 18 | 11 | 212 | 202.3739 | 0.2720 | 199.9673 | 12.0327 | 144.7850 |
| 19 | 12 | 192 | 200.5167 | 0.0591 | 202.6459 | -10.6459 | 113.3354 |
| 20 | 13 | 162 | 192.8607 | -0.7124 | 200.5758 | -38.5758 | 1488.0961 |
| 21 | 14 | 189 | 191.5186 | -0.7754 | 192.1483 | -3.1483 | 9.9117 |
| 22 | 15 | 244 | 201.3946 | 0.2898 | 190.7433 | 53.2567 | 2836.2799 |
| 23 | 16 | 209 | 203.1475 | 0.4361 | 201.6844 | 7.3156 | 53.5182 |
| 24 | 17 | 207 | 204.2669 | 0.5044 | 203.5836 | 3.4164 | 11.6718 |
| 25 | 18 | 211 | 206.0170 | 0.6290 | 204.7713 | 6.2287 | 38.7969 |
| 26 | 19 | 210 | 207.3168 | 0.6961 | 206.6460 | 3.3540 | 11.2492 |
| 27 | 20 | 173 | 201.0103 | -0.0042 | 208.0129 | -35.0129 | 1225.9019 |
| | …… | | | | …… | | |
| 52 | 45 | 255 | 280.9500 | 4.4428 | 287.4375 | -32.4375 | 1052.1900 |
| 53 | 46 | 312 | 290.7142 | 4.9749 | 285.3928 | 26.6072 | 707.9453 |
| 54 | 47 | 296 | 295.7513 | 4.9811 | 295.6891 | 0.3109 | 0.0966 |
| 55 | 48 | 307 | 301.9860 | 5.1065 | 300.7324 | 6.2676 | 39.2823 |
| 56 | 49 | 281 | 301.8740 | 4.5846 | 307.0924 | -26.0924 | 680.8155 |
| 57 | 50 | 308 | 306.7669 | 4.6155 | 306.4586 | 1.5414 | 2.3759 |
| 58 | 51 | 280 | 305.1059 | 3.9878 | 311.3823 | -31.3823 | 984.8515 |
| 59 | 52 | 345 | 316.2750 | 4.7059 | 309.0937 | 35.9063 | 1289.2627 |

# Procedures of Holt's Trend Corrected Exponential Smoothing

- **Step 4**: Find the best combination of α and γ that minimizes SSE (or MSE)
- Example: Use Solver in Excel as an illustration



| 1 | n | alpha | gamma | SSE | MSE | s | |
|---|---|---|---|---|---|---|---|
| 2 | 52 | 0.247 | 0.0951 | 38884.2448 | 777.6849 | 27.887002 | |
| 3 | | | | | | | |
| 4 | | | | | Forecast | | Squared |
| 5 | Time | | | Growth | Made Last | Forecast | Forecast |
| 6 | Period | y | Level | Rate | Period | Error | Error |
| 7 | 0 | | 202.6246 | -0.3682 | | | |
| 8 | 1 | 206 | 203.1805 | -0.2804 | 202.2564 | 3.7436 | 14.0145 |
| 9 | 2 | 245 | 213.2921 | 0.7074 | 202.9001 | 42.0999 | 1772.4001 |
| 10 | 3 | 185 | 206.8413 | 0.0270 | 213.9996 | -28.9996 | 840.9751 |
| 11 | 4 | 169 | 197.5208 | -0.8615 | 206.8683 | -37.8683 | 1434.0063 |
| 12 | 5 | 162 | 188.1039 | -1.6747 | 196.6593 | -34.6593 | 1201.2665 |
| 13 | 6 | 177 | 184.1017 | -1.8960 | 186.4292 | -9.4292 | 88.9096 |
| 14 | 7 | 207 | 188.3260 | -1.3142 | 182.2057 | 24.7943 | 614.7576 |
| 15 | 8 | 216 | 194.1673 | -0.6341 | 187.0117 | 28.9883 | 840.3188 |
| 16 | 9 | 193 | 193.4016 | -0.6466 | 193.5332 | -0.5332 | 0.2843 |
| 17 | 10 | 230 | 201.9486 | 0.2273 | 192.7550 | 37.2450 | 1387.1882 |
| | …… | | | | …… | | |
| 52 | 45 | 255 | 281.4910 | 4.1454 | 290.1732 | -35.1732 | 1237.1566 |
| 53 | 46 | 312 | 292.1440 | 4.7640 | 285.6364 | 26.3636 | 695.0399 |
| 54 | 47 | 296 | 296.6839 | 4.7427 | 296.9080 | -0.9080 | 0.8245 |
| 55 | 48 | 307 | 302.8023 | 4.8734 | 301.4265 | 5.5735 | 31.0637 |
| 56 | 49 | 281 | 301.0910 | 4.2475 | 307.6757 | -26.6757 | 711.5940 |
| 57 | 50 | 308 | 305.9955 | 4.3100 | 305.3386 | 2.6614 | 7.0832 |
| 58 | 51 | 280 | 302.8248 | 3.5989 | 310.3055 | -30.3055 | 918.4227 |
| 59 | 52 | 345 | 315.9460 | 4.5040 | 306.4237 | 38.5763 | 1488.1288 |

# Holt's Trend Corrected Exponential Smoothing

- *p*-step-ahead forecast made at time *T*

$$\hat{y}_{T+p}(T) = \ell_T + pb_T \qquad (p = 1, 2, 3, \ldots)$$

- Example
  - In period 52, the one-period-ahead sales forecast for period 53 is

$$\hat{y}_{53}(52) = \ell_{52} + b_{52} = 315.9460 + 4.5040 = 320.45$$

  - In period 52, the three-period-ahead sales forecast for period 55 is

$$\hat{y}_{55}(52) = \ell_{52} + 3b_{52} = 315.9460 + 3(4.5040) = 329.458$$

# Holt's Trend Corrected Exponential Smoothing

- Example
  - If we observe $y_{53}$ = 330, we can either find a new set of (optimal) α and γ that minimize the SSE for 53 periods, or
  - we can simply revise the estimate for the level and growth rate and recalculate the forecasts as follows:

$$\ell_{53} = \alpha y_{53} + (1-\alpha)(\ell_{52} + b_{52})$$
$$= 0.247(330) + 0.753(315.946 + 4.5040) = 322.8089$$

$$b_{53} = \gamma(\ell_{53} - \ell_{52}) + (1-\gamma)b_{52}$$
$$= 0.095(322.8089 - 315.9460) + 0.905(4.5040) = 4.7281$$

$$\hat{y}_{54}(53) = \ell_{53} + b_{53} = 322.8089 + 4.7281 = 327.537$$

$$\hat{y}_{55}(53) = \ell_{53} + 2b_{53} = 322.8089 + 2(4.7281) = 332.2651$$

# Holt-Winters Methods

- Two Holt-Winters methods are designed for time series that exhibit linear trend
  - Additive Holt-Winters method: used for time series with constant (additive) seasonal variations
  - Multiplicative Holt-Winters method: used for time series with increasing (multiplicative) seasonal variations
- Holt-Winters method is an exponential smoothing approach for handling SEASONAL data.
- The multiplicative Holt-Winters method is the better known of the two methods.

# Multiplicative Holt-Winters Method

- It is generally considered to be best suited to forecasting time series that can be described by the equation:

$$y_t = (\beta_0 + \beta_1 t) \times SN_t \times IR_t$$

  - $SN_t$: seasonal pattern
  - $IR_t$: irregular component
- This method is appropriate when a time series has a linear trend with a multiplicative seasonal pattern for which the level ($\beta_0 + \beta_1 t$), growth rate ($\beta_1$), and the seasonal pattern ($SN_t$) may be slowly changing over time.

# Multiplicative Holt-Winters Method

- Estimate of the level

$$\ell_T = \alpha(y_T / sn_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

- Estimate of the growth rate (or trend)

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

- Estimate of the seasonal factor

$$sn_T = \delta(y_T / \ell_T) + (1-\delta)sn_{T-L}$$

where α, γ, and δ are smoothing constants between 0 and 1,

*L* = number of seasons in a year (*L* = 12 for monthly data, and *L* = 4 for quarterly data)

# Multiplicative Holt-Winters Method

- Point forecast made at time $T$ for $y_{T+p}$

$$\hat{y}_{T+p}(T) = (\ell_T + pb_T)sn_{T+p-L} \qquad (p = 1, 2, 3, \ldots)$$

- MSE and the standard errors at time $T$

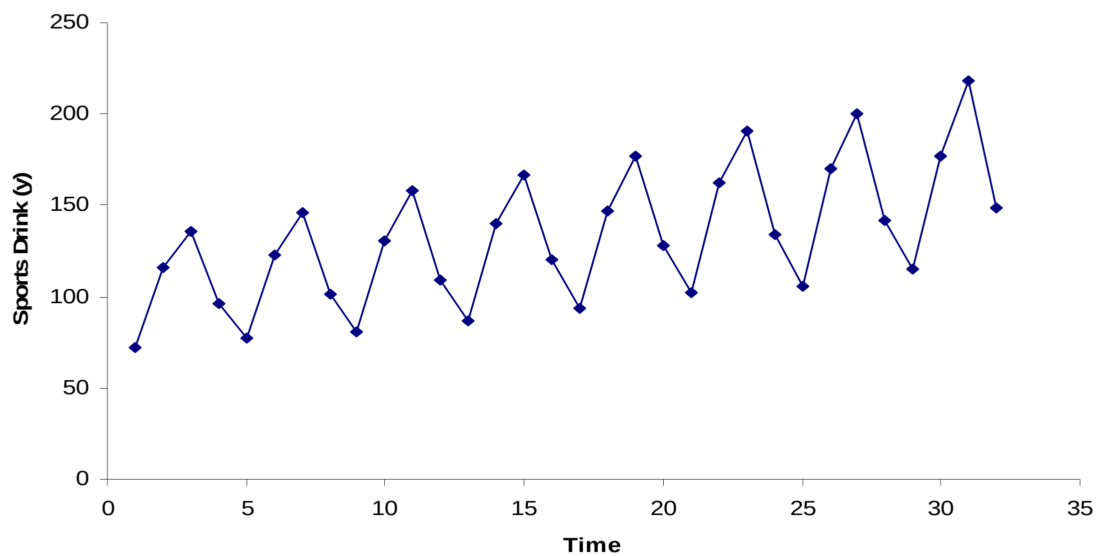
$$SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2$$

$$MSE = \frac{SSE}{T-3}, \qquad s = \sqrt{MSE}$$

# Procedures of Multiplicative Holt-Winters Method

- Use the Sports Drink example as an illustration

| Quarterly sales of Tiger Sports Drink | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Year | | | | | | | |
| **Quarter** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
| 1 | 72 | 77 | 81 | 87 | 94 | 102 | 106 | 115 |
| 2 | 116 | 123 | 131 | 140 | 147 | 162 | 170 | 177 |
| 3 | 136 | 146 | 158 | 167 | 177 | 191 | 200 | 218 |
| 4 | 96 | 101 | 109 | 120 | 128 | 134 | 142 | 149 |



# Procedures of Multiplicative Holt-Winters Method

- Observations:
  - Linear upward trend over the 8-year period
  - Magnitude of the seasonal span increases as the level of the time series increases

  $\Rightarrow$ Multiplicative Holt-Winters method can be applied to forecast future sales

# Procedures of Multiplicative Holt-Winters Method

- **Step 1**: Obtain initial values for the level $\ell_0$, the growth rate $b_0$, and the seasonal factors $sn_{-3}$, $sn_{-2}$, $sn_{-1}$, and $sn_0$, by fitting a least squares trend line to at least four or five years of the historical data.
  - $y$-intercept = $\ell_0$; slope = $b_0$

# Procedures of Multiplicative Holt-Winters Method

- Example
  - Fit a least squares trend line to the first 16 observations
  - Trend line

$$\hat{y}_t = 95.2500 + 2.4706t$$

  - $\ell_0 = 95.2500$; $b_0 = 2.4706$

| SUMMARY OUTPUT | |
|---|---|
| | |
| *Regression Statistics* | |
| Multiple R | 0.403809754 |
| R Square | 0.163062318 |
| Adjusted R Square | 0.103281055 |
| Standard Error | 27.58325823 |
| Observations | 16 |
| | |
| ANOVA | |
| | *df* |
| Regression | 1 |
| Residual | 14 |
| Total | 15 |
| | |
| | *Coefficients* |
| Intercept | 95.25 |
| X Variable 1 | 2.470588235 |

# Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  1. Compute $\hat{y}_t$ for the in-sample observations used for fitting the regression. In this example, $t$ = 1, 2, …, 16.

$$\hat{y}_1 = 95.2500 + 2.4706(1) = 97.7206$$

$$\hat{y}_2 = 95.2500 + 2.4706(2) = 100.1912$$

$$\ldots\ldots$$

$$\hat{y}_{16} = 95.2500 + 2.4706(16) = 134.7794$$

# Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  2. Detrend the data by computing $S_t = y_t / \hat{y}_t$ for each time period that is used in finding the least squares regression equation. In this example, $t$ = 1, 2, …, 16.

$$S_1 = y_1 / \hat{y}_1 = 72 / 97.7206 = 0.7368$$

$$S_2 = y_2 / \hat{y}_2 = 116 / 100.1912 = 1.1578$$

……

$$S_{16} = y_{16} / \hat{y}_{16} = 120 / 134.7794 = 0.8903$$

# Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  3. Compute the average seasonal values for each of the *L* seasons. The *L* averages are found by computing the average of the detrended values for the corresponding season. For example, for quarter 1,

$$\bar{S}_{[1]} = \frac{S_1 + S_5 + S_9 + S_{13}}{4}$$

$$= \frac{0.7368 + 0.7156 + 0.6894 + 0.6831}{4} = 0.7062$$

# Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  4. Multiply the average seasonal values by the normalizing constant

$$CF = \frac{L}{\sum_{i=1}^{L} \bar{S}_{[i]}}$$

such that the average of the seasonal factors is 1. The initial seasonal factors are

$$sn_{i-L} = \bar{S}_{[i]}(CF) \qquad (i = 1, 2, \ldots, L)$$

# Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  4. Multiply the average seasonal values by the normalizing constant such that the average of the seasonal factors is 1.
     - Example

       $CF$ = 4/3.9999 = 1.0000

$$sn_{-3} = sn_{1-4} = \bar{S}_{[1]}(CF) = 0.7062(1) = 0.7062$$

$$sn_{-2} = sn_{2-4} = \bar{S}_{[2]}(CF) = 1.1114(1) = 1.1114$$

$$sn_{-1} = sn_{3-4} = \bar{S}_{[3]}(CF) = 1.2937(1) = 1.2937$$

$$sn_{0} = sn_{4-4} = \bar{S}_{[1]}(CF) = 0.8886(1) = 0.8886$$

# Procedures of Multiplicative Holt-Winters Method

- **Step 3**: Calculate a point forecast of $y_1$ from time 0 using the initial values

$$\hat{y}_{T+p}(T) = (\ell_T + pb_T)sn_{T+p-L} \qquad (T = 0,\ p = 1)$$

$$\begin{aligned}\hat{y}_1(0) &= (\ell_0 + b_0)sn_{1-4} = (\ell_0 + b_0)sn_{-3} \\ &= (95.2500 + 2.4706)(0.7062) \\ &= 69.0103\end{aligned}$$

# Procedures of Multiplicative Holt-Winters Method

- **Step 4**: Update the estimates $\ell_T$, $b_T$, and $sn_T$ by using some predetermined values of smoothing constants.
- Example: let $\alpha = 0.2$, $\gamma = 0.1$, and $\delta = 0.1$

$$\ell_1 = \alpha(y_1 / sn_{1-4}) + (1-\alpha)(\ell_0 + b_0)$$
$$= 0.2(72/0.7062) + 0.8(95.2500 + 2.4706) = 98.5673$$

$$b_1 = \gamma(\ell_1 - \ell_0) + (1-\gamma)b_0$$
$$= 0.1(98.5673 - 95.2500) + 0.9(2.4706) = 2.5553$$

$$sn_1 = \delta(y_1 / \ell_1) + (1-\delta)sn_{1-4}$$
$$= 0.1(72/98.5673) + 0.9(0.7062) = 0.7086$$

$$\hat{y}_2(1) = (\ell_1 + b_1)sn_{2-4}$$
$$= (98.5673 + 2.5553)(1.1114) = 112.3876$$

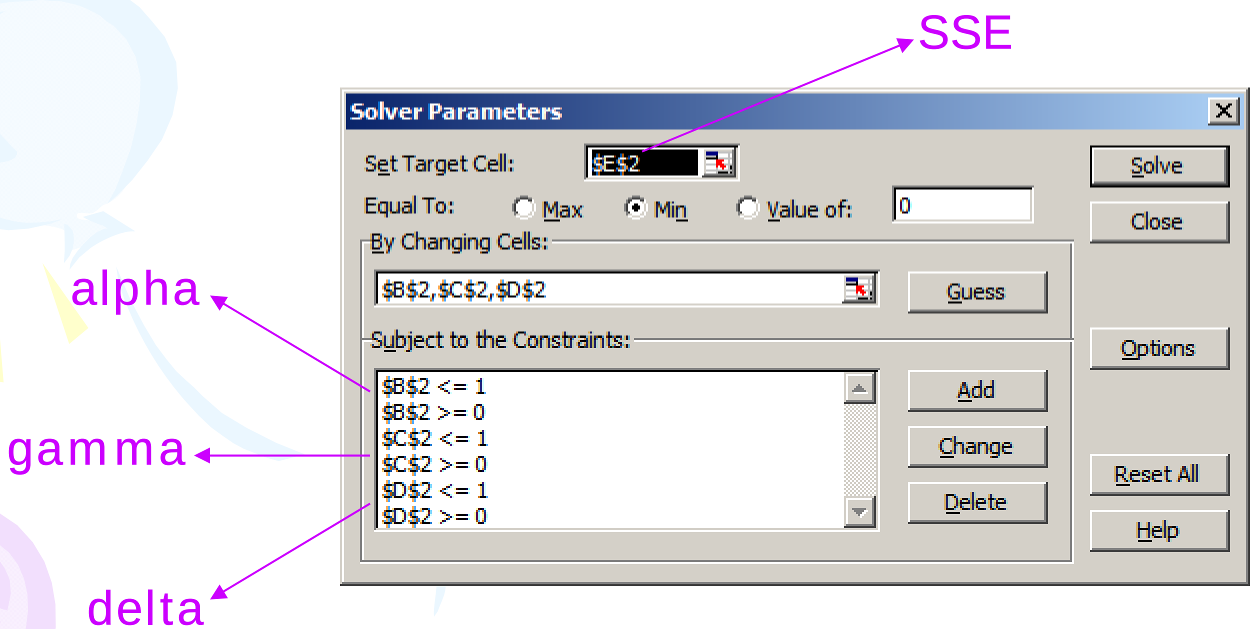
$$
\begin{aligned}
\ell_2 &= \alpha\left(y_2/sn_{2-4}\right)+(1-\alpha)(\ell_1+b_1) \\
&= 0.2(116/1.1114)+0.8(98.5673+2.5553) \\
&= 101.7727 \\
b_2 &= \gamma(\ell_2-\ell_1)+(1-\gamma)b_1 \\
&= 0.1(101.7727-98.5673)+0.9(2.5553) \\
&= 2.62031 \\
sn_2 &= \delta(y_2/\ell_2)+(1-\delta)sn_{2-4} \\
&= 0.1(116/101.7727)+0.9(1.1114) \\
&= 1.114239 \\
\hat{y}_3(2) &= (\ell_2+b_2)sn_{3-4} \\
&= (101.7727+2.62031)(1.2937) \\
&= 135.053
\end{aligned}
$$

$$\begin{aligned}
\ell_4 &= \alpha\left(y_4 / sn_{4-4}\right) + (1-\alpha)(\ell_3 + b_3) \\
&= 0.2(96/0.8886) + 0.8(104.5393 + 2.6349) \\
&= 107.3464 \\
b_4 &= \gamma(\ell_4 - \ell_3) + (1-\gamma)b_3 \\
&= 0.1(107.3464 - 104.5393) + 0.9(2.6349) \\
&= 2.65212 \\
sn_4 &= \delta(y_4/\ell_4) + (1-\delta)sn_{4-4} \\
&= 0.1(96/107.3464) + 0.9(0.8886) \\
&= 0.889170 \\
\hat{y}_5(4) &= (\ell_4 + b_4)sn_{5-4} \\
&= (107.3464 + 2.65212)(0.7086) \\
&= 77.945
\end{aligned}$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | n | alpha | gamma | delta | SSE | MSE | s | |
| 2 | 32 | 0.2 | 0.1 | 0.1 | 177.3223 | 6.1146 | 2.4728 | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | Forecast | | Squared |
| 6 | | | | Growth | Seasonal | Made Last | Forecast | Forecast |
| 7 | Time | y | Level | Rate | Factor | Period | Error | Error |
| 8 | -3 | | | | 0.7062 | | | |
| 9 | -2 | | | | 1.1114 | | | |
| 10 | -1 | | | | 1.2937 | | | |
| 11 | 0 | | 95.25 | 2.4706 | 0.8886 | | | |
| 12 | 1 | 72 | 98.56729 | 2.5553 | 0.7086 | 69.0103 | 2.9897 | 8.9384 |
| 13 | 2 | 116 | 101.7726 | 2.6203 | 1.1142 | 112.3876 | 3.6124 | 13.0494 |
| 14 | 3 | 136 | 104.5393 | 2.6349 | 1.2944 | 135.0531 | 0.9469 | 0.8967 |
| 15 | 4 | 96 | 107.3464 | 2.6521 | 0.8892 | 95.2350 | 0.7650 | 0.5853 |
| 16 | 5 | 77 | 109.731 | 2.6254 | 0.7079 | 77.9478 | -0.9478 | 0.8984 |
| 17 | 6 | 123 | 111.9629 | 2.5860 | 1.1127 | 125.1919 | -2.1919 | 4.8043 |
| 18 | 7 | 146 | 114.1974 | 2.5509 | 1.2928 | 148.2750 | -2.2750 | 5.1755 |
| 19 | 8 | 101 | 116.1165 | 2.4877 | 0.8872 | 103.8091 | -2.8091 | 7.8911 |
| 20 | 9 | 81 | 117.7668 | 2.4040 | 0.7059 | 83.9641 | -2.9641 | 8.7858 |
| 21 | 10 | 131 | 119.6835 | 2.3552 | 1.1109 | 133.7108 | -2.7108 | 7.3482 |
| 22 | 11 | 158 | 122.0734 | 2.3587 | 1.2930 | 157.7754 | 0.2246 | 0.0504 |
| 23 | 12 | 109 | 124.1164 | 2.3271 | 0.8863 | 110.4005 | -1.4005 | 1.9615 |
| 24 | 13 | 87 | 125.8035 | 2.2631 | 0.7045 | 89.2593 | -2.2593 | 5.1044 |
| 25 | 14 | 140 | 127.6589 | 2.2224 | 1.1094 | 142.2642 | -2.2642 | 5.1268 |
| 26 | 15 | 167 | 129.7369 | 2.2079 | 1.2924 | 167.9337 | -0.9337 | 0.8718 |
| | | ...... | | | | ...... | | |
| 38 | 27 | 200 | 156.1396 | 2.1752 | 1.2903 | 202.0396 | -2.0396 | 4.1601 |
| 39 | 28 | 142 | 158.5505 | 2.1988 | 0.8908 | 140.9508 | 1.0492 | 1.1008 |
| 40 | 29 | 115 | 161.2803 | 2.2519 | 0.7047 | 113.1314 | 1.8686 | 3.4918 |
| 41 | 30 | 177 | 162.8178 | 2.1804 | 1.1046 | 180.9529 | -3.9529 | 15.6252 |
| 42 | 31 | 218 | 165.7889 | 2.2595 | 1.2928 | 212.8988 | 5.1012 | 26.0220 |
| 43 | 32 | 149 | 167.8899 | 2.2437 | 0.8905 | 149.7057 | -0.7057 | 0.4981 |

# Procedures of Multiplicative Holt-Winters Method

- **Step 5**: Find the most suitable combination of $\alpha$, $\gamma$, and $\delta$ that minimizes SSE (or MSE)
- Example: Use Solver in Excel as an illustration



| 1 | n | alpha | gamma | delta | SSE | MSE | s | |
|---|---|---|---|---|---|---|---|---|
| 2 | 32 | 0.3356 | 0.0455 | 0.1342 | 168.4747 | 5.8095 | 2.4103 | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | Forecast | | Squared |
| 6 | | | | Growth | Seasonal | Made Last | Forecast | Forecast |
| 7 | Time | y | Level | Rate | Factor | Period | Error | Error |
| 8 | -3 | | | | 0.7062 | | | |
| 9 | -2 | | | | 1.1114 | | | |
| 10 | -1 | | | | 1.2937 | | | |
| 11 | 0 | | 95.25 | 2.4706 | 0.8886 | | | |
| 12 | 1 | 72 | 99.14144 | 2.5353 | 0.7089 | 69.0103 | 2.9897 | 8.9384 |
| 13 | 2 | 116 | 102.5816 | 2.5765 | 1.1140 | 113.0035 | 2.9965 | 8.9789 |
| 14 | 3 | 136 | 105.1469 | 2.5760 | 1.2937 | 136.0431 | -0.0431 | 0.0019 |
| 15 | 4 | 96 | 107.8277 | 2.5808 | 0.8888 | 95.7226 | 0.2774 | 0.0769 |
| 16 | 5 | 77 | 109.8084 | 2.5534 | 0.7079 | 78.2674 | -1.2674 | 1.6064 |
| 17 | 6 | 123 | 111.7076 | 2.5236 | 1.1123 | 125.1717 | -2.1717 | 4.7164 |
| 18 | 7 | 146 | 113.7703 | 2.5027 | 1.2923 | 147.7768 | -1.7768 | 3.1569 |
| 19 | 8 | 101 | 115.3868 | 2.4623 | 0.8870 | 103.3468 | -2.3468 | 5.5075 |
| 20 | 9 | 81 | 116.7014 | 2.4100 | 0.7060 | 83.4207 | -2.4207 | 5.8597 |
| | ...... | | | | | ...... | | |
| 38 | 27 | 200 | 155.9042 | 2.2691 | 1.2906 | 202.1107 | -2.1107 | 4.4552 |
| 39 | 28 | 142 | 158.5811 | 2.2876 | 0.8915 | 140.9173 | 1.0827 | 1.1721 |
| 40 | 29 | 115 | 161.7496 | 2.3278 | 0.7044 | 113.1540 | 1.8460 | 3.4078 |
| 41 | 30 | 177 | 162.7095 | 2.2655 | 1.1038 | 181.5085 | -4.5085 | 20.3262 |
| 42 | 31 | 218 | 166.2957 | 2.3256 | 1.2934 | 212.9210 | 5.0790 | 25.7957 |
| 43 | 32 | 149 | 168.1213 | 2.3028 | 0.8908 | 150.3283 | -1.3283 | 1.7643 |

# Multiplicative Holt-Winters Method

- $p$-step-ahead forecast made at time $T$

$$\hat{y}_{T+p}(T) = (\ell_T + pb_T)sn_{T+p-L} \qquad (p = 1, 2, 3, \ldots)$$

- Example

$$\hat{y}_{33}(32) = (\ell_{32} + b_{32})sn_{33-4} = (168.1213 + 2.3028)(0.7044) = 120.0467$$

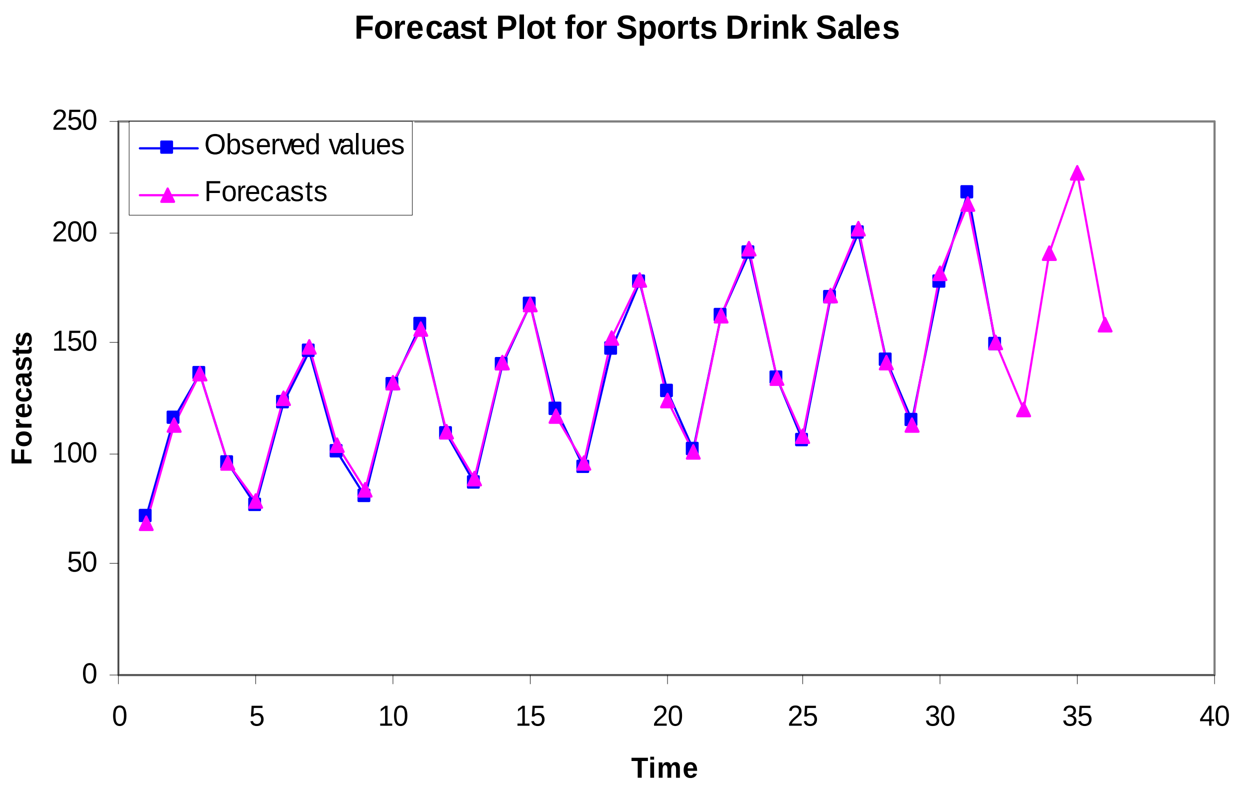
$$\hat{y}_{34}(32) = (\ell_{32} + 2b_{32})sn_{34-4} = [168.1213 + 2(2.3028)](1.1038) = 190.6560$$

$$\hat{y}_{35}(32) = (\ell_{32} + 3b_{32})sn_{35-4} = [(168.1213 + 3(2.3028)](1.2934) = 226.3834$$

$$\hat{y}_{36}(32) = (\ell_{32} + 4b_{32})sn_{36-4} = [(168.1213 + 4(2.3028)](0.8908) = 157.9678$$

# Multiplicative Holt-Winters Method

- Example



# Additive Holt-Winters Method

- It is generally considered to be best suited to forecasting a time series that can be described by the equation:

$$y_t = (\beta_0 + \beta_1 t) + SN_t + IR_t$$

  - $SN_t$: seasonal pattern
  - $IR_t$: irregular component
- This method is appropriate when a time series has a linear trend with a constant (additive) seasonal pattern such that the level ($\beta_0 + \beta_1 t$), growth rate ($\beta_1$), and the seasonal pattern ($SN_t$) may be slowly changing over time.

# Additive Holt-Winters Method

- Estimate of the level

$$\ell_T = \alpha(y_T - sn_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

- Estimate of the growth rate (or trend)

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

- Estimate of the seasonal factor

$$sn_T = \delta(y_T - \ell_T) + (1-\delta)sn_{T-L}$$

where $\alpha$, $\gamma$, and $\delta$ are smoothing constants between 0 and 1,

$L$ = number of seasons in a year ($L$ = 12 for monthly data, and $L$ = 4 for quarterly data)

# Additive Holt-Winters Method

- Point forecast made at time $T$ for $y_{T+p}$

$$\hat{y}_{T+p}(T) = \ell_T + pb_T + sn_{T+p-L} \qquad (p = 1, 2, 3, \ldots)$$

- MSE and the standard error $s$ at time $T$

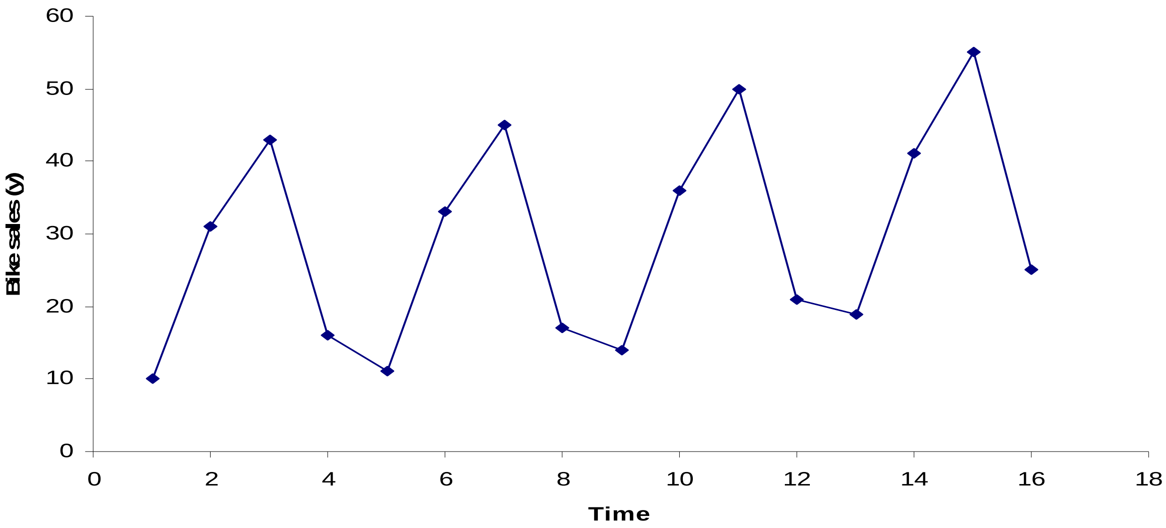
$$SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2$$

$$MSE = \frac{SSE}{T-3}, \qquad s = \sqrt{MSE}$$

# Procedures of Additive Holt-Winters Method

- Consider the Mountain Bike example,

| Quarterly sales of the TRK-50 Mountain Bike | | | | |
|---|---|---|---|---|
| | Year | | | |
| **Quarter** | **1** | **2** | **3** | **4** |
| 1 | 10 | 11 | 14 | 19 |
| 2 | 31 | 33 | 36 | 41 |
| 3 | 43 | 45 | 50 | 55 |
| 4 | 16 | 17 | 21 | 25 |



# Procedures of Additive Holt-Winters Method

- Observations:
  - Linear upward trend over the 4-year period
  - Magnitude of seasonal span is almost constant as the level of the time series increases

  ⇒ Additive Holt-Winters method can be applied to forecast future sales

# Procedures of Additive Holt-Winters Method

- **Step 1**: Obtain initial values for the level $\ell_0$, the growth rate $b_0$, and the seasonal factors $sn_{-3}$, $sn_{-2}$, $sn_{-1}$, and $sn_0$, by fitting a least squares trend line to <u>at least four or five years</u> of the historical data.
  - $y$-intercept = $\ell_0$; slope = $b_0$

# Procedures of Additive Holt-Winters Method

- Example
  - Fit a least squares trend line to all 16 observations
  - Trend line

$$\hat{y}_t = 20.85 + 0.980882\, t$$

  - $\ell_0 = 20.85;\ b_0 = 0.9809$

| SUMMARY OUTPUT | |
|---|---|
| | |
| *Regression Statistics* | |
| Multiple R | 0.320508842 |
| R Square | 0.102725918 |
| Adjusted R Square | 0.038634912 |
| Standard Error | 14.28614022 |
| Observations | 16 |
| | |
| ANOVA | |
| | *df* |
| Regression | 1 |
| Residual | 14 |
| Total | 15 |
| | |
| | *Coefficients* |
| Intercept | 20.85 |
| Time | 0.980882353 |

# Procedures of Additive Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  1. Compute $\hat{y}_t$ for each time period that is used in finding the least squares regression equation. In this example, $t = 1, 2, \ldots, 16$.

$$\hat{y}_1 = 20.85 + 0.980882(1) = 21.8309$$

$$\hat{y}_2 = 20.85 + 0.980882(2) = 22.8118$$

$$\ldots\ldots$$

$$\hat{y}_{16} = 20.85 + 0.980882(16) = 36.5441$$

# Procedures of Additive Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  2. Detrend the data by computing $S_t = y_t - \hat{y}_t$ for each observation used in the least squares fit. In this example, $t = 1, 2, \ldots, 16$.

$$S_1 = y_1 - \hat{y}_1 = 10 - 21.8309 = -11.8309$$
$$S_2 = y_2 - \hat{y}_2 = 31 - 22.8112 = 8.1882$$
$$\ldots\ldots$$
$$S_{16} = y_{16} - \hat{y}_{16} = 25 - 36.5441 = -11.5441$$

# Procedures of Additive Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  3. Compute the average seasonal values for each of the *L* seasons. The *L* averages are found by computing the average of the detrended values for the corresponding season. For example, for quarter 1,

$$\bar{S}_{[1]} = \frac{S_1 + S_5 + S_9 + S_{13}}{4}$$

$$= \frac{(-11.8309) + (-14.7544) + (-15.6779) + (-14.6015)}{4} = -14.2162$$

# Procedures of Additive Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  4. Compute the average of the $L$ seasonal factors. The average should be 0.

# Procedures of Additive Holt-Winters Method

- **Step 3**: Calculate a point forecast of $y_1$ from time 0 using the initial values

$$\hat{y}_{T+p}(T) = \ell_T + pb_T + sn_{T+p-L} \qquad (T = 0, p = 1)$$

$$\hat{y}_1(0) = \ell_0 + b_0 + sn_{1-4} = \ell_0 + b_0 + sn_{-3}$$

$$= 20.85 + 0.9809 + (-14.2162) = 7.6147$$

# Procedures of Additive Holt-Winters Method

- **Step 4**: Update the estimates $\ell_T$, $b_T$, and $sn_T$ by using some predetermined values of smoothing constants.
- Example: let $\alpha = 0.2$, $\gamma = 0.1$, and $\delta = 0.1$

$$\ell_1 = \alpha(y_1 - sn_{1-4}) + (1-\alpha)(\ell_0 + b_0)$$
$$= 0.2(10 - (-14.2162)) + 0.8(20.85 + 0.9808) = 22.3079$$

$$b_1 = \gamma(\ell_1 - \ell_0) + (1-\gamma)b_0$$
$$= 0.1(22.3079 - 20.85) + 0.9(0.9809) = 1.0286$$

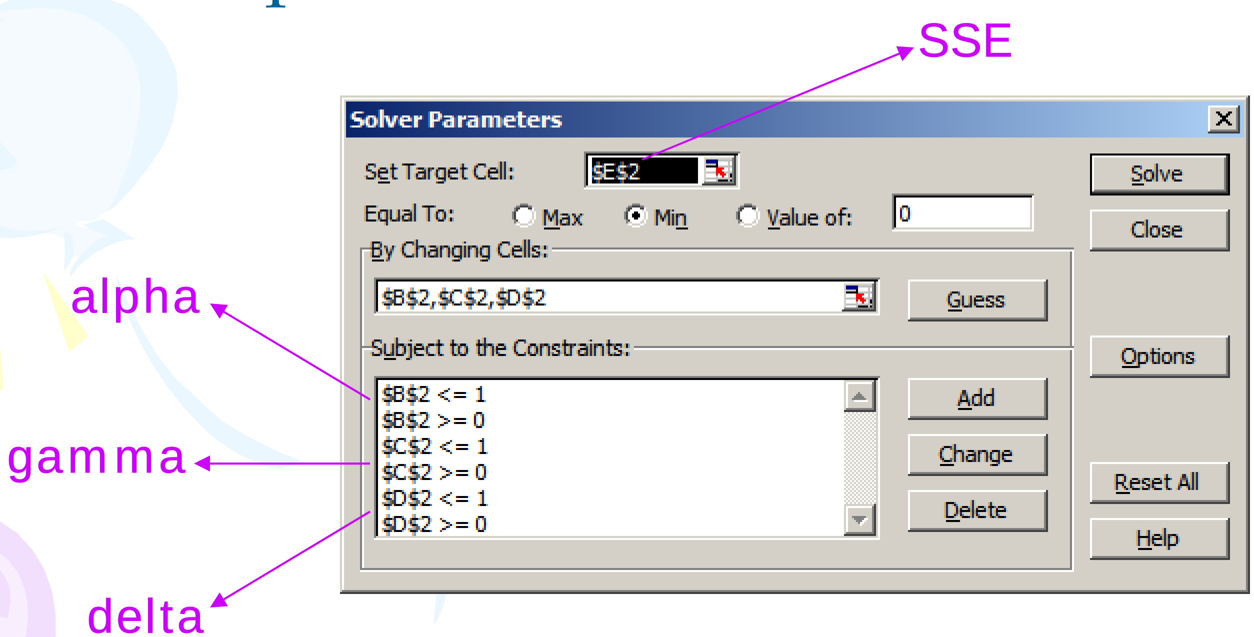
$$sn_1 = \delta(y_1 - \ell_1) + (1-\delta)sn_{1-4}$$
$$= 0.1(10 - 22.3079) + 0.9(-14.2162) = -14.0254$$

$$\hat{y}_2(1) = \ell_1 + b_1 + sn_{2-4} = \ell_1 + b_1 + sn_{-2}$$
$$= 22.3079 + 1.0286 + 6.5529 = 29.8895$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | n | alpha | gamma | delta | SSE | MSE | s | |
| 2 | 16 | 0.2000 | 0.1000 | 0.1000 | 25.2166 | 1.9397 | 1.3927 | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | Forecast | | Squared |
| 6 | | | | Growth | Seasonal | Made Last | Forecast | Forecast |
| 7 | Time | y | Level | Rate | Factor | Period | Error | Error |
| 8 | -3 | | | | -14.2162 | | | |
| 9 | -2 | | | | 6.5529 | | | |
| 10 | -1 | | | | 18.5721 | | | |
| 11 | 0 | | 20.85 | 0.9809 | -10.9088 | | | |
| 12 | 1 | 10 | 22.30794 | 1.0286 | -14.0254 | 7.6147 | 2.3853 | 5.6896 |
| 13 | 2 | 31 | 23.55864 | 1.0508 | 6.6418 | 29.8895 | 1.1105 | 1.2333 |
| 14 | 3 | 43 | 24.57314 | 1.0472 | 18.5575 | 43.1815 | -0.1815 | 0.0329 |
| 15 | 4 | 16 | 25.87801 | 1.0729 | -10.8057 | 14.7115 | 1.2885 | 1.6603 |
| 16 | 5 | 11 | 26.56583 | 1.0344 | -14.1794 | 12.9256 | -1.9256 | 3.7079 |
| 17 | 6 | 33 | 27.35185 | 1.0096 | 6.5424 | 34.2420 | -1.2420 | 1.5427 |
| 18 | 7 | 45 | 27.97764 | 0.9712 | 18.4040 | 46.9190 | -1.9190 | 3.6825 |
| 19 | 8 | 17 | 28.72023 | 0.9483 | -10.8972 | 18.1431 | -1.1431 | 1.3067 |
| 20 | 9 | 14 | 29.37074 | 0.9186 | -14.2985 | 15.4892 | -1.4892 | 2.2176 |
| 21 | 10 | 36 | 30.12295 | 0.9019 | 6.4759 | 36.8317 | -0.8317 | 0.6918 |
| 22 | 11 | 50 | 31.1391 | 0.9133 | 18.4497 | 49.4289 | 0.5711 | 0.3262 |
| 23 | 12 | 21 | 32.0214 | 0.9102 | -10.9096 | 21.1553 | -0.1553 | 0.0241 |
| 24 | 13 | 19 | 33.00502 | 0.9176 | -14.2692 | 18.6331 | 0.3669 | 0.1346 |
| 25 | 14 | 41 | 34.04291 | 0.9296 | 6.5240 | 40.3985 | 0.6015 | 0.3618 |
| 26 | 15 | 55 | 35.28807 | 0.9612 | 18.5759 | 53.4222 | 1.5778 | 2.4894 |
| 27 | 16 | 25 | 36.18131 | 0.9544 | -10.9368 | 25.3396 | -0.3396 | 0.1153 |

# Procedures of Additive Holt-Winters Method

- **Step 5**: Find the most suitable combination of α, γ, and δ that minimizes SSE (or MSE)
- Example: Use Solver in Excel as an illustration



| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | n | alpha | gamma | delta | SSE | MSE | s | |
| 2 | 16 | 0.5606 | 0.0000 | 0.0000 | 18.7975 | 1.4460 | 1.2025 | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | Forecast | | Squared |
| 6 | | | | Growth | Seasonal | Made Last | Forecast | Forecast |
| 7 | Time | y | Level | Rate | Factor | Period | Error | Error |
| 8 | -3 | | | | -14.2162 | | | |
| 9 | -2 | | | | 6.5529 | | | |
| 10 | -1 | | | | 18.5721 | | | |
| 11 | 0 | | 20.85 | 0.9809 | -10.9088 | | | |
| 12 | 1 | 10 | 23.16818 | 0.9809 | -14.2162 | 7.6147 | 2.3853 | 5.6896 |
| 13 | 2 | 31 | 24.31613 | 0.9809 | 6.5529 | 30.7020 | 0.2980 | 0.0888 |
| 14 | 3 | 43 | 24.80977 | 0.9809 | 18.5721 | 43.8691 | -0.8691 | 0.7553 |
| 15 | 4 | 16 | 26.41755 | 0.9809 | -10.9088 | 14.8818 | 1.1182 | 1.2503 |
| 16 | 5 | 11 | 26.17496 | 0.9809 | -14.2162 | 13.1823 | -2.1823 | 4.7622 |
| 17 | 6 | 33 | 26.75847 | 0.9809 | 6.5529 | 33.7088 | -0.7088 | 0.5024 |
| 18 | 7 | 45 | 27.00412 | 0.9809 | 18.5721 | 46.3114 | -1.3114 | 1.7198 |
| 19 | 8 | 17 | 27.94229 | 0.9809 | -10.9088 | 17.0762 | -0.0762 | 0.0058 |
| 20 | 9 | 14 | 28.5268 | 0.9809 | -14.2162 | 14.7070 | -0.7070 | 0.4998 |
| 21 | 10 | 36 | 29.47369 | 0.9809 | 6.5529 | 36.0606 | -0.0606 | 0.0037 |
| 22 | 11 | 50 | 31.00029 | 0.9809 | 18.5721 | 49.0266 | 0.9734 | 0.9474 |
| 23 | 12 | 21 | 31.94061 | 0.9809 | -10.9088 | 21.0723 | -0.0723 | 0.0052 |
| 24 | 13 | 19 | 33.0867 | 0.9809 | -14.2162 | 18.7053 | 0.2947 | 0.0868 |
| 25 | 14 | 41 | 34.28034 | 0.9809 | 6.5529 | 40.6205 | 0.3795 | 0.1440 |
| 26 | 15 | 55 | 35.91533 | 0.9809 | 18.5721 | 53.8333 | 1.1667 | 1.3612 |
| 27 | 16 | 25 | 36.34264 | 0.9809 | -10.9088 | 25.9874 | -0.9874 | 0.9749 |

# Additive Holt-Winters Method

- $p$-step-ahead forecast made at time $T$

$$\hat{y}_{T+p}(T) = \ell_T + pb_T + sn_{T+p-L} \qquad (p = 1, 2, 3, \ldots)$$

- Example

$$\hat{y}_{17}(16) = \ell_{16} + b_{16} + sn_{17-4} = 36.3426 + 0.9809 - 14.2162 = 23.1073$$

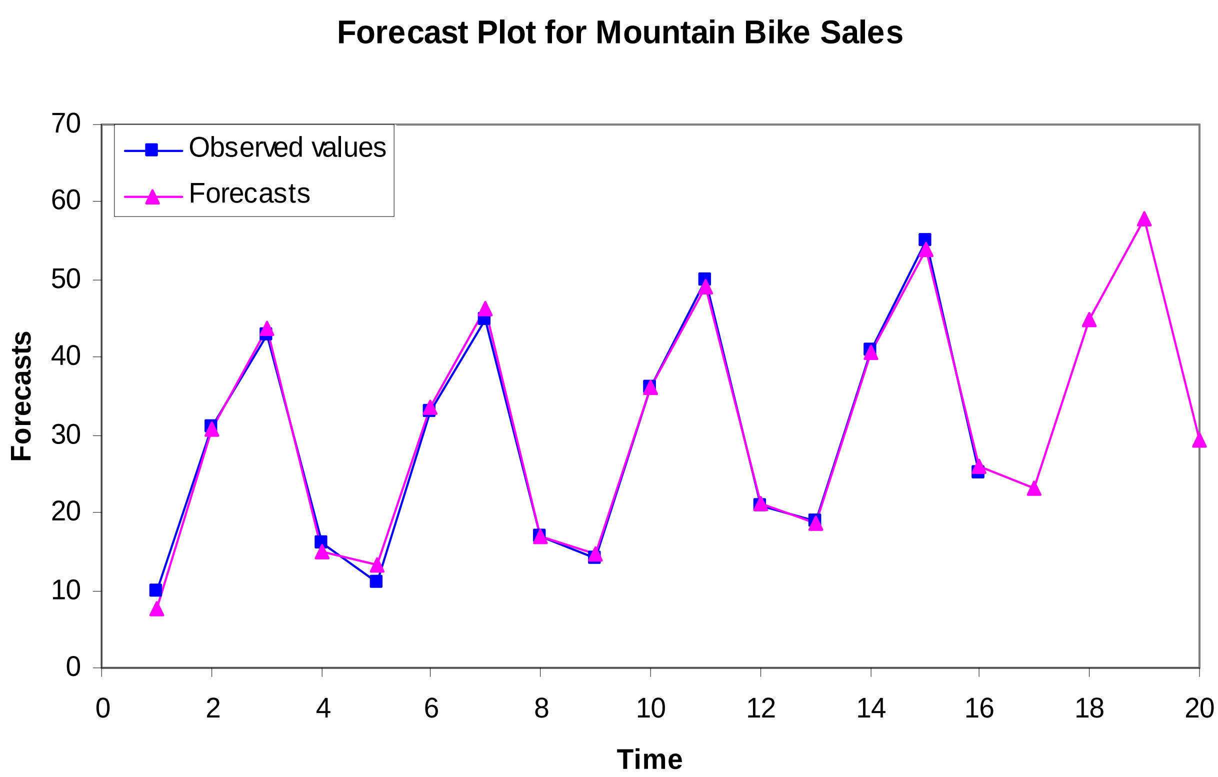
$$\hat{y}_{18}(16) = \ell_{16} + 2b_{16} + sn_{18-4} = 36.3426 + 2(0.9809) + 6.5529 = 44.8573$$

$$\hat{y}_{19}(16) = \ell_{16} + 3b_{16} + sn_{19-4} = 36.3426 + 3(0.9809) + 18.5721 = 57.8573$$

$$\hat{y}_{20}(16) = \ell_{16} + 4b_{16} + sn_{20-4} = 36.3426 + 4(0.9809) - 10.9088 = 29.3573$$

# Additive Holt-Winters Method

- Example



# Chapter Summary

- Simple Exponential Smoothing
  - No trend, no seasonal pattern
- Holt's Trend Corrected Exponential Smoothing
  - Trend, no seasonal pattern
- Holt-Winters Methods
  - Both trend and seasonal pattern
    - Multiplicative Holt-Winters method
    - Additive Holt-Winters Method