CHAPTER 06

Set Theory and Basic Probability

Chapter Contents

You should read this chapter if you need to learn about:

Set

"A well-defined collection of distinct objects is called set"

The objects in a set may be the numbers, people, letters, books, rivers etc. Sets are usually denoted by capital letters such as A, B, C etc.

EXAMPLE

- Set of vowels in English alphabets
- Set of books in a library
- Set of students in a college etc.

Finite and Infinite Sets

"A set consisting of finite number of elements is called finite set"

EXAMPLE

- Set of vowels
- Set of months of a year
- Set of days in a week etc.

On the other hand "***a set consisting of infinite number of elements is called infinite sets***"

EXAMPLE

- Set of points on a line
- Set of stars on the sky
- Set of odd numbers
- Set of even numbers etc.

Null Set or Empty Set

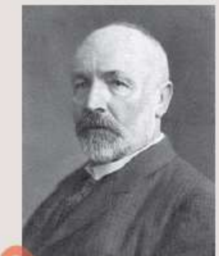
"A set that contains no elements is called an empty set or null set"

A null set is denoted by the symbol ϕ (phi) or by $\{ \}$.

EXAMPLE

- Number of male students in a girl's college
- Set of first year statistics students older than 200 years etc.

Historical Note



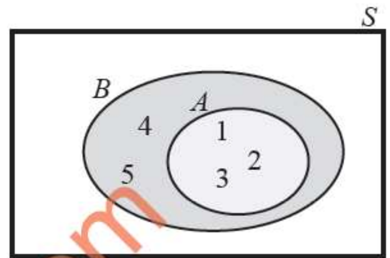
Georg Cantor

Richard Dedekind

The modern study of set theory was initiated by two German Mathematicians Georg Cantor and Richard Dedekind in the 1870s.

Sub-Set

"If each element of a set A is also the elements of set B then A is said to be the subset of B written as: $A \subset B$"



EXAMPLE

If $A = \{1,2,3\}$
And $B = \{1,2,3,4,5\}$
Then $A \subset B$

Proper Sub-Set

We call "A", a proper sub-set of "B" if:

- "A" is a sub-set of "B"
- $A \neq B$

Written as $A \subseteq B$

EXAMPLE

If $A = \{1,2,3\}$
And $B = \{1,2,3,4,5\}$
Then $A \subseteq B$

> TIP
>
> Every set is a subset of itself and the null set is a subset of every set.

Improper Sub-Set

We call "A", an improper sub-set of "B" if:

- "A" is a sub-set of "B"
- $A = B$

EXAMPLE

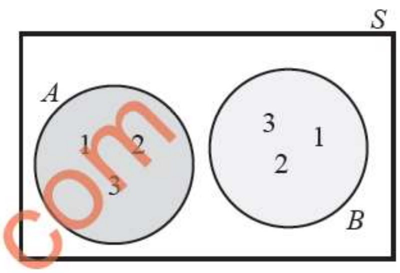
If $A = \{1,2,3\}$
And $B = \{1,2,3\}$
Then A is an improper sub set of B.

Equal Sets

"Two sets "A" and "B" are said to be equal, if they contain exactly the same elements"

In other words

If $A \subset B$ and $B \subset A$ then A = B



If $A = \{1,2,3\}$
And $B = \{3,1,2\}$
Then $A = B$

Power Set

"The set of all possible sub-sets of a set is called power set and is denoted by P (A)"

The number of subsets in power set may be counted by 2^n.

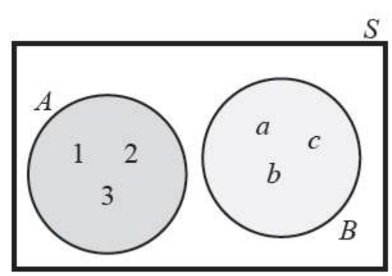
If $A = \{1,2,3\}$ then power set contains $2^3 = 8$ subsets i.e.

$$P(A) = \{\{\ \},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$$

Disjoint Sets

"If there is no element common in between the two "A" and "B", then they are called disjoint sets"

Disjoint sets are also called **mutually exclusive** sets.

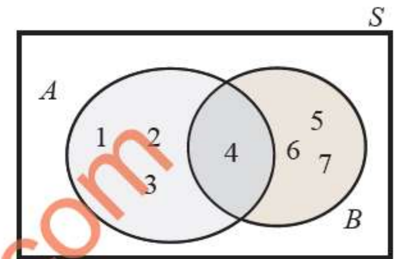


If $A = \{1,2,3\}$ and $B = \{a,b,c\}$

Then A and B are disjoint sets.

Overlapping sets

"If at least one element is common in between two sets such that they are not subsets of each other then they are called overlapping sets"



If $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$

Then A and B are overlapping sets.

Universal Set

"The set which is consisted of all the elements specified for some discussion is called universal set". It is denoted by U or S.

Product Set OR Cartesian product of Sets

The Cartesian product of sets "A" and "B" denoted by $A \times B$ (read as "A" cross "B") is the set of elements that contains all the ordered pairs (x, y) where $x \in A$ and $y \in B$

If $A = \{H, T\}$ and $B = \{1, 2\}$

$\Rightarrow A \times B = \{(H,1),(H,2),(T,1),(T,2)\}$

Historical Note

The Cartesian product is named after Rene Descartes, a French Mathematician

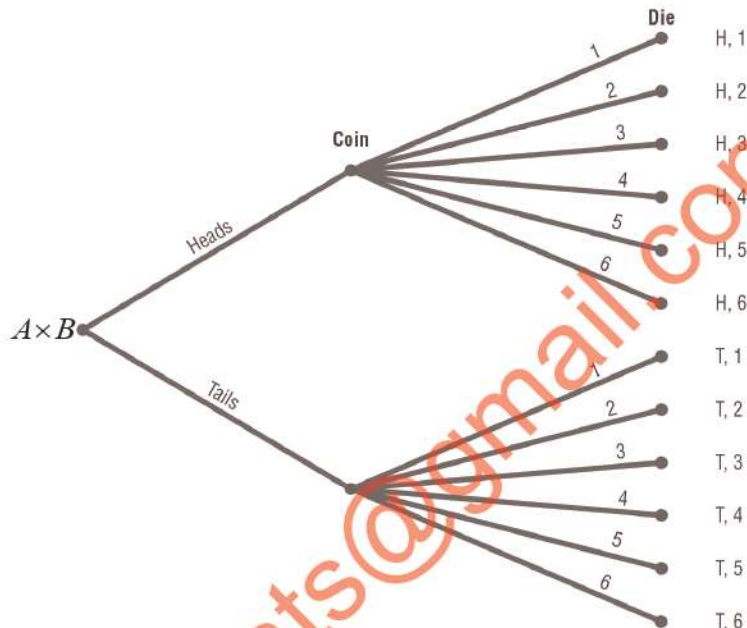
Tree diagram

"A systematic method of finding Cartesian product through a diagram is called tree diagram"

EXAMPLE

If for a coin, $A=\{H,T\}$ and for a die, $B=\{1,2,3,4,5,6\}$

$$\Rightarrow A\times B=\begin{Bmatrix}(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)\\(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\end{Bmatrix}$$



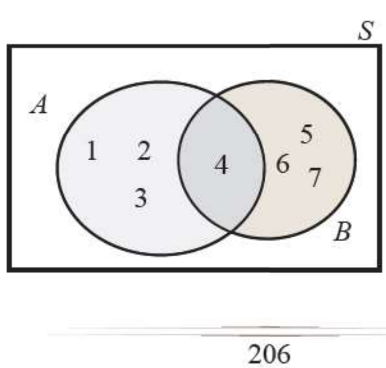
Venn diagram

"The simple and effective way of representing the relationships between sets diagrammatically is called Venn diagram".

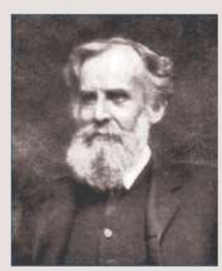
In Venn diagram the universal set U (or S) is represented by a rectangle and the sub sets are represented by circles inside the rectangles e.g.

EXAMPLE

If $A=\{1,2,3,4\}$ and $B=\{4,5,6,7\}$ then, they can be represented by the Venn diagram as:



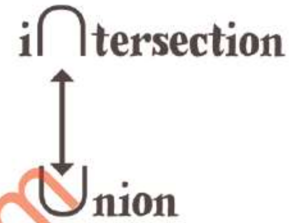
Historical Note



In 1880, British Philosopher John Venn introduced the Venn Diagrams.

Operations on Sets

Like algebraic operation such as addition, subtraction, multiplication and division in mathematics, we have basic operations on sets i.e.:



- Union of two sets
- Intersection of two set
- Difference of two sets
- Complement of a set

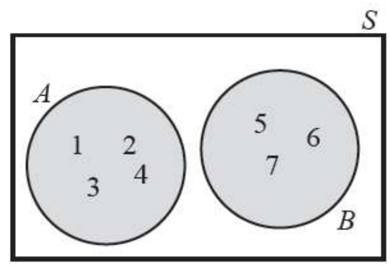
Union of sets

The union of two sets "A" and "B" is the set of all elements that belongs to "A" or to "B" or to both "A" and "B". The union of two sets "A" and "B" is denoted by $A \cup B$.

EXAMPLE

If $A = \{1,2,3,4\}$ and $B = \{5,6,7\}$

Then $A \cup B = \{1,2,3,4,5,6,7\}$

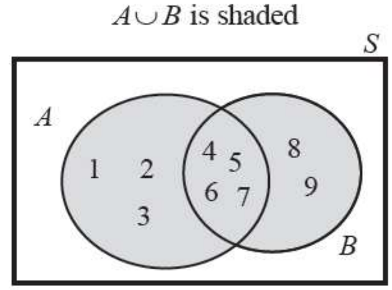


$A \cup B$ is shaded

EXAMPLE

If $A = \{1,2,3,4,5,6,7\}$ and $B = \{4,5,6,7,8,9\}$

Then $A \cup B = \{1,2,3,4,5,6,7,8,9\}$



$A \cup B$ is shaded

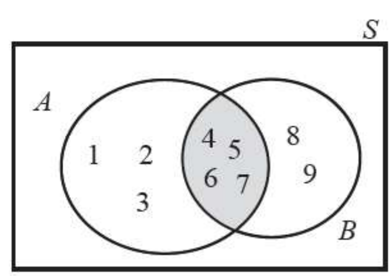
Intersection of sets

The intersection of two sets "A" and "B" is the set of elements that belongs to both "A" and "B". The intersection of two sets "A" and "B" is denoted by $A \cap B$

EXAMPLE

If $A = \{1,2,3,4,5,6,7\}$
And $B = \{4,5,6,7,8,9\}$

Then $A \cap B = \{4,5,6,7\}$



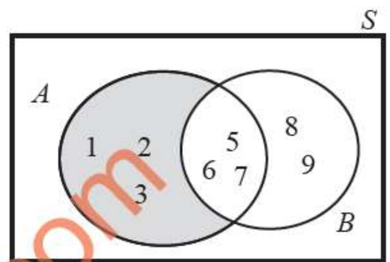
$A \cap B$ is shaded

Difference of sets

The difference of sets "A" and "B" is the set of elements that belongs to "A" but do not belongs to "B". The difference of two sets "A" and "B" is denoted by $A-B$ or $A-(A\cap B)$ or $A\cap\overline{B}$

If $A=\{1,2,3,4,5,6,7\}$
And $B=\{4,5,6,7,8,9\}$

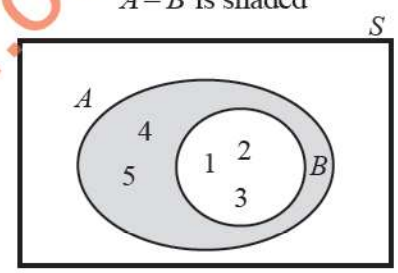
Then $A-B=\{1,2,3\}$



$A-B$ is shaded

If $A=\{1,2,3,4,5\}$
And $B=\{1,2,3\}$

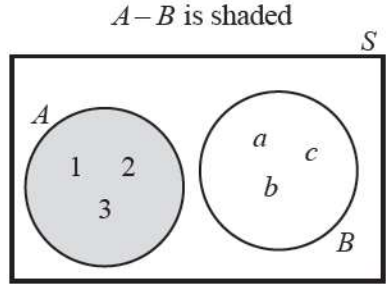
Then $A-B=\{4,5\}$



$A-B$ is shaded

If $A=\{1,2,3\}$
And $B=\{a,b,c\}$

Then $A-B=\{1,2,3\}$



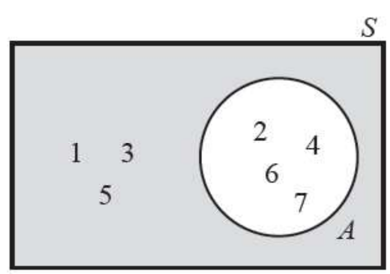
$A-B$ is shaded

Complement of a Set

The complement of a set "A" is the set of elements that belongs to "S" but do not belongs to "A"
The complement of set "A" is denoted by $\overline{A}$ or A^c

If $S=\{1,2,3,4,5,6\}$
And $A=\{2,4,6,7\}$

Then $\overline{A}=S-A=\{1,3,5\}$



$\overline{A}$ is shaded

Experiment

"An experiment is a process in which we obtain results"

Random Experiment

In our daily life, we perform many activities which have a **fixed result** no matter any number of times they are repeated. For example given any triangle, without knowing the three angles, we can definitely say that the sum of measure of angles is 180°. We also perform many experimental activities, where the **result may not be same**, when they are repeated under identical conditions. For example, when a coin is tossed it may turn up a head or a tail, but we are not sure which one of these results will actually be obtained. Such experiments are called random experiments.

A random experiment satisfies the following three **properties**:

- It can be repeated any number of times.
- It has more than one possible outcome.
- It is not possible to predict the outcome in advance.

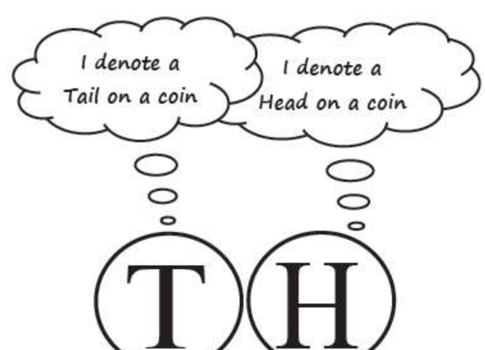
Hence we may define the random experiment as *"An experiment that generates uncertain results under similar conditions, is called random experiment"*

EXAMPLE

- Tossing of a coin
- Rolling of a dice
- Drawing a card from a pack of 52 playing cards etc.

Trail

"A single performance of an experiment is called a trial"



Outcome

"A possible result of a random experiment is called outcome"

If we toss a coin then "H" or "T" may be the outcomes.

Sample space

"A set consisting of all possible outcomes of a random experiment is called a sample space". It is denoted by "S" and each element of a sample space is called a sample point.

> TIP
>
> *Number of sample points in a sample space for coin tossing experiment can be determined by 2^n, where "n" is the number of coin. And for die rolling experiment 6^n, where "n" is the number of dice.*

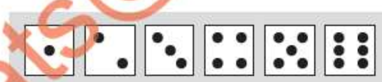
- If a coin is tossed
  Then $S = \{H, T\}$

Head *Tail*

- If two coins are tossed
  Then $S = \{HH, HT, TH, TT\}$

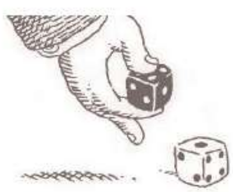
- If three coins are tossed
  Then $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

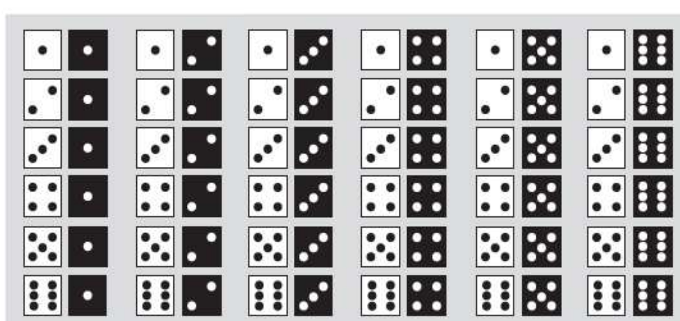
- If a dice is rolled
  Then $S = \{1, 2, 3, 4, 5, 6\}$



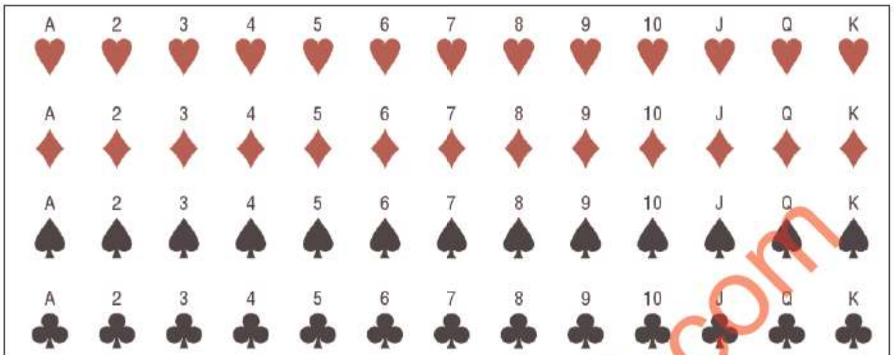
- If two dice are rolled, then

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$



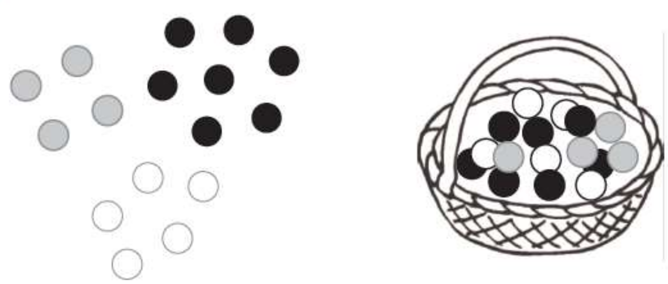


- If we draw a card from a deck of 52 playing cards then the sample points in the sample space are:



Hearts *Diamonds* *Spades* *Clubs*

- If we draw a ball from a basket having 3 different color balls then the sample points in the sample space may be as follows:



Event

"Any sub set from a sample space is called an event"

Events are usually denoted by A, B, C etc.

If we toss two coins
Then $S=\{HH,HT,TH,TT\}$
Now if $A=\{HH\}$, then "A" is called an event.

- *Each element of a sample space "S" is called sample point.*
- *Total number of sample points in sample space is denoted by n(S)*
- *Favorable cases of an event "A" is denoted by n(A)*

Simple Event

"If an event contains only one sample point from the sample space then it is called simple event"

If we toss two coins then $S=\{HH,HT,TH,TT\}$
If $A=\{HH\}$, then "A" is called a simple event.

Compound Event

"If an event contains two or more sample points from the sample space then this is called a compound event"

If we toss two coins then $S=\{HH,HT,TH,TT\}$
If $A=\{HT,TH\}$, then "A" is called a compound event.

The Certain or Sure Event

"An event consisting of the sample space itself is called the sure event"

Impossible event

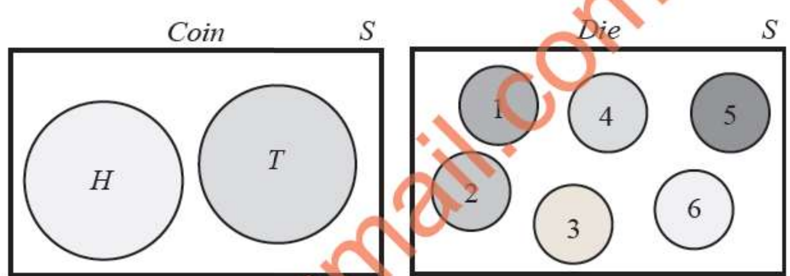
"An event consisting of the null set is called the impossible event"

Mutually Exclusive (Disjoint) Events

"Events in a same sample space are said to be mutually exclusive if they cannot occur together"

For two mutually exclusive events "A" and "B" $A \cap B = \phi$

If we toss a coin then "H" and "T" are mutually exclusive because if "H" occurs then "T" cannot take place; similarly 1, 2, 3, 4, 5 and 6 are mutually exclusive when a dice is rolled. In other words they **exclude** each other.

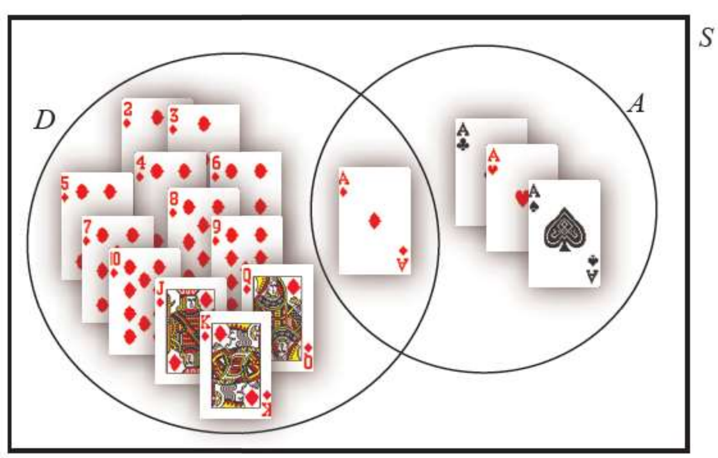


Not Mutually Exclusive (Overlapping) Events

"Events in a same sample space are said to be not mutually exclusive if they can occur together"

For two not mutually exclusive events "A" and "B" $A \cap B \neq \phi$

If a card is drawn at random from a pack of 52 playing cards then it may be at the same time an "Ace" and a "Diamond"; therefore "Ace" and "Diamond" are not mutually exclusive.



Equally likely Events

"Events are said to be equally likely if they have the same chances of occurrence"

If we toss a fair coin then "H" and "T" are equally likely; because they have the same chances of occurences.

Exhuastive Events

"Two or more events defined in the same sample sapce are said to be exhaustive if their union is equal to the sample space"

If $S=\{1,2,3,4,5,6\}$
Let $A=\{1,3,5\}$ and $B=\{2,4,6\}$
Then $A\cup B=\{1,2,3,4,5,6\}=S$
Therefore "A" and "B" are exhaustive events.

REMEMBER

An event "A" and its compliment "$\overline{A}$" are always exhaustive i.e. $A\cup\overline{A}=S$

Counting Techniques

Sometimes it is very difficult **to list all the sample points** of a sample space; therefore we use some mathematical techniques for finding the number of sample points of the sample space. These techniques are called counting techniques i.e.

- Factorial
- Rule of Multiplication
- Permutation
- Combination

Factorial

"The product of first "n" natural numbers is called Factorial and is denoted by n!"

$2!=2\times 1=2$
$5!=5\times 4\times 3\times 2\times 1=120$

In general $n!=n(n-1)(n-2)(n-3).....3..2.1$

0!=1
1!=1

Rule of Multiplication

"If a selection operation can be performed in "m" ways and a second selection operation can be performed in "n" ways; then the two operations can be performed together in " $"m\times n"$ " ways"

- A coin is tossed and a die is rolled; here operation one i.e. the coin gives $\{H,T\}$ and the second operation i.e. the die gives $\{1,2,3,4,5,6\}$; hence the two operations can be performed in $2\times 6=12$ ways.
- If a man has 3 suits and 5 ties; then he can wear a suit and a tie in $3\times 5=15$ ways.

Historical Note

The first book on permutations and combinations is written by Swiss mathematician, Jacob Bernoulli in 1713 A.D

Permutation

"A permutation is an arrangement of "r" objects taken from "n" distinct objects in a particular order"

It is denoted by nPr and is given by: $nPr=\dfrac{n!}{(n-r)!}$

Instead of nPr we can also use $^{n}P_{r}$ or $P(n,r)$

EXAMPLE 6.01

How many different permutations can be formed from the letters A, B, C when two letters are taken at a time?

Solution

Here $n=3$ and $r=2$

Therefore $nPr=\dfrac{n!}{(n-r)!}\Rightarrow {}_3P_2=\dfrac{3!}{(3-2)!}=6$

ABC
ACB
BAC
BCA
CAB
CBA

EXAMPLE 6.02

In how many ways "3" persons can be seated on "4" chairs?

Solution Here $n = 4$ and $r = 3$

Therefore $nPr = \frac{n!}{(n-r)!} \Rightarrow {}_4P_3 = \frac{4!}{(4-3)!} = 24$

EXAMPLE 6.03

In how many ways can president, vice-president, secretary and treasure be selected from nine members of a committee?

Solution Here $n = 9$ and $r = 4$

Therefore $nPr = \frac{n!}{(n-r)!} \Rightarrow {}_9P_4 = \frac{9!}{(9-4)!} = 3024$

EXAMPLE 6.04

In how many ways 2 lottery tickets are drawn from 16 for the 1st and 2nd prizes?

Solution Here $n = 16$ and $r = 2$

Therefore $nPr = \frac{n!}{(n-r)!} \Rightarrow {}_{16}P_2 = \frac{16!}{(16-2)!} = 240$

EXAMPLE 6.05

In how many ways can two different books out of 5 books be arranged on a shelf?

Solution Here $n = 5$ and $r = 2$

Therefore $nPr = \frac{n!}{(n-r)!} \Rightarrow {}_5P_2 = \frac{5!}{(5-2)!} = 20$

EXAMPLE 6.06

In how many ways can 5 different books be arranged on a shelf?

Total number of permutation of "n" distinct objects taking all "n" at a time is equal to "n!"

Solution Here $n = 5$

Therefore $\quad Number\ of\ permutation = n! \Rightarrow 5! = 120$

EXAMPLE 6.07

In how many ways can four people be lined up to get on a bus?

Solution Here $n = 4$

Therefore $\quad Number\ of\ permutation = n! \Rightarrow 4! = 24$

EXAMPLE 6.08

How many different words can be formed from the letters of the word "BOXER" if:

1) All the letters are taken at a time
2) Three letters are taken at a time

Solution

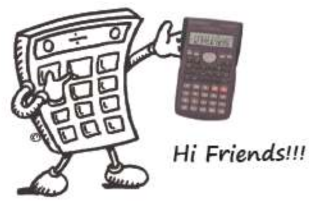
1) All the letters are taken at a time:

Here $n = 5$

Therefore $\quad Number\ of\ permutation = n! \Rightarrow 5! = 120$

2) Three letters are taken at a time

Here $n = 5$ and $r = 3$



Therefore $\quad nPr = \frac{n!}{(n-r)!} \Rightarrow {}_5P_3 = \frac{5!}{(5-3)!} = 60$

EXAMPLE 6.09

Find the number of arrangements of 8 distinct books on a shelf taken:

1) Taken all books at a time
2) Three books are taken at a time

Solution

1) All the letters are taken at a time:

Here $n = 8$

Therefore $\quad \textit{Number of permutation} = n! \Rightarrow 8! = 40320$

2) Three letters are taken at a time

Here $n = 8$ and $r = 3$

Therefore $\quad nPr = \dfrac{n!}{(n-r)!} \Rightarrow {}_8P_3 = \dfrac{8!}{(8-3)!} = 336$

If we arrange objects in a circle then there is no starting point to it, therefore we fixed one object and the remaining objects are arranged as in linear permutation. The formula for arranging "n" objects in a circle is (n−1)!

EXAMPLE 6.10

In how many ways can 4 people be seated at round table?

Solution Here $n = 4$

Therefore

$\textit{Number of circular permutation} = (n-1)!$

$= (4-1)! = 3! = 6$

Group Permutation

The number of distinct permutations of "n" things when "n_1" are alike, "n_2" are alike but different from the first group; "n_3" are alike but different from the first and second group and so on; for "k" groups, is:

$$P = \frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!} \qquad \text{Where } n = \sum_{i=1}^{k} n_i$$

EXAMPLE 6.11

How many possible permutations can be formed from the letters of the word "STATISTICS"?

Solution Here $n = 10$

n_1 = number of "S" = 3
n_2 = number of "T" = 3
n_3 = number of "A" = 1
n_4 = number of "I" = 2
n_5 = number of "C" = 1

$$\text{Therefore } P = \frac{n!}{n_1! \times n_2! \times n_3! \times n_4 \times !n_5!} = \frac{10!}{3! \times 3! \times 1! \times 2! \times 1!} = 50400$$

EXAMPLE 6.12

How many different ways 3 red, 3 yellow and 3 blue balls are arranged in a string with 9 sockets?

Solution Here $n = 9$

n_1 = number of red balls = 3
n_2 = number of yellow balls = 3
n_3 = number of blue balls = 3

$$\text{Therefore } P = \frac{n!}{n_1! \times n_2! \times n_3!} = \frac{9!}{3! \times 3! \times 3!} = 1680$$

EXAMPLE 6.13

In how many possible orders can two boys and three girls be born to a family having five children?

Solution Here $n = 5$

n_1 = number boys = 2
n_2 = number of girls = 3

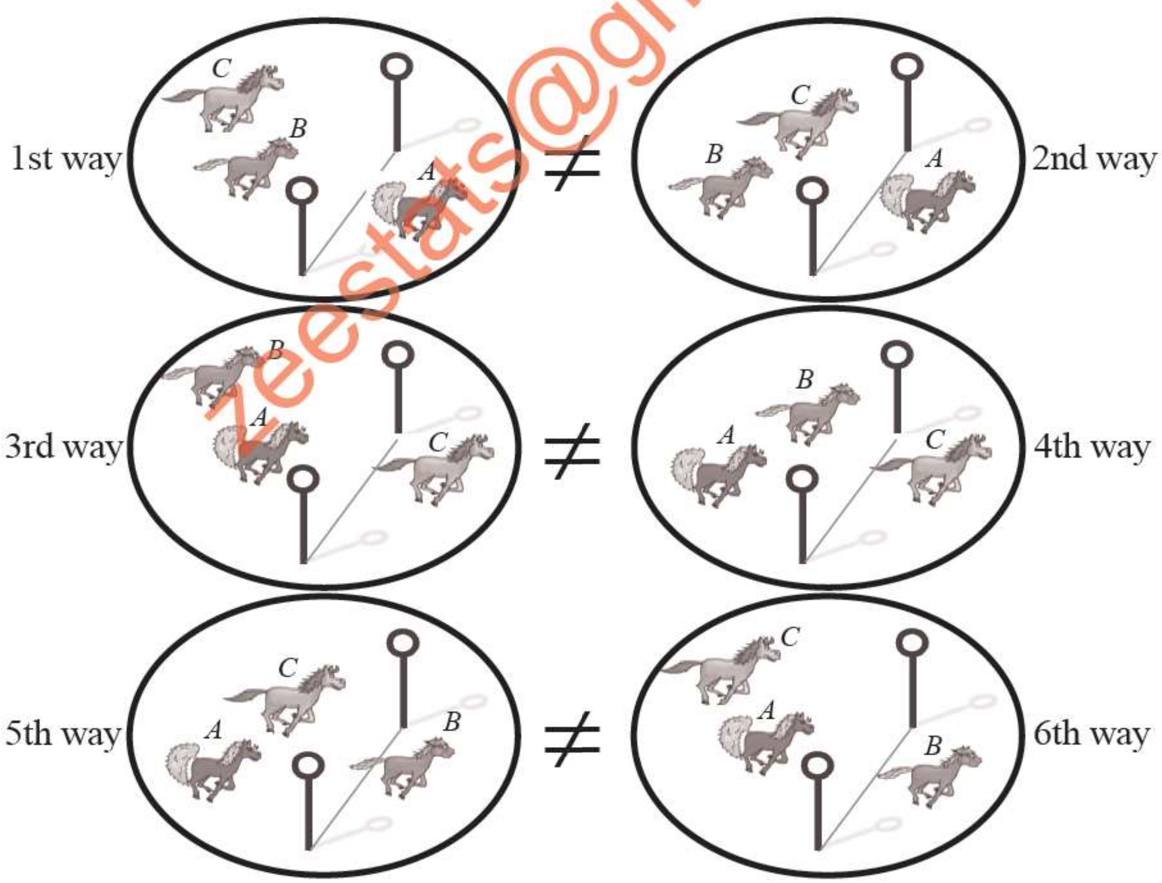
$$\text{Therefore } P = \frac{n!}{n_1! \times n_2!} = \frac{5!}{2! \times 3!} = 10$$

Test Yourself

1) How many permutations can be formed out of the letters of the word "MISSISSIPPI"?
2) Make permutations of A, B, C, D.
3) In how many ways can 4 people be seated at round table?
4) Fine $^{7}P_{3}$, $^{4}P_{2}$, $^{12}P_{5}$, $^{10}P_{8}$
5) Find the number of arrangements of 6 distinct books on a shelf taken:
   (i) Taken all books at a time (ii) Three books are taken at a time
6) In how many ways can 8 people be lined up to get on a bus?

The Order is important in Permutation!!!

There are **six different ways** in which three horses can finish a race as shown in the figure:
(Assume that there are no ties and that every horse finishes)



Combination

"A combination is a selection of "r" objects taken from "n" distinct objects without regarding any order"

It is denoted by nCr and is given by:

$$nCr = \frac{n!}{n!(n-r)!}$$

Instead of nCr we can also use ${}^{n}C_{r}$, $C(n,r)$ or $\begin{pmatrix} n \\ r \end{pmatrix}$

EXAMPLE 6.14

How many combinations of the letters A, B, C can be made if two letters are taken at a time?

Solution Here $n=3$ and $r=2$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{3}C_{2} = \frac{3!}{2!(3-2)!} = 3$$

ABC
BCA
CAB

EXAMPLE 6.15

In how many ways can a team of 11 players be chosen from a total of 15 players?

Solution Here $n=15$ and $r=11$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{15}C_{11} = \frac{15!}{11!(15-11)!} = 1365$$

EXAMPLE 6.16

In how many ways can we select a committee of 4 people from a group of 10 people?

Solution Here $n = 10$ and $r = 4$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{10}C_4 = \frac{10!}{4!(10-4)!} = 210$$

EXAMPLE 6.17

In how many ways can we select a set of 6 books from 10 different books?

Solution Here $n = 10$ and $r = 6$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{10}C_6 = \frac{10!}{6!(10-6)!} = 210$$

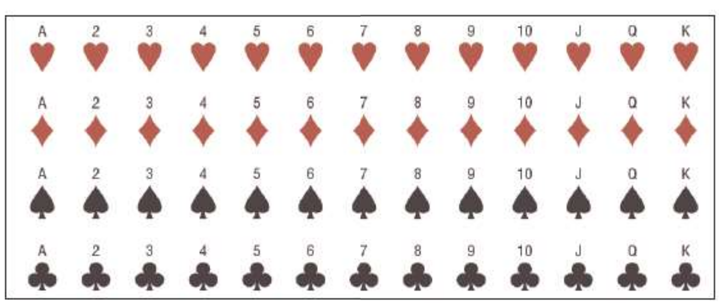
EXAMPLE 6.18

In how many ways can we select a card from a pack of 52 playing cards?

Solution Here $n = 52$ and $r = 1$

Therefore $$nCr = \frac{n!}{r!(n-r)!} \Rightarrow {}_{52}C_1 = \frac{52!}{1!(52-1)!} = 52$$

The 52 ways shown in the following figure:



EXAMPLE 6.19

A bag contains 7 balls; in how many ways can we select 3 balls?

Solution Here $n = 7$ and $r = 3$

Therefore $${}_{n}C_{r} = \frac{n!}{r!(n-r)!} \Rightarrow {}_{7}C_{3} = \frac{7!}{3!(7-3)!} = 35$$

EXAMPLE 6.20

A basket contains 5 white and 4 black balls; in how many ways can we select 3 white and 2 black balls?

Solution Here

White	Black	Total
5	4	9

"3" white balls can be selected out of "5" in ${}_{5}C_{3} = \frac{5!}{3!(5-3)!} = 10$ ways

"2" black balls can be selected out of "4" in ${}_{4}C_{2} = \frac{4!}{2!(4-2)!} = 6$ ways

Hence the number ways in which "3" white and "2" black balls are selected = $10 \times 6 = 60$

EXAMPLE 6.21

In how many ways can a consonant and a vowel be chosen out of the letters of the word SCHOLAR?

Solution Here

SCHOLAR →

Consonants	Vowels	Total
5	2	7

A consonant can be selected out of "5" in ${}_{5}C_{1} = \frac{5!}{1!(5-1)!} = 5$ ways

A vowel can be selected out of "2" in ${}_{2}C_{1} = \frac{2!}{1!(2-1)!} = 2$ ways

Hence the number ways in which a consonant and a vowel is selected = $5 \times 2 = 10$

EXAMPLE 6.22

From 4 black, 5 white and 6 gray balls; how many selection of 9 balls are possible if 3 balls of each color are to be selected?

Solution Here

Black	White	Gray	Total
4	5	6	15

"3" black balls can be selected out of "4" in ${}_4C_3 = \frac{4!}{3!(4-3)!} = 4$ ways

"3" white balls can be selected out of "5" in ${}_5C_3 = \frac{5!}{3!(5-3)!} = 10$ ways

"3" gray balls can be selected out of "6" in ${}_6C_3 = \frac{6!}{3!(6-3)!} = 20$ ways

Hence the number ways in which 3 balls of each color are to be selected= 4x10x20=800

EXAMPLE 6.23

A committee of 5 persons is to be selected out of 6 men and 2 women. Fine the number of ways in which more men are selected than women?

Solution Here

Committee →

Men	Women	Total
6	2	8

Now here more men can be selected in "3" mutually exclusive ways i.e.

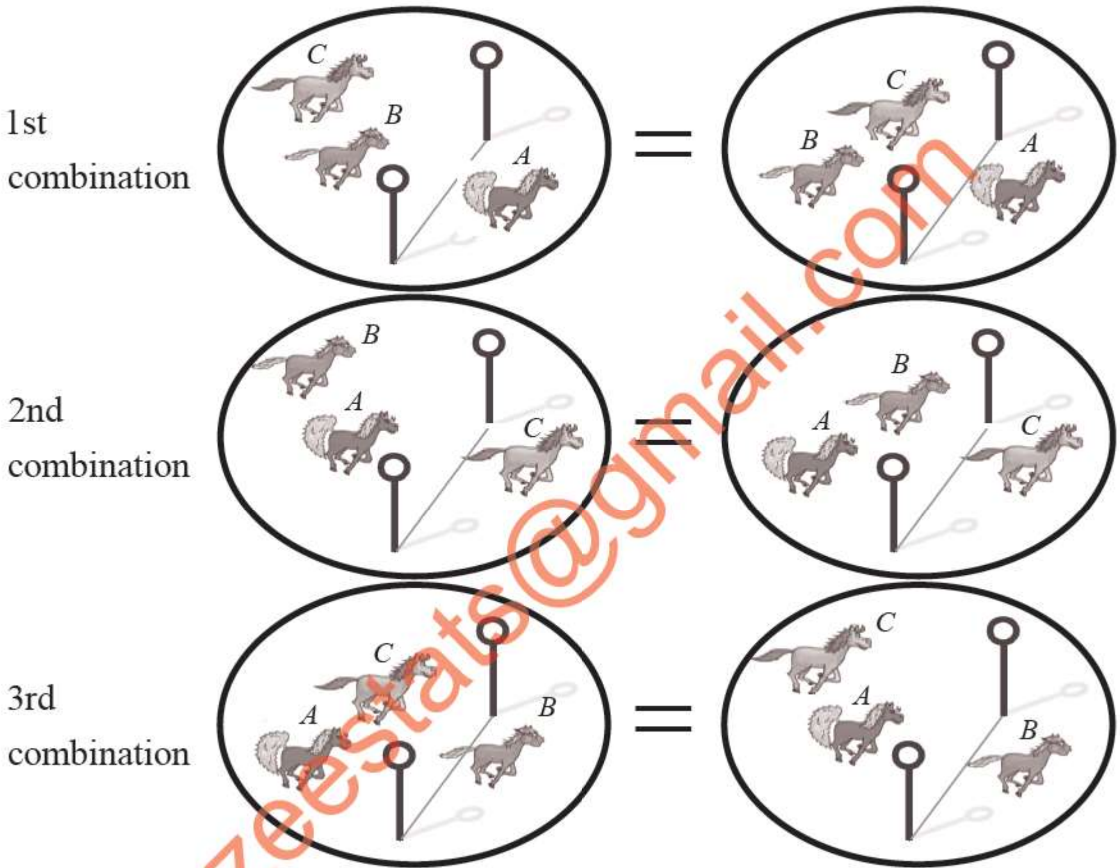
$$\begin{pmatrix} 5\,men \\ 0\,women \end{pmatrix} \text{or} \begin{pmatrix} 4\,men \\ 1\,women \end{pmatrix} \text{or} \begin{pmatrix} 3\,men \\ 2\,women \end{pmatrix}$$

$$\Rightarrow \binom{6}{5}\binom{2}{0} \text{ or } \binom{6}{4}\binom{2}{1} \text{ or } \binom{6}{3}\binom{2}{2}$$

Since the three ways are mutually exclusive; therefore the number of ways in which more men than women can be chosen are $= \binom{6}{5}\binom{2}{0} + \binom{6}{4}\binom{2}{1} + \binom{6}{3}\binom{2}{2} = 56$

The Order Doesn't matter in Combination!!!

There are **three different combinations** in which three horses can finish a race as shown in the figure: (Assume that there are no ties and that every horse finishes)



Test Yourself

1) In how many ways can we select a set of 3 tables from 9 different tables?
2) A bag contains 6 balls; in how many ways can we select 4 balls?
3) A bag contains 9 white and 8 black balls; in how many ways can we select 6 white and 4 black balls?
4) Fine ${}^{7}C_3$, ${}^{4}C_2$, ${}^{12}C_5$, ${}^{10}C_8$
5) In how many ways can a consonant and a vowel be chosen out of the letters of the word CHOSEN?
6) In how many ways can a team of 11 players be chosen from a total of 13 players?

Origin of Probability!!!

Probability theory had its origin in the 16th century when an Italian physician and mathematician **J.Cardan** wrote the first book on the subject, "**The Book on Games of Chance**". Cardan was an astrologer, philosopher, physician, mathematician, and gambler. This book was published in 1663 after his death. It contained techniques on how to cheat and how to catch others at cheating.

In 1654, a professional gambler named **Chevalier de Mere** approached the well known French Philosopher and Mathematician **Blaise Pascal** for certain dice problem. Pascal became interested in these problems, studied them and discussed them with another French mathematician, **Pierre de Fermat**. Both Pascal and Fermat solved the problems independently. This work was the beginning of Probability Theory.

Outstanding contributions to probability theory were also made by **J. Bernoulli**, **De Moivre**, **Pierre Laplace**, **Lagrange** **Chebyshev**, **Markov, Bayes, Huygens** and **Kolmogorov**.

The equation of the normal curve was first published in 1733 by **De Moivre**. The same result was later developed by two mathematical astronomers **Laplace** and **Gauss**.

Historical Note

De Moivre

Laplace

Chebyshev

Bayes

Historical Note

J. Cardano

B. Pascal

Fermat

J. Bernoulli

Gauss

Huygens

Markov

Lagrange

Kolmogorov

Probability

In our daily life we often make the statements such as:

- It will probably rain today
- I will probably go abroad this year
- He is almost certain that he will win this game

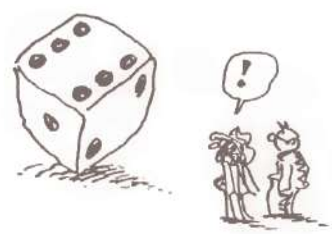
All these statements are related with uncertainty and can be measured numerically by means of "probability". Thus we may simply define probability as ***"the numeric measure of uncertainty is called probability"***.

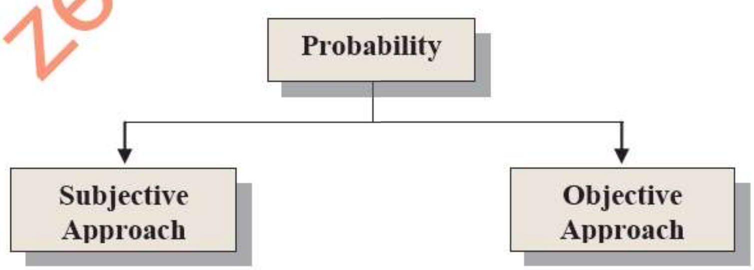
Though probability started with gambling, it has been used extensively in the fields of Physical Sciences, Commerce, Biological Sciences, Medical Sciences, Weather Forecasting, etc.

Definition of Probability

Usually probability of an event is defined by adopting any of the following two approaches:

1) Subjective approach
2) Objective approach





Subjective Approach

In subjective approach the probability of an event is defined as ***"the measure of believe in the occurrence of an event by a particular person"***. Probability in this sense is purely subjective, and is based on whatever evidence is available to the person.

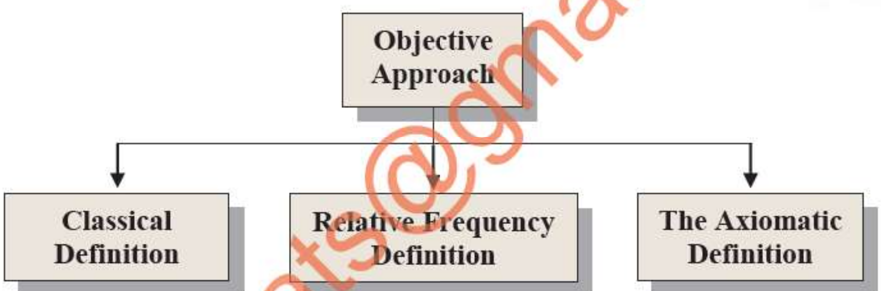
For example:

- A sports-writer may say that there is a 70% probability that Australia will win the world cup.
- A physician might say that, there is a 30% chance the patient will need an operation etc.

Objective Approach

In Objective approach, the probability of an event is defined in the following three ways:

- Classical or Priori or Theoretical Definition of Probability
- Relative Frequency or Empirical or Experimental Definition of Probability
- The Axiomatic Definition of Probability



Classical Definition

"If a random experiment can produce "n" mutually exclusive and equally likely outcomes, and if "m" of these outcomes are favorable to the occurrence of an event "A", then the probability of the event "A" is equal to the ratio m/n" If we take P(A) as "the probability of A" then:

$$P(A) = \frac{m}{n} = \frac{No.\,of\ \ favourable\ outcomes}{No.\,of\ \ possible\ outcomes}$$

- For example, when a fair Coin is tossed, then we know in **advance** that the possible outcomes are Head and Tail. Since the Head and Tail are **equally likely**, therefore, the probability of each is 1/2 or 0.5.

Historical Note

The classical definition was formulated by the French mathematician P.S. Laplace

Relative Frequency Definition

"If "m" is the number of occurrences of an event "A" in large number of trials "n", then the probability of "A" is the relative frequency of "m" and "n" as the number of trials grows infinitely large" If we take P(A) as "the probability of A" then:

$$P(A) = \lim_{n\to\infty}\left(\frac{m}{n}\right)$$

- For example, if a coin has been **loaded** (unfair), then the probability of Head and Tail will not be equal to 0.5 i.e. the Head and Tail are **not equally likely**. Thus for experiments not having equally likely outcomes if we flip the coin 10 times, say, and observe 4 heads, then, based on this information, we say that the chance of observing a head will be 4/10 or 0.4, which is not the same as 0.5. If, however, we flip the coin a large number of times, we would expect about 50 percent of the flips result in a head.

The Axiomatic Definition

Let S be a sample space with the sample points A_1, A_2 ... A_i ...A_n. To each sample point, we assign a real number, denoted by $P(A_i)$, and called the probability of A_i, that must satisfy the following basic axioms:

- **Axiom 1:** *For any event A_i* $0 \le P(A_i) \le 1$
- **Axiom 2:** $P(S) = 1$
- **Axiom 3:** *If A_i and A_j are mutually exclusive events, Then* $P(A_i \cup A_j) = P(A_i) + (A_j)$

In this case $P(A_i)$ is defined by the formula:

$$P(A_i) = \frac{n(A_i)}{n(S)} = \frac{No.\ of\ sample\ points\ in\ the\ event\ A_i}{No.\ of\ sample\ points\ in\ the\ sample\ space}$$

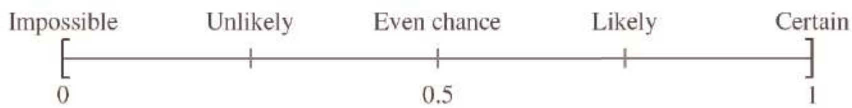
Historical Note

The Axiomatic definition was introduced in 1933 by the Russian mathematician. A.N. Kolmogorov

Subjective probability is purely subjective i.e. that two or more persons faced with the same evidence may arrive at different probabilities. On the other hand, objective probability relates to those situations where everyone will arrive at the same conclusion.

Range of Probability

If the probability of an event is **1**, the event is **certain** to occur. If the probability of an event is **0**, the event is **impossible**. A probability of **0.5** indicates that an event has an **even** chance of occurring. The following graph shows the possible range of probabilities and their meanings.



> **REMEMBER**
>
> The closer the probability is to 1, the more likely is an event will occur. Similarly, The closer the probability is to 0, the less likely is an event will occur.

EXAMPLE 6.24

A fair coin is tossed only once what is the probability that a Head will appear?

Solution Since a coin is tossed

Therefore $S=\{H,T\}\Rightarrow n(S)=2$

Let "A" denotes the event of getting "a Head"

Then $A=\{H\}\Rightarrow n(A)=1$

Hence $P(A)=\dfrac{n(A)}{n(S)}=\dfrac{1}{2}=0.50$

> **TIP**
>
> Probabilities should be expressed as reduced fractions or rounded to two or three decimal places. When the probability of an event is an extremely small decimal, it is permissible to round the decimal to the first nonzero digit after the point. For example, 0.0000587 would be 0.00006

EXAMPLE 6.25

Two fair coins are tossed simultaneously, what is the probability that at least one head will appear?

Solution Since two coins are tossed

Therefore $S=\{HH,HT,TH,TT\}\Rightarrow n(S)=4$

Let "A" denotes the event of getting "at least one Head"

Then $A=\{HH,HT,TH\}\Rightarrow n(A)=3$

Hence $P(A)=\dfrac{n(A)}{n(S)}=\dfrac{3}{4}$

EXAMPLE 6.26

A die is rolled find the probability of getting a six?

Solution Since a die is rolled

Therefore $S=\{1,2,3,4,5,6\} \Rightarrow n(S)=6$

Let "A" denotes the event of getting "a six"

Then $A=\{6\} \Rightarrow n(A)=1$

Hence $P(A)=\dfrac{n(A)}{n(S)}=\dfrac{1}{6}$

REMEMBER

Probabilities can be expressed as fractions, decimals, or percentages. If you ask, "What is the probability of getting a head when a coin is tossed?" typical responses can be any of the following three. "1/2" "0.5" "50%" These answers are all equivalent.

EXAMPLE 6.27

Two dice are rolled, find the probability that the sum is:

(1) Exactly "5" (2) At least "9" (3) At most "4"
(4) Even (5) Less than "3

Solution Since two dice are rolled therefore:

$$S=\left\{\begin{array}{cccccc}(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6)\\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6)\\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6)\\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6)\\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6)\\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\end{array}\right\}$$

1) The sum is "Exactly "5"

Let "A" be an event of getting "sum is exactly 5"

Then $A=\left\{\begin{array}{l}(1,4),(2,3),\\ (3,2),(4,1)\end{array}\right\} \Rightarrow n(A)=4$

Hence $P(A)=\dfrac{n(A)}{n(S)}=\dfrac{4}{36}$

$$S=\left\{\begin{array}{cccccc}(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6)\\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6)\\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6)\\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6)\\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6)\\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\end{array}\right\}$$

2) The sum is "At least "9"

Let "B" be an event of getting "sum is at least 9" then

$$B=\begin{Bmatrix}(3,6),(4,5),(5,4),(6,3),(4,6),\\(5,5),(6,4),(5,6),(6,5),(6,6)\end{Bmatrix}$$

$$\Rightarrow n(B)=10$$

Hence $P(B)=\frac{n(B)}{n(S)}=\frac{10}{36}$

$$S=\begin{Bmatrix}(1,1)&(1,2)&(1,3)&(1,4)&(1,5)&(1,6)\\(2,1)&(2,2)&(2,3)&(2,4)&(2,5)&(2,6)\\(3,1)&(3,2)&(3,3)&(3,4)&(3,5)&(3,6)\\(4,1)&(4,2)&(4,3)&(4,4)&(4,5)&(4,6)\\(5,1)&(5,2)&(5,3)&(5,4)&(5,5)&(5,6)\\(6,1)&(6,2)&(6,3)&(6,4)&(6,5)&(6,6)\end{Bmatrix}$$

3) The sum is "At most "4"

Let "C" be an event of getting "sum is at most 4" then

$$C=\begin{Bmatrix}(1,1),(1,2),(1,3),\\(2,1),(2,2),(3,1)\end{Bmatrix}\Rightarrow n(C)=6$$

Hence $P(C)=\frac{n(C)}{n(S)}=\frac{6}{36}$

$$S=\begin{Bmatrix}(1,1)&(1,2)&(1,3)&(1,4)&(1,5)&(1,6)\\(2,1)&(2,2)&(2,3)&(2,4)&(2,5)&(2,6)\\(3,1)&(3,2)&(3,3)&(3,4)&(3,5)&(3,6)\\(4,1)&(4,2)&(4,3)&(4,4)&(4,5)&(4,6)\\(5,1)&(5,2)&(5,3)&(5,4)&(5,5)&(5,6)\\(6,1)&(6,2)&(6,3)&(6,4)&(6,5)&(6,6)\end{Bmatrix}$$

4) The sum is "Even"

Let "D" be an event of getting "sum is even" then

$$D=\begin{Bmatrix}(1,1),(1,3),(2,2),\\(3,1),(1,5),(2,4),\\(3,3),(4,2),(5,1),\\(2,6),(3,5),(4,4),\\(5,3),(6,2),(4,6),\\(5,5),(6,4),(6,6)\end{Bmatrix}\Rightarrow n(D)=18$$

$$S=\begin{Bmatrix}(1,1)&(1,2)&(1,3)&(1,4)&(1,5)&(1,6)\\(2,1)&(2,2)&(2,3)&(2,4)&(2,5)&(2,6)\\(3,1)&(3,2)&(3,3)&(3,4)&(3,5)&(3,6)\\(4,1)&(4,2)&(4,3)&(4,4)&(4,5)&(4,6)\\(5,1)&(5,2)&(5,3)&(5,4)&(5,5)&(5,6)\\(6,1)&(6,2)&(6,3)&(6,4)&(6,5)&(6,6)\end{Bmatrix}$$

Hence $P(D)=\frac{n(D)}{n(S)}=\frac{18}{36}$

5) The sum is "Less than "3"

Let "D" be an event of getting "is less than 3" then

$E=\{(1,1)\}\Rightarrow n(E)=1$

Hence $P(E)=\dfrac{n(E)}{n(S)}=\dfrac{1}{36}$

$$S=\begin{Bmatrix}(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6)\\(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6)\\(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6)\\(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6)\\(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6)\\(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\end{Bmatrix}$$

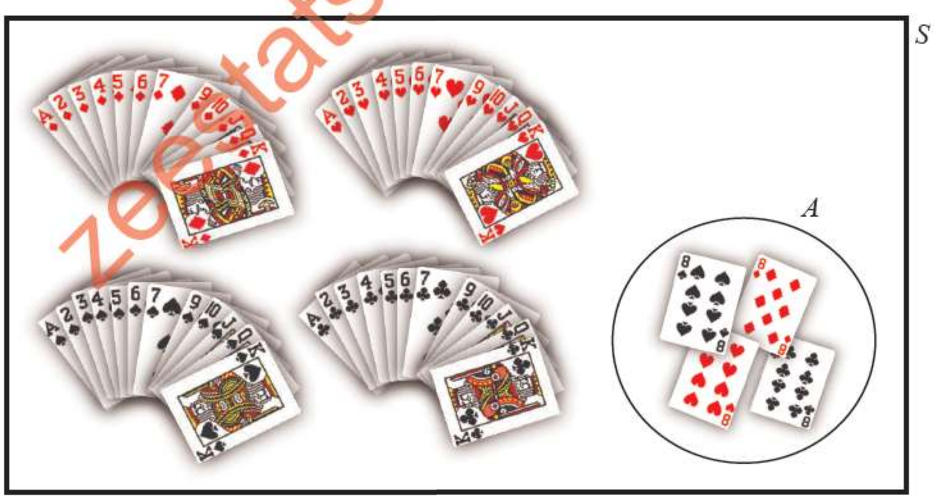
EXAMPLE 6.28

A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card drawn is "8"?

Solution Since a card is drawn therefore

$S=\{\text{the pack of 52 cards}\}\Rightarrow n(S)=\binom{52}{1}=52$

Eights	Others	Total
4	48	52



Let "A" be the event that "the card is eight" Then $n(A)=\binom{4}{1}=4$

Hence $P(A)=\dfrac{n(A)}{n(S)}=\dfrac{4}{52}=\dfrac{1}{13}$

EXAMPLE 6.29

A basket contains 5 white and 4 black balls; what is the probability of selecting 3 white balls?

White	Black	Total
5	4	9

Solution Since "3" balls are selected out of "9"

Therefore $n(S)=\binom{9}{3}=84$

Let "W" be the event of "selecting 3 white balls"

Then $n(W)=\binom{5}{3}=10$

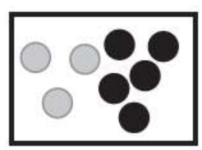
Hence $P(W)=\frac{n(W)}{n(S)}=\frac{10}{84}$

EXAMPLE 6.30

A box contains 3 gray and 5 black balls. If 4 balls are drawn together from the box then find the probability of getting:

(i) At least 2 black balls (ii) At most 2 gray balls.

Solution Since 4 balls are drawn from 8 balls:

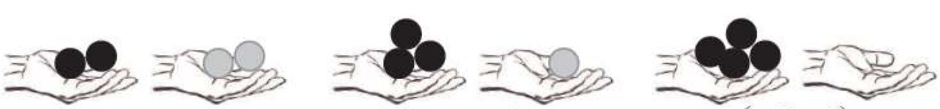


Gray	Black	Total
3	5	8

Therefore $n(S)=\binom{8}{4}=70$

Let "A" be an event of getting "at least 2 black balls i.e. two or more black balls:

Now "A" can occur in the following mutually exclusive ways:



$\binom{2\,black}{2\,gray}$ *OR* $\binom{3\,black}{1\,gray}$ *OR* $\binom{4\,black}{0\,gray}$

$$\therefore n(A)=\binom{5}{2}\binom{3}{2} \text{ or } \binom{5}{3}\binom{3}{1} \text{ or } \binom{5}{4}\binom{3}{0} \Rightarrow n(A)=\binom{5}{2}\binom{3}{2}+\binom{5}{3}\binom{3}{1}+\binom{5}{4}\binom{3}{0}=65$$

Hence $P(A)=\dfrac{n(A)}{n(S)}=\dfrac{65}{70}=\dfrac{13}{14}$

Let "B" be an event of getting "at most 2 black balls i.e. two or less black balls:

Now "B" can occur in the following mutually exclusive ways:

$$\begin{pmatrix} 2\,black \\ 2\,gray \end{pmatrix} \quad OR \quad \begin{pmatrix} 1\,black \\ 3\,gray \end{pmatrix}$$

$$\therefore n(B)=\binom{5}{2}\binom{3}{2} \text{ or } \binom{5}{1}\binom{3}{3} \Rightarrow n(B)=\binom{5}{2}\binom{3}{2}+\binom{5}{1}\binom{3}{3}=60$$

Hence $P(B)=\dfrac{n(B)}{n(S)}=\dfrac{60}{70}=\dfrac{6}{7}$

Test Yourself

1) A fair coin is tossed only once what is the probability that a Tail will appear?
2) Two fair coins are tossed, what is the probability that at least two head will appear?
3) A die is rolled find the probability of getting a four?
4) Two dice are rolled, find the probability that the sum is:

   (i) Exactly "4" (ii) At least "10" (iii) At most "5"
   (iv) Odd (v) Less than "2"

5) A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card drawn is "picture"?
6) A basket contains 6 white and 3 black balls; what is the probability of selecting 4 white balls?
7) A box contains 4 gray and 6 black balls. If 4 balls are drawn together from the box then find the probability of getting:

   (i) At least 2 black balls (ii) At most 3 gray balls.

Addition Rule of prbability for Mutually Exclusive Events

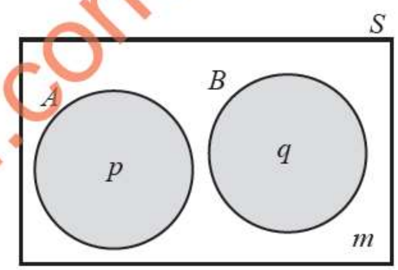
Statement: Let "A" and "B" are two mutually exclusive events then the probability that "A" or "B" occurs is equal to the probability that "A" occurs plus the probability that "B" occurs i.e.

$$P(A \text{ or } B) = P(A) + P(B)$$

OR

$$P(A \cup B) = P(A) + P(B)$$

Proof: To prove the theorem, consider the two Mutually Exclusive events "A" and "B" in the Venn-diagram:



$n(A \cup B) = p + q$ is shaded

It is clear from the Venn-diagram that:

$n(S) = m$
$n(A) = p$
$n(B) = q$
$n(A \cup B) = p + q$

Now

$$P(A) = \frac{n(A)}{n(S)} = \frac{p}{m}$$
$$P(B) = \frac{n(B)}{n(S)} = \frac{q}{m}$$

Therefore

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{p+q}{m} = \frac{p}{m} + \frac{q}{m} = P(A) + P(B)$$

$\Rightarrow P(A \cup B) = P(A) + P(B)$ Hence proved

EXAMPLE 6.31

Suppose that we roll a pair of dice, what is the probability of getting a sum of 5 or a sum of 11?

Solution Since a pair of dice is rolled therefore:

$$S=\begin{Bmatrix}(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6)\\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6)\\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6)\\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6)\\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6)\\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\end{Bmatrix}$$

Let "A" be an event of getting "sum is exactly 5" then

$$A=\begin{Bmatrix}(1,4),(2,3),\\ (3,2),(4,1)\end{Bmatrix}\Rightarrow n(A)=4$$

$$\Rightarrow P(A)=\frac{n(A)}{n(S)}=\frac{4}{36}$$

$$S=\begin{Bmatrix}(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6)\\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6)\\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6)\\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6)\\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6)\\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\end{Bmatrix}$$

Let "B" be an event of getting "sum is 11"

Then $B=\{(5,6),(6,5)\}\Rightarrow n(B)=2$

$$\Rightarrow P(B)=\frac{n(B)}{n(S)}=\frac{2}{36}$$

Now we have to find $P(A\ or\ B)$ and since the two events "A" and "B" are mutually exclusive (because they cannot occur together)

$$\therefore P(A\ or\ B)=P(A\cup B)=P(A)+P(B)=\frac{4}{36}+\frac{2}{36}=\frac{6}{36}$$

EXAMPLE 6.32

A card is drawn from a well-shuffled deck of 52 cards; find the probability that the card is a red or black queen?

Solution Since a card is drawn therefore

$$S=\{the\ pack\ of\ 52\ cards\} \Rightarrow n(S)=\binom{52}{1}=52$$

Red Queens	Black Queens	Others	Total
2	2	48	52

Let "R" be the event that "red queen"

Then $n(R)=\binom{2}{1}=2$

$$P(R)=\frac{n(R)}{n(S)}=\frac{2}{52}$$

Let "B" be the event that "black queen"

Then $n(B)=\binom{2}{1}=2$

$$P(B)=\frac{n(B)}{n(S)}=\frac{2}{52}$$

Now we have to find $P(R\ or\ B)$ and since the two events "R" and "B" are mutually exclusive (because they cannot occur together)

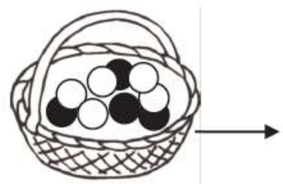
$$\therefore P(R\ or\ B)=P(R\cup B)=P(R)+P(B)=\frac{2}{52}+\frac{2}{52}=\frac{4}{52}$$

EXAMPLE 6.33

A basket contains 5 white and 4 black balls; what is the probability that a ball drawn at random is white or black balls?

Solution Since a ball is drawn out of "9"

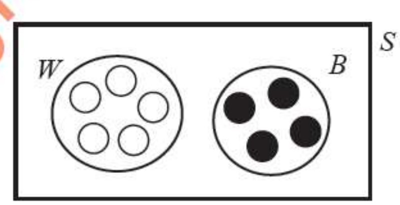
Therefore $n(S)=\binom{9}{1}=9$



White	Black	Total
5	4	9

Let "W" be the event of "drawing a white ball"

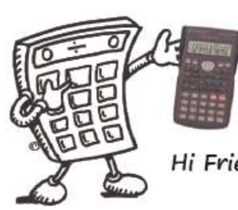
Then $n(W)=\binom{5}{1}=5$



Therefore $P(W)=\frac{n(W)}{n(S)}=\frac{5}{9}$

Let "B" be the event of "drawing a black ball"

Then $n(B)=\binom{4}{1}=4$



Therefore $P(B)=\frac{n(B)}{n(S)}=\frac{4}{9}$

Now we have to find $P(W\ or\ B)$ and since the two events "W" and "B" are mutually exclusive (because they cannot occur together)

$$\therefore P(W\ or\ B)=P(W\cup B)=P(W)+P(B)=\frac{5}{9}+\frac{4}{9}=1$$

Addition Rule of prbability for Not Mutually Exclusive Events

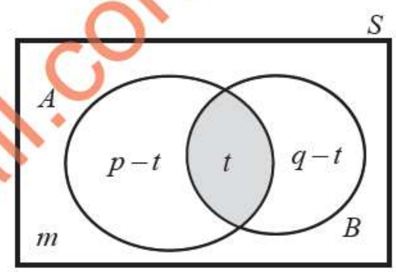
Statement: *Let "A" and "B" are two not mutually exclusive events then the probability of event "A" or "B" or "both" occuring is equal to the probability that "A" occurs plus the probabilitly that "B" occurs minus the probability that "both" events "A" and "B" occur together i.e.*

$$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$$

OR

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: To prove the theorem, consider the two Not Mutually Exclusive events "A" and "B" in the Venn-diagram:



$n(A \cap B) = t$ is shaded

$n(A \cup B) = p - t + t + q - t = p + q - t$

It is clear from the Venn-diagram that:

$n(S) = m$, $n(A) = p$, $n(B) = q$

$n(A \cup B) = p + q - t$

$n(A \cap B) = t$

Now

$$P(A) = \frac{n(A)}{n(S)} = \frac{p}{m}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{q}{m}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{t}{m}$$

Therefore

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{p+q-t}{m} = \frac{p}{m} + \frac{q}{m} - \frac{t}{m} = P(A) + P(B) - P(A \cap B)$$

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$ Hence proved

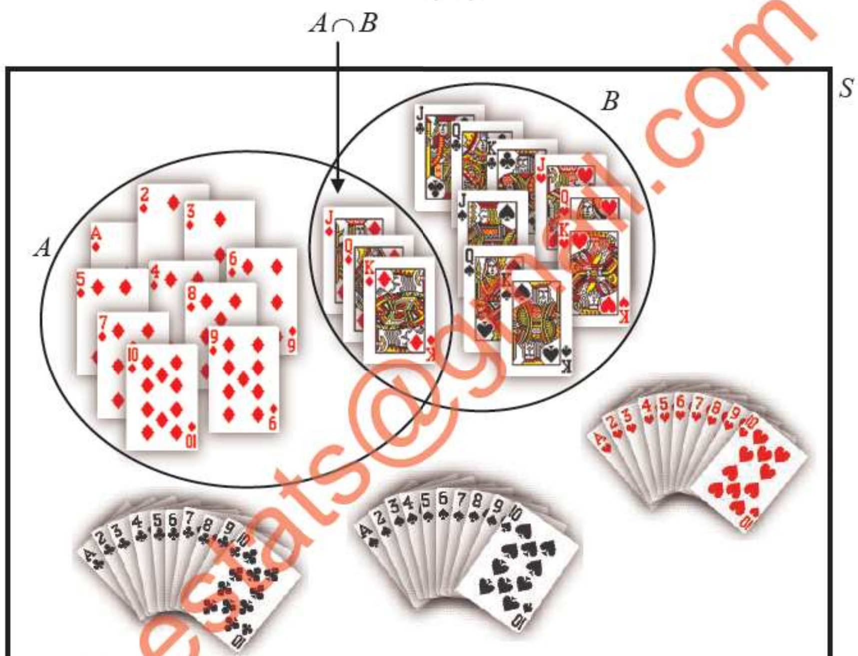
EXAMPLE 6.34

If a card is selected at random from a deck of 52 plyaing cards, what is the proability that the card is a diamond or a picture card or both?

Solution Since a card is drawn, therefore

$$S = \{the\ pack\ of\ 52\ cards\} \Rightarrow n(S) = \binom{52}{1} = 52$$

Diamonds	Picture	Others	Total
13	12	37	52



Let "A" be the event that "a diamond card"

Then $n(A) = \binom{13}{1} = 13$

$$P(A) = \frac{n(A)}{n(S)} = \frac{13}{52}$$

Let "B" be the event that "a picture card"

Then $n(B) = \binom{12}{1} = 12$

$$P(B) = \frac{n(B)}{n(S)} = \frac{12}{52}$$

Since the two events "A" and "B" are not mutually exclusive (because they can occur together), therefore $n(A \cap B) = 3$

Now the probability of both "A" and "B" occur together is: $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{3}{52}$

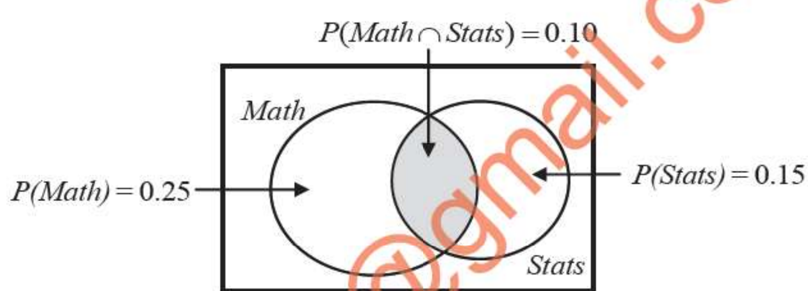
Hence $P(A\ or\ B\ or\ both) = P(A \cup B) = P(A) + P(B) - P(A \cup B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}$

EXAMPLE 6.35

In a certain college 25% of the students failed math, 15% of the students failed stats and 10% of the students failed both mant and stats. A student is selected at random; what is the prbability the he/she failed math or stats?

Solution Given that

25% of students who failed Math $\Rightarrow P(Math) = 0.25$
15% of students who failed Stats $\Rightarrow P(Stats) = 0.15$
10% of students who failed both Math **and** Stats $\Rightarrow P(Math \cap Stats) = 0.10$



Now since the two subjects are **not mutually exclusive**, therefore

$$P(\text{a student failed Math or Stats}) = P(Math \cup Stats)$$
$$= P(Math) + P(Stats) - P(Math \cap Stats)$$
$$= 0.25 + 0.15 - 0.10 = 0.30$$

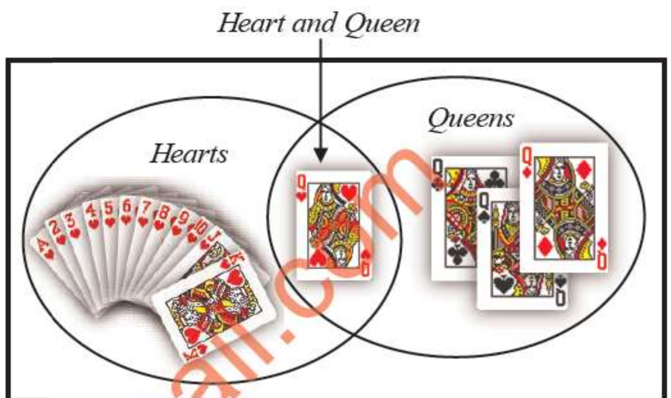
Test Yourself

1) Suppose that we roll a pair of dice, what is the probability of getting a sum of 5 or a sum of 11?
2) A card is drawn from a well-shuffled deck of 52 cards; find the probability that the card is a red or black King?
3) A basket contains 7 white and 3 black balls; what is the probability that a ball drawn at random is white or black balls?
4) If a card is selected at random from a deck of 52 plyaing cards, what is the proability that the card is a Heart or a picture card or both?
5) A customer enters a food store. The probability that the customer buys bread is 0.60, milk is 0.50 and both bread and milk is 0.30. What is the probability that the customer would buy either bread or milk or both?

Understand the meaning of the words "AND" and "OR"!!!

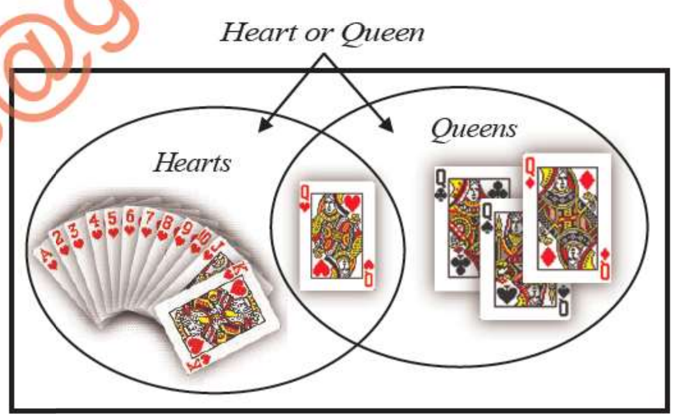
The word **"AND"** has a single meaning.

- For example, if you were asked to find the probability of getting **a queen and a heart** when you were drawing a single card from a deck, you would be looking for the queen of hearts. Here the word **"and"** means **"at the same time."**

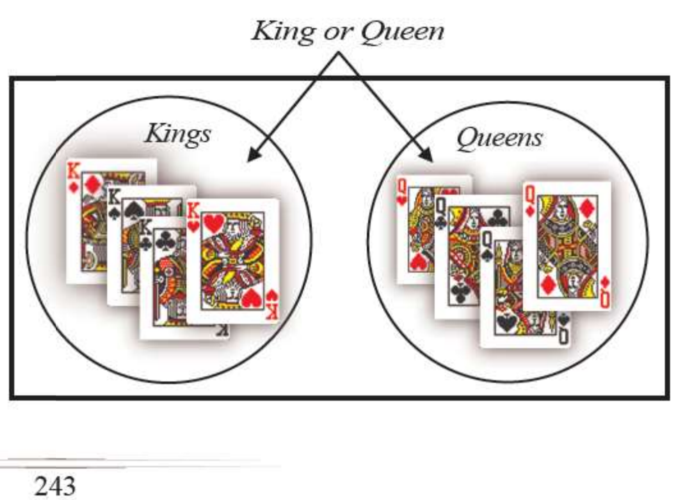


The word **"OR"** has two meanings.

- For example, if you were asked to find the probability of selecting **a queen or a heart** when one card is selected from a deck, you would be looking for **one of the 4 queens or one of the 13 hearts**. In this case, **the queen of hearts would be included in both cases and counted twice**. In this case, both events can occur at the same time; we say that this is an example of the **inclusive or.**



- On the other hand, if you were asked to find the probability of getting **a queen or a king**, you would be looking for **one of the 4 queens or one of the 4 kings**. In this case, both events cannot occur at the same time, and we say that this is an example of the **exclusive or**.



The Rule of Complimentation

The probability that an event "A" will **not** occur, denoted by $P(\overline{A})$ is equal to one minus the probability that "A" will occur i.e.

$$P(\overline{A}) = 1 - P(A)$$

REMEMBER

$$P(A) + P(\overline{A}) = 1$$

In other words, *"If the probability of an event or the probability of its complement is known, then the other can be found by subtracting the probability from 1".*

EXAMPLE 6.36

A coin is tossed 5 times, what is the probability that at least one tail occurs?

Solution Since a coin is tossed 5 times therefore $n(S) = 2^5 = 32$

Let "A" is the event of getting at least one tail (i.e. one, two, three, four or five tails)

So "$\overline{A}$" is the event of getting no tail (i.e. HHHHH)

$\Rightarrow n\left(\overline{A}\right) = 1$

Now $P(\overline{A}) = \dfrac{n(\overline{A})}{n(S)} = \dfrac{1}{32}$

Hence $P(A) = 1 - P(\overline{A}) = 1 - \dfrac{1}{32} = \dfrac{31}{32}$

Conditioanl Probability

The probability that event "A" will occure; once event "B" has already occrured is called conditional prbability of "A" given "B" denoted by P(A/B) and is given as:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}; \qquad P(B) > 0$$

Similarly

$$P(B/A) = \frac{P(A \cap B)}{P(A)}; \qquad P(A) > 0$$

Historical Note

The conditional probability was first introduced by Fermat, a French Mathematician.

EXAMPLE 6.37

Two fair dice are thrown, let "A" denotes " the sum of dots is 10" and "B" denotes "the two dice show the same number, then find

(i) $P(A/B)$ (ii) $P(B/A)$

Solution Since two fair dice are rolled therefore:

$$S=\begin{Bmatrix}(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6)\\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6)\\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6)\\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6)\\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6)\\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\end{Bmatrix}$$

Let "A" be an event of getting "sum is 10" then

$$A=\{(4,6),(5,5),(6,4)\}\Rightarrow n(A)=3$$

$$\Rightarrow P(A)=\frac{n(A)}{n(S)}=\frac{3}{36}$$

Let "B" be an event of getting "same numbers" then

$$B=\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}\Rightarrow n(B)=6$$

$$\Rightarrow P(B)=\frac{n(B)}{n(S)}=\frac{6}{36}$$

Also $A\cap B=\{(5,5)\}=n(A\cap B)=1$

$$\Rightarrow P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{1}{36}$$

$$S=\begin{Bmatrix}(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6)\\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6)\\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6)\\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6)\\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6)\\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\end{Bmatrix}$$

A B

(i) $P(A/B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{1/36}{6/36}=\dfrac{1}{6}$

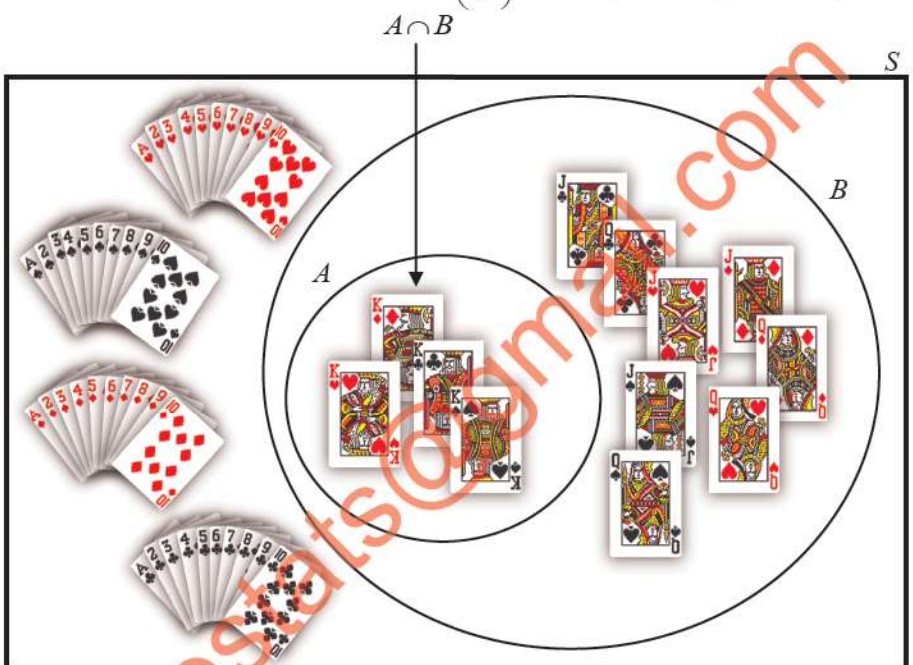
(ii) $P(B/A)=\dfrac{P(A\cap B)}{P(A)}=\dfrac{1/36}{3/36}=\dfrac{1}{3}$

EXAMPLE 6.38

A card is selected at random from a pack, what is the probability that the card is a King given that it is a picture card?

Solution $S = \{\text{the pack of 52 cards}\} \Rightarrow n(S) = \binom{52}{1} = 52$

Kings	Picture	Others	Total
4	12	36	52



Let "A" be the event that "a king"

Then $n(A) = \binom{4}{1} = 4$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

Let "B" be the event that "a picture card"

Then $n(B) = \binom{12}{1} = 12$

$$P(B) = \frac{n(B)}{n(S)} = \frac{12}{52}$$

Since the two events "A" and "B" are not mutually exclusive (because they can occur together), therefore $n(A \cap B) = 4$

Now the probability of both "A" and "B" occur together is: $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{4}{52}$

Hence the probability that the card is a King **given that** it is a picture card is:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{4/52}{12/52} = \frac{1}{3}$$

Test Yourself

1) In a certain college 25% of the students failed math, 15% of the students failed stats and 10% of the students failed both mant and stats. A student is selected at random:

   (i) If he failed statistics, what is the probability that he failed math?
   (ii) If he failed math, what is the probability that he failed statistics?

2) Two fair dice are thrown, let "A" denotes " the sum of dots is 9" and "B" denotes "the two dice show odd number, then find

   (i) $P(A/B)$ (ii) $P(B/A)$

3) A card is selected at random from a pack, what is the probability that the card is a queen given that it is a picture card?

Independent Events

"Two events are independent if the occurrence of one of the events does not affect the probability of the occurrence of the other event".

The folloing are some examples of **independent events**:

- Rolling a die and getting a 6, and then rolling a second die and getting a 3.
- Drawing a card from a deck and getting a queen, replacing it, and drawing a second card and getting a queen.

Dependent Events

"Two events are dependent if the occurrence of one of the events affects the probability of the occurrence of the other event".

The following are some examples of **dependent events**:

- Drawing a card from a deck, not replacing it, and then drawing a second card.
- Selecting a ball from an urn, not replacing it, and then selecting a second ball.

Multiplication Rule of probability for Independent Events

"If "A" and "B" are two independent events, then the probability that both of them occur is equal to the probability of "A" occurs multiply by the probability of "B" occurs" i.e.

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B)$$

TIP

To find the probability of two events occurring in sequence, you can use the Multiplication Laws.

EXAMPLE 6.39

A pair of dice is thrown **twice**. What is the probability of getting a total of 6 on **first** throw **and** a total of 9 on the **second**?

Solution Let "A_1" be an event of getting "total of 6" by a pair of dice in **first** throw:

$$A_1 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

$$\Rightarrow n(A_1) = 5$$

$$\Rightarrow P(A_1) = \frac{n(A_1)}{n(S)} = \frac{5}{36}$$

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

A_1

And let "A_2" be an event of getting "total of 9" in **second** throw.

$$A_2 = \{(3,6),(4,5),(5,4),(6,3)\}$$

$$\Rightarrow n(A_2) = 4$$

$$\Rightarrow P(A_2) = \frac{n(A_2)}{n(S)} = \frac{4}{36}$$

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

A_2

Now we have to find $P(A_1 \text{ and } A_2)$ and since the two events "A_1" and "A_2" are independent, because, they belong to two different throw:

$$\therefore P(A_1 \text{ and } A_2) = P(A_1) \cdot P(A_2) = \left(\frac{5}{36}\right) \cdot \left(\frac{4}{36}\right) = \frac{5}{324}$$

EXAMPLE 6.40

Two cards are drawn in succession from a pack of playing cards and the card drawn in first attempt is being replaced in the pack before the second attempt. Find the probability that both the drawn cards are queens.

Solution Let "Q_1" be an event of getting "a queen" on a **first** draw.

Attempt No.	Cards		
	Queen	Others	Total
1^{st}	4	48	52

$$\therefore P(Q_1) = \frac{\binom{4}{1}}{\binom{52}{1}} = \frac{4}{52}$$

And let "Q_2" be an event of getting "a queen" on **second** draw such that the first card is being replaced:

Attempt No.	Cards		
	Queen	Others	Total
2^{nd}	4	48	52

$$\therefore P(Q_2) = \frac{\binom{4}{1}}{\binom{52}{1}} = \frac{4}{52}$$

Now we have to find $P(Q_1 \text{ and } Q_2)$ and since the two events "Q_1" and "Q_2" are independent, because, the first card is being replaced:

$$\therefore P(Q_1 \text{ and } Q_2) = P(Q_1) \cdot P(Q_2) = \left(\frac{4}{52}\right) \cdot \left(\frac{4}{52}\right) = \frac{1}{169}$$

TIP

With replacement means that the events are independent (the probability don't change)

Multiplication Rule of probability for Dependent Events

" If "A" and "B" are two dependent events, then the probability that both of them occur is equal to the probability of "A" occurs multiply by the conditional probability of "B" given that "A" has already occurred" i.e.

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B/A)$$

Similarly

$$P(A \text{ and } B) = P(A \cap B) = P(B) \cdot P(A/B)$$

EXAMPLE 6.41

Two cards are drawn in succession from a pack of playing cards and the card drawn in first attempt is not being replaced in the pack before the second attempt. Find the probability that both the drawn cards are queens.

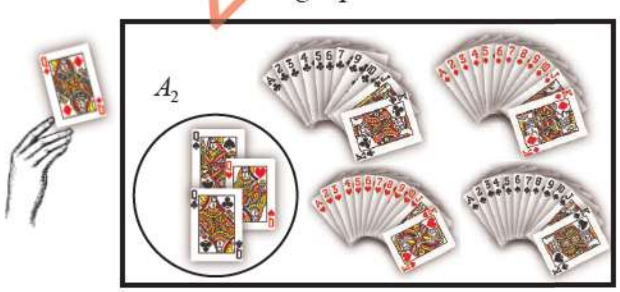
Solution Let "Q_1" be an event of getting "a queen" on a **first** draw:

Attempt No.	Cards		
	Queen	Others	Total
1^{st}	4	48	52

$$\therefore P(Q_1) = \frac{\binom{4}{1}}{\binom{52}{1}} = \frac{4}{52}$$

And let "Q_2" be an event of getting "a queen" on **second** draw such that the first card is not being replaced.

Attempt No.	Cards		
	Queen	Others	Total
2^{nd}	3	48	51



$$\therefore P(Q_2/Q_1) = \frac{\binom{3}{1}}{\binom{51}{1}} = \frac{3}{51}$$

(Because "Q_1" has already occurred)

Now we have to find $P(Q_1 \text{ and } Q_2)$ and since the two events "Q_1" and "Q_2" are dependent, because, the first card is being replaced:

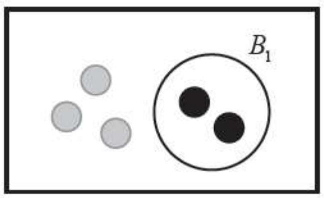
$$\therefore P(Q_1 \text{ and } Q_2) = P(Q_1) \cdot P(Q_2/Q_1) = \left(\frac{4}{52}\right) \cdot \left(\frac{3}{51}\right) = \frac{1}{221}$$

EXAMPLE 6.42

A box contains 3 gray and 2 black balls. Two balls are drawn in succession. Find the probability that both balls drawn are black when the balls are **not replaced** after being drawn.

Solution Let "B_1" be an event of getting "a black ball" on a **first** draw:

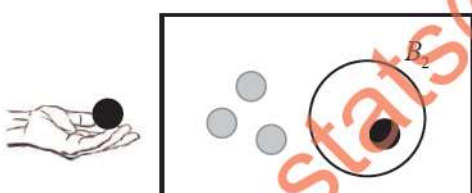
Draw No.	Balls		
	Gray	Black	Total
1^{st}	3	2	5



$$\therefore P(B_1) = \frac{\binom{2}{1}}{\binom{5}{1}} = \frac{2}{5}$$

And let "B_2" be an event of getting "a black ball" on **second** draw, such that the first ball is not being replaced:

Draw No.	Balls		
	Gray	Black	Total
2^{nd}	3	1	4



$$\therefore P(B_2/B_1) = \frac{\binom{1}{1}}{\binom{4}{1}} = \frac{1}{4}$$

(Because "B_1" has already occurred)

Now we have to find $P(B_1 \text{ and } B_2)$ and since the two events "B_1" and "B_2" are dependent, because, the first ball is not being replaced:

$$\therefore P(B_1 \text{ and } B_2) = P(B_1) \cdot P(B_2/B_1) = \left(\frac{2}{5}\right) \cdot \left(\frac{1}{4}\right) = \frac{1}{10}$$

EXAMPLE 6.43

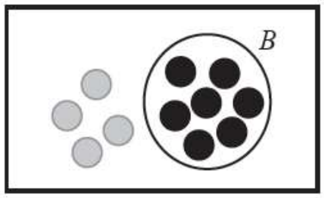
Two drawings each of 3 balls are made from a box containing 4 gray and 7 black balls; the balls are **not being replaced** before the second draw. Find the probability that first drawing gives 3 black balls and second 3 gray balls.

TIP

Without replacement means that the events are dependent (the probability changes)

Solution Let "B" be an event of getting "3 black balls" on a **first** draw:

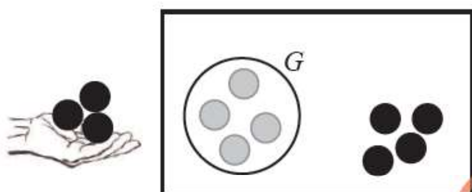
Draw No.	Balls		
	Gray	Black	Total
1^{st}	4	7	11



$$\therefore P(B) = \frac{\binom{7}{3}}{\binom{11}{3}} = \frac{35}{165}$$

And let "G" be an event of getting "3 white balls" on **second** draw, such that the first ball is not being replaced:

Draw No.	Balls		
	Gray	Black	Total
2^{nd}	4	4	8



$$\therefore P(G/B) = \frac{\binom{4}{3}}{\binom{8}{3}} = \frac{4}{56}$$

(Because "B" has already occurred)

Now we have to find $P(B\ and\ G)$ and since the two events "B" and "G" are dependent, because, the first ball is not being replaced:

$$\therefore P(B\ and\ G) = P(B) \cdot P(G/B) = \left(\frac{35}{165}\right) \cdot \left(\frac{4}{56}\right) = \frac{1}{66}$$

Test Yourself

1) Find the probability of drawing a picture card on each of two consecutive draws from a standard pack **with replacement** of the first card.
2) Two drawings each of 4 balls are made from a box containing 5 white and 8 black balls; the balls are **not being replaced** before the second draw. Find the probability that first drawing gives 4 black balls and second 4 white balls.

WARNING!

If two events are independent, it doesn't mean that they can't occur at the same time. Many people make the mistake of thinking of independent events as being totally separate from each other. In probability, two independent events can occur at the same time they just don't affect each other in terms of probabilities as discussed in examples 6.38 and 6.39

EXAMPLE 6.44

Let "A" and "B" be the two possible out comes of an experiment and suppose:
$P(A)=0.33$, $P(A\cap B)=0.25$, $P(B)=p$ and $P(A\cup B)=0.75$

1) Find "p", if "A" and "B" are not mutually exclusive.
2) Find "p", if "A" and "B" are independent.

Solution

1) Find "p", if "A" and "B" are not mutually exclusive.

If "A" and "B" are not mutually exclusive then:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$\Rightarrow 0.75=0.33+p-0.25$$
$$\Rightarrow p=0.67$$

2) Find "p", if "A" and "B" are independent.

If "A" and "B" are independent then:

$$P(A\cap B)=P(A)\cdot P(B)$$
$$\Rightarrow P(B)=\frac{P(A\cap B)}{P(B)} \qquad \Rightarrow p=\frac{0.25}{0.33}=0.76$$

Sometimes there is confusion between independent events and mutually exclusive events. Term 'independent' is defined in terms of 'probability of events' whereas mutually exclusive is defined in term of events (subset of sample space). Moreover, mutually exclusive events never have an outcome common, but independent events, may have common outcome. Clearly, 'independent' and 'mutually exclusive' do not have the same meaning. In other words, two independent events having non-zero probabilities of occurrence can not be mutually exclusive, and conversely, i.e. two mutually exclusive events having non-zero probabilities of occurrence can not be independent.

EXAMPLE 6.45

Let "A" and "B" be the two possible out comes of an experiment and suppose:
$P(A) = 0.60$, $P(B) = p$ and $P(A \cup B) = 0.92$

1) Find "p", if "A" and "B" are mutually exclusive.
2) Find "p", if "A" and "B" are independent.

Solution

1) Find "p", if "A" and "B" are mutually exclusive.

If "A" and "B" are mutually exclusive then:

$$P(A \cup B) = P(A) + P(B)$$
$$\Rightarrow 0.92 = 0.60 + p$$
$$\Rightarrow p = 0.32$$

2) Find "p", if "A" and "B" are independent.

If "A" and "B" are independent then:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \because \text{Independent events are not M.E}$$
$$\Rightarrow P(A \cup B) = P(A) + P(B) - \{P(A) \cdot P(B)\} \qquad \because P(A \cap B) = P(A) \cdot P(B)$$
$$\Rightarrow 0.92 = 0.60 + p - \{0.60 \cdot p\}$$
$$\Rightarrow p = 0.80$$

TECHNICAL STUFF

- Two events A and B are independent if:

  $P(A/B) = P(A)$ or $P(B/A) = P(B)$

- Two events A and B are dependent if:

  $P(A/B) \neq P(A)$ or $P(B/A) \neq P(B)$

- Mutually Exclusive Events are always dependent.

- Two dependent events A and B cannot be mutually exclusive, unless $P(A/B) = 0$

Interesting in Playing Cards

- There are 52 cards in a deck of playing cards. There are four suits (**clubs**, **diamonds**, **hearts** and **spades**) in it, each have 13 cards. The clubs and spades are **black** in color while hearts and diamonds are **red** in color. Total black cards are 26 and total red cards are also 26.

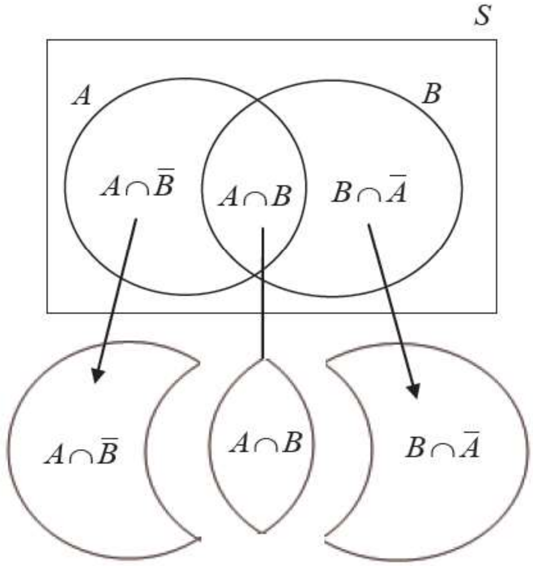
> TIP
> - No. of spots on cards 365 (days in year)
> - Cards in pack 52 (weeks in year)
> - No. of Suits 4 (weeks in month)
> - No. of Picture cards 12 (months in year)

- There are **four** aces.

- Number of **picture cards** is 12 that include "4" jacks, "4" queens and "4" kings from each suit.

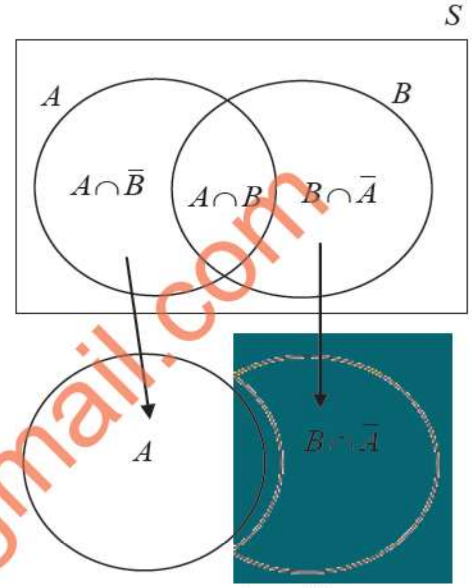
- Number of **face cards** is 16 that include "4" aces, "4" jacks, "4" queens and "4" kings from each suit.

Results: Consider the Venn-diagrams:



Result #01

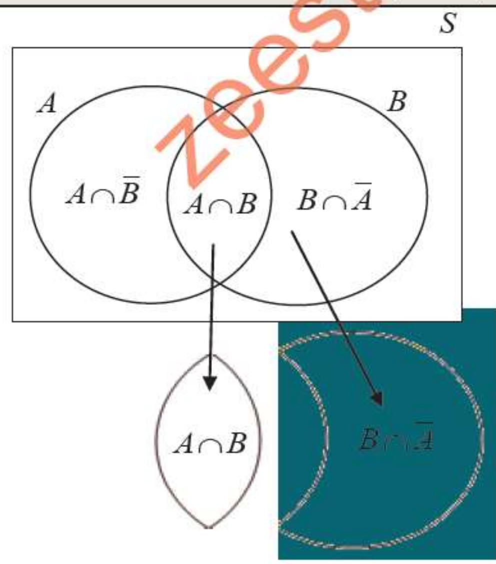
$$A\cup B = A\cup\left(B\cap\bar{A}\right)$$
$$\Rightarrow P(A\cup B) = P(A)+P\left(B\cap\bar{A}\right)$$



$$A\cup B = A\cup\left(B\cap\bar{A}\right)$$

Result #02

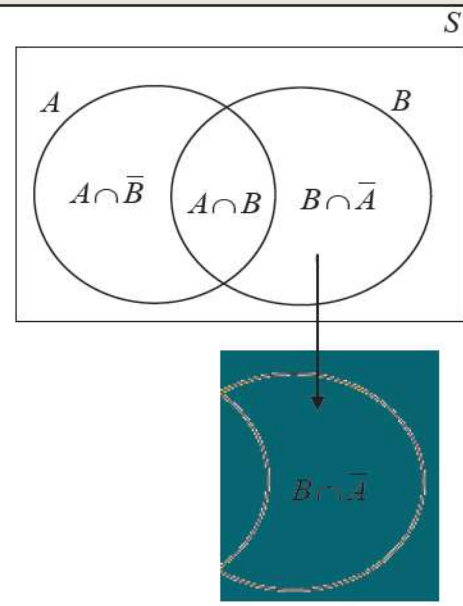
$$B = (A\cap B)\cup\left(B\cap\bar{A}\right)$$
$$\Rightarrow P(B) = P(A\cap B)+P\left(B\cap\bar{A}\right)$$



$$B = (A\cap B)\cup\left(B\cap\bar{A}\right)$$

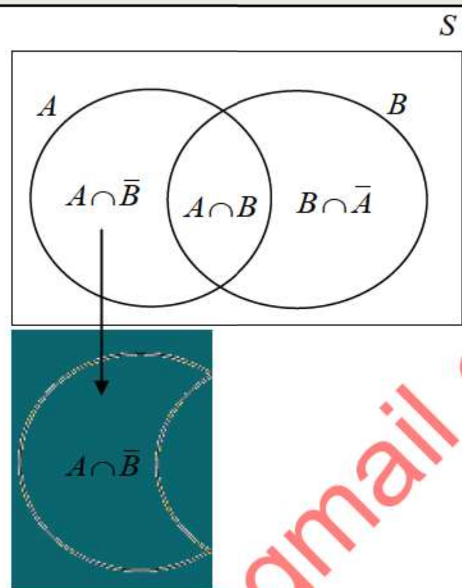
Result #03

$$B\cap\bar{A} = B-(A\cap B)$$
$$\Rightarrow P\left(B\cap\bar{A}\right) = P(B)-P(A\cap B)$$



$$B\cap\bar{A} = B-(A\cap B)$$

Result #04 $A\cap\bar{B}=A-(A\cap B)\Rightarrow P(A\cap\bar{B})=P(A)-P(A\cap B)$



$A\cap\bar{B}=A-(A\cap B)$

Important!!!

While reading probability problems, pay special attention to key phrases that translate into mathematical symbols. The following table lists various phrases and their corresponding mathematical equivalents:

Math Symbol	Phrases
$>$	"greater than" or "more than" or "exceed" or "better than" or "taller than" or "above"
$<$	"less than" or "smaller than" or "below" or "under" or "fewer than"
$\geq$	"at least" or "greater than or equal to" or "no less than"
$\leq$	"at most" or "less than or equal to" or "no more than"
$=$	"exactly" or "equal" or "is"