Chapter 13

STATISTICAL INFERENCE TESTING OF HYPOTHESES

13.1 INTRODUCTION

Statistical inference consists of estimation of parameters and testing of hypotheses. Estimation has already been discussed in the previous chapter and in this chapter our lesson is about the testing of hypotheses. Point estimation and interval estimation as discussed earlier have their own fields of application. Sometimes there is a situation in which the point estimation and the interval estimation are either not required or the estimation of parameters does not provide any inference. For example, the following situations require inference which is not possible by methods of estimation.

(i) The contents of a medicine have been changed to improve the effectiveness of the medicine. In this situation both the point estimation and the interval estimation fail to answer the question about the improvement of the medicine. In this case we have to take help from the sample data to decide whether or not the medicine has been improved.

(ii) A manufacturer of tires claims that the average life of his tires is at least 15000 kilometers. The life of tires is an important factor to settle the price of the tires. It is a big information if we prove with reasonable amount of confidence that the life of the tires is not more than 15000 kilometers. The answer is not provided by a point estimate or by an interval estimate of the life of the tires. What we shall have to do is that we shall examine the claim of the manufacturer on the basis of the experiment conducted on the sample of tires. A certain procedure will be adopted to reach some conclusion. This is what we shall call the test of hypothesis about the life of tires.

13.2 STATISTICAL HYPOTHESES

Any opinion or idea may be formed about the population under study. Consider the following statements: Average consumption of sugar per month for a consumer is 1 kg; Intelligent parents have intelligent children, tall fathers have tall sons, average life of the people of Pakistan is higher than that of India, proper greasing increases the life of ceiling fans, use of coffee increases chances of heart attack, one variety of seed is better than the other, a medicine of allergy gives relief to at least 80 % of the people, more than 25 % people are literate in our country, only 60 % people will go to the polling stations for voting. These statements are the questions in different fields of life and these questions are to be answered after proper experimentation. These questions have come up in the process of investigations. This is how the hypotheses are generated during various studies. When an assumption is explained in the form of a statement about the distribution of a population or populations, it is called a statistical hypothesis. In simple words, a statistical hypothesis is a statement about the unknown value of the population parameter. The statement may be true or false.

13.2.1 NULL HYPOTHESIS

The hypothesis which is to be tested is called *null hypothesis*. It is denoted by H_0. It is a starting point in the investigations. A statement which we hope will be rejected is taken as a hypothesis. Modern approach is different. Today any hypothesis we wish to test is called null hypothesis and is denoted by H_0. In this book we shall follow the old convention. Any hypothesis will be called null hypothesis only when we hope to reject it. Thus the null hypothesis is framed for possible rejection. Tall fathers have tall children. We shall assume that tall fathers do not have tall children. This will be considered as null hypothesis and will be denoted by H_0. We are hoping that H_0 will be rejected on the basis of sample data. Use of coffee increases chances of heart attack. To start with we shall assume that heart attack has no link with the use of coffee. This will be taken as H_0 and we hope it will be rejected by the sample data.

13.2.2 ALTERNATIVE HYPOTHESIS

The hypothesis which is accepted when the null hypothesis has been rejected is called the alternative hypothesis. It is denoted by H_1 or H_A. Whatever we are expecting from the sample data is taken as the alternate hypothesis. "More than 25% people are literate in our country". We are hoping to get this result from the sample. It will be taken as an alternate hypothesis H_1 and null hypothesis H_0 will be that 25 % or less than that are literate. To be more specific, H_0 will be 25 % or less are literate and H_1 will be more than 25 % are literate. It is written as:

H_0: $p \leq 0.25$ (25 % or less) H_1: $p > 0.25$ (more than 25 %)

To keep the things simple, we can write H_0 in the form of equality as H_0: $p = 0.25$ but it is important to write H_1 with proper direction of inequality. Thus we write H_1: $p > 0.25$.

In this case the H_1 contains the inequality *more than* (>). We shall explain later that H_1 may be written with inequality *less than* (<) or *not equal* (≠). In general, if the hypothesis about the population parameter θ is θ_0, then H_1 can be written in three different ways.

For $H_0 : \theta = \theta_0$, $H_1 : \theta \neq \theta_0$ $H_1 : \theta > \theta_0$ $H_1 : \theta < \theta_0$

But this is the simple approach which is allowed for the students. Another way of writing the above hypotheses H_0 and H_1 is

(a) $H_0 : \theta = \theta_0$, $H_1 : \theta \neq \theta_0$ (b) $H_0 : \theta \leq \theta_0$, $H_1 : \theta > \theta_0$ (c) $H_0 : \theta \geq \theta_0$, $H_1 : \theta < \theta_0$

The alternative hypothesis H_1 never contains the sign of equality. Thus H_1 will not contain '=', '≤' or '≥' signs. The equality sign '=' and inequalities like '≤' and '≥' are used for writing H_0.

13.2.3 SIMPLE HYPOTHESIS

If a hypothesis has a single value for the population parameter, it is called *simple hypothesis.* The breaking strength of copper wire is 10 kg. Here H_0: $\mu = 10$ kg has a single specified value. H_0 is simple hypothesis, similarly $\mu_1 - \mu_2 = 10$ and $p = 0.6$ are simple hypotheses.

13.2.4 COMPOSITE HYPOTHESIS

The hypothesis is called *composite* if it specifies a *range* of values for the parameter. The hypothesis $\mu \geq 10$ is a composite hypothesis. Similarly the hypotheses $(\mu_1 - \mu_2) \geq 10$ and $p \leq 0.6$ are composite.

13.2.5 ACCEPTANCE AND REJECTION OF NULL HYPOTHESIS

The given hypothesis is tested with the help of the sample data. A simple random sample has the full freedom of giving any value to its statistic. The sample is not aware of our plans. We decide about our hypothesis on the basis of the sample statistic. If the sample does not support the null hypothesis, we reject it on probability basis and accept the alternative hypothesis. If the sample does not oppose the hypothesis, the hypothesis is accepted. But here *'accept'* does not mean the *acceptance* of null hypothesis but only means that the sample has not strongly opposed it. "Not opposed" does not mean that the sample has strongly supported the hypothesis. The support of the sample in favour of the hypothesis cannot be established. When the hypothesis is *rejected,* it is rejected with a high probability. Thus *rejection* of H_0: is a strong decision and it leads us to the acceptance of H_1. But acceptance of H_1 is not like the acceptance of H_0. The *acceptance* of null hypothesis does not give us a certain strong decision. It is a situation which may require some further investigations. At this stage, many factors are to be taken into account. The sample size and certain other things not yet discussed help us to do something more about the null hypothesis before it is finally accepted. Thus *rejection* is a decision but not necessarily true and *acceptance* is not a decision in any sense of the word.

There is a modern approach in which the terms *rejection* and *acceptance* are not used. This modern approach is beyond the level of this book. But it remains true in its place that *acceptance* of a null hypothesis is a weak decision whereas rejection is a strong evidence of the sample against the null hypothesis. When the null hypothesis is rejected, it means the sample has done some statistical work but when the null hypothesis is accepted, it means the sample is almost silent. This behaviour of the sample should not be used in favour of the null hypothesis.

13.2.6 TEST STATISTIC

A statistic is calculated from the sample. To begin with we assume that the hypothesis about the population parameter is true. We compare the value of the statistic with the hypothetical value of the parameter. If the difference between them is small, the hypothesis is accepted and if the difference between them is large, the hypothesis is rejected. A statistic on which the decision can be based whether to accept or reject a hypothesis is called *test statistic*. Some of the test statistics to be discussed in this book are 'Z', 't' and χ^2 (Chi–square)

13.2.7 ACCEPTANCE AND REJECTION REGIONS

The values of the test statistic which we think do not agree with the given hypothesis are called the critical region or rejection region. The values of the test statistic which support the hypothesis form the acceptance region. The rejection region is equal to α and the acceptance region is denoted by $(1 - \alpha)$. These two regions are separate from each other and both regions combined together make the complete sampling distribution of the statistic. These regions are separated by a value (or values), which is called critical value (or values).

13.2.8 TWO-TAILED TEST

When the rejection region is taken on both ends of the sampling distribution, the test is called *two-sided test* or *two-tailed test*. When we are using a two-sided test, half of the rejection region equal to $\alpha/2$ is taken on the right side and the other half equal to $\alpha/2$ is taken on the left side of the sampling distribution. Suppose the sampling distribution of the statistic is a normal distribution and we have to test the hypothesis H_0: $\theta = \theta_0$ against the alternative hypothesis H_1: $\theta \neq \theta_0$ which is two-sided. H_0 is rejected when the calculated value of Z is greater than $Z_{\alpha/2}$ or it is less than $-Z_{\alpha/2}$. Thus the critical region is $Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$, it can also be written as $-Z_{\alpha/2} < Z < Z_{\alpha/2}$

When H_0 is rejected, then H_1 is accepted . *Two-sided test* is shown in Fig.13.1.

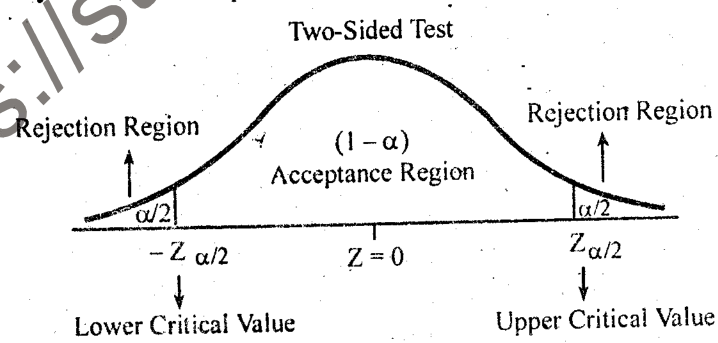

Figure 13.1

13.2.9 ONE–TAILED TEST

When the alternative hypothesis H_1 is one-sided like $\theta > \theta_0$ or $\theta < \theta_0$, then the rejection region is taken only on one side of the sampling distribution. It is called *one-tailed test or one-sided test.* When H_1 is *one-sided* to the right like $\theta > \theta_0$, the entire rejection region equal to α is taken in the right end of the sampling distribution.

The test is called *one-sided* to the right. The hypothesis H_0 is rejected if the calculated value of a statistic, say Z falls in the rejection region. The critical value is Z_α which has the area equal to α to its right. The rejection region and acceptance region are shown in Fig.13.2. The null hypothesis H_0 is rejected when Z(calculated) > Z_α.

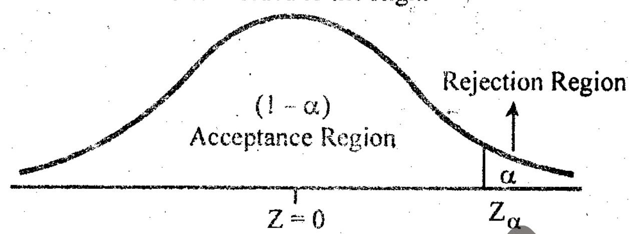

Figure 13.2

If the alternative hypothesis is one-sided to the left like $\theta < \theta_0$, the entire rejection region equal to α is taken on the left tail of the sampling distribution. The test is called one-sided or one-tailed to the left. The critical value is $-Z_\alpha$ which cuts off the area equal to α to its left. The critical region is $Z < -Z_\alpha$ and is shown in Fig.13.3.

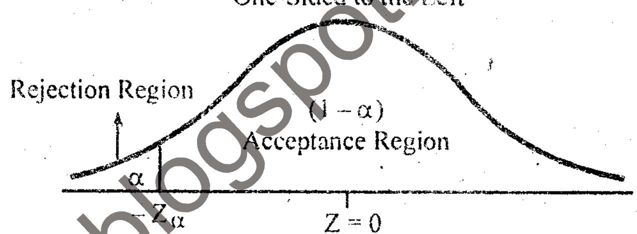

Figure 13.3

For some important values of α, the critical values of Z for two-tailed and one tailed tests are given below:

Critical values of Z

α	Two – sided test	One-sided to the right	One-sided to the left
0.10 (10 %)	– 1.645 and + 1.645	+ 1.282	– 1.282
0.05 (5 %)	– 1.96 and + 1.96	+ 1.645	– 1.645
0.02 (2 %)	– 2.326 and + 2.326	+ 2.054	– 2.054
0.01 (1 %)	– 2.575 and + 2.575	+ 2.326	– 2.326

13.3 ERRORS IN TESTING OF HYPOTHESIS

The null hypothesis H_0 is accepted or rejected on the basis of the value of the test-statistic which is a function of the sample. The test statistic may land in acceptance region or rejection region. If the calculated value of test-statistic, say Z, is small (insignificant) i.e., Z is close to zero or we can say Z lies between $-Z_{\alpha/2}$ and $Z_{\alpha/2}$ is a two-sided alternative test (H_1: $\theta \neq \theta_0$), the hypothesis is accepted. If the calculated value of the test-statistic Z is large (significant), H_0 is rejected and H_1 is accepted. In this rejection plan or acceptance plan, there is the possibility of making any one of the two errors which are called Type I and Type II-errors.

13.3.1 TYPE I ERROR

The null hypothesis H_0 may be true but it may be rejected. This is an error and is called *Type I error*. When H_0 is true, the test-statistic, say Z, can take any value between $-\infty$ to $+\infty$. But we reject H_0 when Z lies in the rejection region while the rejection region is also included in the interval $-\infty$ to ∞. In a two-sided H_1 (like $\theta \neq \theta_0$), the hypothesis is rejected when Z is less than $-Z_{\alpha/2}$ or Z is greater than $Z_{\alpha/2}$. When H_0 is true, Z can fall in the rejection region with a probability equal to the rejection region α. Thus it is possible that H_0 is rejected while H_0 is true. This is called *Type I error*. The probability is $(1 - \alpha)$ that H_0 is accepted when H_0 is true. It is called correct decision. We can say that *Type I error* has been committed when:

(i) an intelligent student is not promoted to the next class.

(ii) a good player is not allowed to play the match.

(iii) an innocent person is punished.

(iv) a driver is punished for no fault of him.

(v) a good worker is not paid his salary in time.

These are the examples from practical life. These examples are quoted to make a point clear to the students.

α (ALPHA)

The probability of making *Type I error* is denoted by α(alpha). When a null hypothesis is rejected, we may be wrong in rejecting it or we may be right in rejecting it. We do not know that H_0 is true or false. Whatever our decision will be, it will have the support of probability. A true hypothesis has some probability of rejection and this probability is denoted by α. This probability is also called the size of *Type I error* and is denoted by α.

13.3.2 TYPE II ERROR

The null hypothesis H_0 may be false but it may be accepted. It is an error and is called Type II error. The value of the test-statistic may fall in the acceptance region when H_0 is in fact false. Suppose the hypothesis being tested is H_0: $\theta = \theta_0$ and H_0 is false and true value of θ is θ_1 or θ_{true}. If the difference between θ_0 and θ_1 is very large then the chance is very small that θ_0(wrong) will be accepted. In this case the true sampling distribution of the statistic will be quite away from the sampling distribution under H_0. There will be hardly any test-statistic which will fall in the acceptance region of H_0. When the true distribution of the test-statistic overlaps the acceptance region of H_0, then H_0 is accepted though H_0 is false. If the difference between θ_0 and θ_1 is small, then there is a high chance of accepting H_0. This action will be an error of Type II.

β (BETTA)

The probability of making *Type II error* is denoted by β. Type II error is committed when H_0 is accepted while H_1 is true. The value of β can be calculated only when we happen to know the true value of the population parameter being tested.

13.3.3 RELATION BETWEEN α AND β

Suppose we have to test $H_0: \mu = \mu_0$ against the alternative $H_1: \mu > \mu_0$. A random sample of size n is selected from the population and the sample mean $\bar{X}$ is calculated. The sample size n is large and therefore the sampling distribution of $\bar{X}$ is normal with mean μ. To start with we assume that $H_0: \mu = \mu_0$ is true and $\bar{X}$ has the distribution as shown on left side of the fig. 13.4.

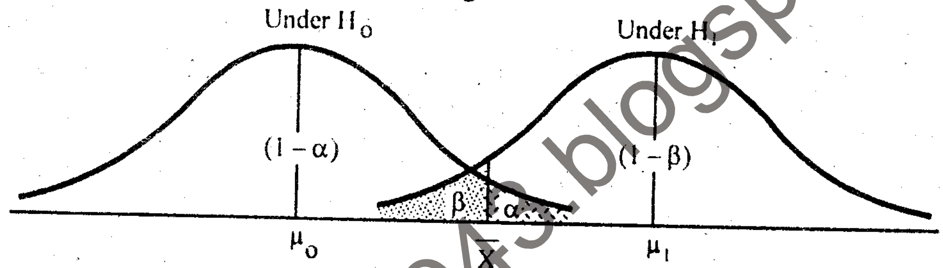

Figure 13.4

Fig.13.4. has two sampling distributions one is on the left side and the other is on the right side. When the null hypothesis $H_0: \mu = \mu_0$ is being tested, there are the following four possibilities.

(i) H_0 is true and $\bar{X}$ falls in the area marked (1 – α) in the Fig.13.4. The hypothesis H_0 is accepted and this is called correct decision. Probability of this correct decision is (1 – α). We may or may not make this decision.

(ii) H_0 is true and $\bar{X}$ falls in the area marked α. This is the area of the distribution on the left side. Now H_0 is true but it will be rejected because $\bar{X}$ falls in the rejection region. This is an error of Type I and this error will be committed with the probability of α. We do not know whether we have committed α error or not.

(iii) H_0 is false. The true value of μ is say μ_1 and the true distribution of $\bar{X}$ is the distribution on the right side in Fig. 13.4. Now suppose $\bar{X}$ falls in the area marked (1 – β). This is outside the acceptance region of the distribution on the left side. Thus $H_0: \mu = \mu_0$ is rejected and the probability of this action is (1 – β).

It is called correct decision when H_0 is false. In fact, $\bar{X}$ belongs to some distribution. When we take a hypothesis H_0, this is an assumption about the mean of the distribution of $\bar{X}$. If true distribution of $\bar{X}$ is on the right side, then some area of this distribution is falling on the acceptance region of the hypothetical distribution on the left side. This area is marked as β.

(iv) H_0 is false and the value of $\bar{X}$ falls in the area marked β. In this case H_0 is accepted because $\bar{X}$ has fallen in the acceptance region of the first distribution. Thus H_0 being false may be accepted with probability of β.

If the distribution on the right side is shifted to the right, β will decrease and if this distribution is shifted to the left, β will increase. Thus the value of β depends upon the true value of population mean μ. In a certain given situation when n is fixed the value of β increases when α is decreased. Thus if we want to decrease α, we shall do it at the risk of increasing β. α -error and β-error are also called α-risk and β-risk respectively. Which risk do we want to keep at minimum level? This depends upon the costs of committing α-error and β-error. Suppose we are hesitant of rejecting H_0 when it is true, then we shall take α at a small level. In most of the tests, α is fixed at a small level like 0.01 (1 %) or 0.05 (5 %).

The following table shows four possible decisions in a certain test of hypothesis.

	H_0 is True	H_0 is False
H_0 is Accepted	Correct decision	Type II error
H_0 is Rejected	Type I error	Correct decision

When we are testing a hypothesis, our decision will fall in any one of the above four boxes. The four possible decisions in terms of probabilities are shown below in a tabular form.

	H_0 True	H_0 False
H_0 is Accepted	$(1-\alpha)$	β
H_0 is Rejected	α	$(1-\beta)$

It may be noted that α is an area in the right tail of the distribution under H_0 and β is the area in the left tail of the distribution under H_1. Thus $\alpha + \beta \neq 1$ in general. In some special case and that too very rarely, $\alpha + \beta$ may be equal to 1. Level of α is usually small. Thus probability is small that our decision will fall in the box marked α. But when our decision has fallen in the box marked α, it is a powerful decision against H_0.

13.4 LEVEL OF SIGNIFICANCE

The α-risk is the probability of rejecting a true null hypothesis. It is also called the significance level or level of significance of the test. It is denoted by α and its level is usually 1 % or 5 %. The value of α is usually decided before the selection of the sample.

13.5 FARMULATING H_0 AND H_1 AND MAKING CRITICAL REGION

Now, when we have discussed different terms used in the testing of hypothesis, we are in a position to discuss a point which is quite confusing sometimes. The question is how to formulate the null hypothesis H_0 and the alternative hypothesis H_1. We elaborate this point here and we shall repeat here certain points already discussed in this chapter about framing of H_0 and H_1. Let us consider some cases.

(i) A machine has been producing components with mean length of 3 cm. which is the required standard. A new machinery has been installed and it is required to test the hypothesis that the mean length of the components is the same. It is obvious that in this case the H_0 and H_1 will be:

$H_0 : \mu = 3$ cm. $\quad H_1 : \mu \neq 3$ cm.

H_1 contains the inequality '≠' which means that the rejection region is taken in both ends of the sampling distribution.

The test-statistic used is $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$.

The null hypothesis H_0 is rejected if $Z < -Z_{\alpha/2}$ or $Z > Z_{\alpha/2}$. It is called *two-tailed test* with rejection region on both sides. H_0 is rejected when sample mean $\bar{X}$ is sufficiently larger than 3 cm. or sufficiently smaller than 3.

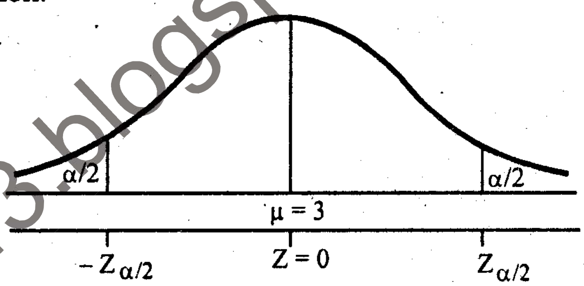

Figure 13.5

(ii) Suppose that we want to test whether the mean μ of a normal distribution exceeds a specified value μ_0. We set up the null and alternative hypotheses as follows: $\quad H_0 : \mu = \mu_0 \qquad H_1 : \mu > \mu_0$

The null hypothesis H_0 and the alternative hypothesis H_1 in this case can also be written as $\quad H_0 : \mu \leq \mu_0 \qquad H_1 : \mu > \mu_0$

H_1 is complement of H_0 and the area of the distribution under H_0 and H_1 makes the complete distribution. In this case, the region of rejection is taken in the right tail of the distribution.

The test-statistic is

$Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$. The null hypothesis H_0 is rejected when the calculated value of Z is greater than the critical value Z_α.

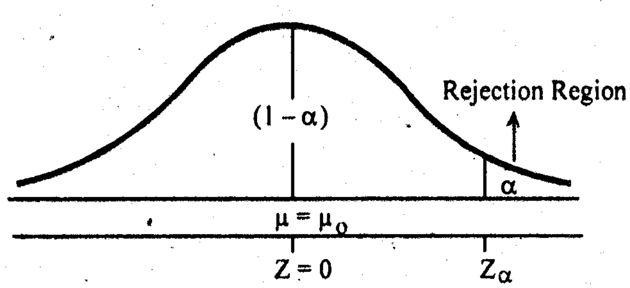

Figure 13.6

(iii) At least 60 % of the people are in favour of English as medium of instructions. The sampling distribution of proportion $\hat{p}$ is divided into two parts (1) at least 60 % (2) less than 60 %.

We have a serious doubt about the statement and we hope to disprove it. The proportion of the people $p \geq 0.6$ is to be tested. The idea or suggestion of at least 60 % ($p \geq 0.6$) will be rejected if the sample gives the result well below 60 %. The rejection region is decided by H_1 which is one-sided to the left. Thus we frame H_0 and H_1 as: $H_o: p \geq 0.6 \quad H_1: p < 0.6$

In this case the entire critical region lies in the left tail. If H_1: $p < 0.6$ is true then the sample proportion $\hat{p}$ should lie in the rejection region.

The test statistic used here is

$Z = \dfrac{\hat{p} - p}{\sqrt{\dfrac{p\,q}{n}}}$. The hypothesis H_0 is rejected if $Z < -Z_\alpha$.

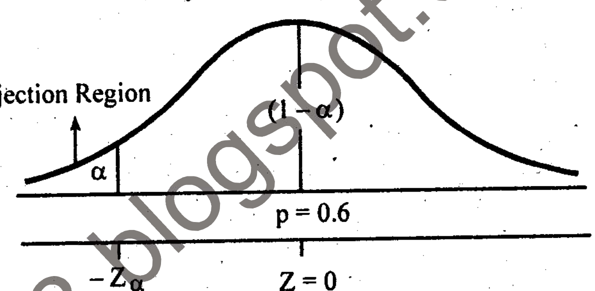

Figure 13.7

Example 13.1.

Indicate the type of errors committed in the following cases:

(i) H_0: $\mu = 500$, H_1: $\mu \neq 500$. H_0 is rejected while H_0 is true.

(ii) H_0: $\mu = 500$, H_1: $\mu < 500$. H_0 is accepted while true value of $\mu = 600$.

Answer:

(i) The hypothesis $\mu = 500$ is true and it has been rejected. Type I error has been committed.

(ii) H_0 is false and has been accepted. Type II error has been committed.

13.6 GENERAL PROCEDURE FOR TESTING OF HYPOTHESIS

Following are the main steps involved in the testing of a hypothesis about the population parameter.

1. Formulating Null hypothesis H_0:

First of all we have to identify the problem and then we frame the hypothesis which we think shall be rejected. Suppose the population parameter is θ about which we have to frame the hypothesis. We specify a value θ_0 for the unknown parameter. The null hypothesis H_0 can be written in three ways as shown below:

(i) $H_o: \theta = \theta_0$ (ii) $H_o: \theta \leq \theta_0$ (iii) $H_o: \theta \geq \theta_0$

In some particular situation any one of the above three forms of H_0 is taken. The important thing about H_0 is that H_0 always contains some form of an equality sign such as '=', '≥', or '≤ '. As H_0 always contains sign of equality of some type, some people always write H_0 as $H_0: \theta = \theta_0$ and they do not write the inequality contained in H_0.

Alternative hypothesis H_1:

The alternative hypothesis H_1 is the opposite or complement of H_0. H_0 and H_1 combined together make the entire sampling distribution. Both H_0 and H_1 are equally important and they are to be defined properly and clearly. As H_1 is complement of H_0, therefore H_1 stands decided when H_0 has been fixed. For example, for each value of H_0, the corresponding value of H_1 is given below:

(i) If $H_0: \theta = \theta_0$ then $H_1: \theta \neq \theta_0$

(ii) If $H_0: \theta \leq \theta_0$ then $H_1: \theta > \theta_0$

(iii) If $H_0: \theta \geq \theta_0$ then $H_1: \theta < \theta_0$

2. Level of significance α:

It is the probability of rejecting H_0 when H_0 is true. It is denoted by α. It makes the size of the critical region.

3. Test-statistic:

The *test statistic* depends upon the shape of the sampling distribution of the statistic. If the sampling distribution is a normal distribution, the test-statistic to be used is Z and if it is a t-distribution, the test-statistic to be used is t. Other test statistics are F and χ^2 (chi-square).

4. Critical region:

Critical region or rejection region is decided by H_1. The size of critical region is equal to α.

(i) If the alternative hypothesis is $H_1: \theta \neq \theta_0$ the rejection region is taken in both ends of the sampling distribution. Each side has rejection region equal to $\alpha/2$. It is called two-sided rejection region. The rejection regions are separated by the two critical values.

(ii) When H_1 is $\theta > \theta_0$, then rejection region of size α is taken only in the right side. It is called *one-sided to the right.* The rejection region is separated from the acceptance region by a critical value of test-statistic.

(iii) When H_1 is $\theta < \theta_0$, the rejection region of size α is taken only on the left side. It is called *one-sided to the left.*

5. Computations:

The relevant test-statistic is calculated from the sample data. The calculated value is to be compared with the tabulated value.

6. Conclusion:

If the calculated value of test-statistic lies in the rejection region, the null hypothesis H_0 is rejected and H_1 is accepted. If the calculated value of the test-statistic falls in the acceptance region, we say that H_0 is accepted but it is not acceptance in the real sense of the word. The word acceptance only means that the sample has not provided sufficient information against the null hypothesis.

13.7 HYPOTHESIS TESTING - POPULATION MEAN μ, σ KNOWN (LARGE SAMPLE)

Suppose a population has the mean μ which is unknown and the standard deviation σ which is known. A large sample of size n is selected from the population and sample mean $\bar{X}$ is calculated. We are required to test a hypothesis that the population mean μ has the specified value μ_o. The steps of the procedure are listed below:

1. We frame the null hypothesis H_0 and the alternative hypothesis H_1. Three different forms of H_0 and H_1 are possible which are:

   (a) H_0: $\mu = \mu_0$ and H_1: $\mu \neq \mu_0$ (b) H_0: $\mu \leq \mu_0$ and H_1: $\mu > \mu_0$

   (c) H_0: $\mu \geq \mu_0$ and H_1 : $\mu < \mu_0$
2. Level of significance α is decided.
3. **Test-statistic:**

   • When sample size is large, the sampling distribution of $\bar{X}$ has the normal distribution with mean μ and the standard error $\sigma/\sqrt{n}$. The population may or may not be normal. The test-statistic to be used is Z where $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$
4. **Critical region:**

   The critical region depends upon the alternative hypothesis. There are three possible rejection plans. We discuss all the three turn by turn.

(a) When H_1 is $\mu \neq \mu o$, the rejection region equal to $\alpha/2$ in size is taken on both ends of the sampling distribution as shown in Fig. 13.8. The critical values of Z which separates the

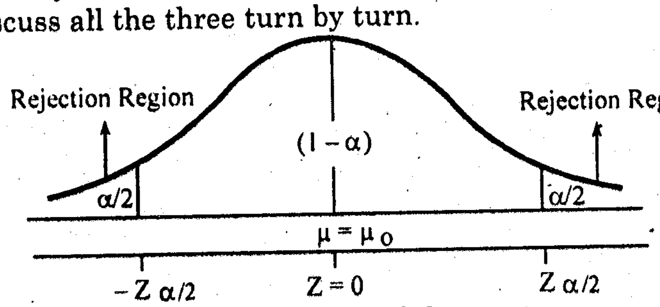

Figure 13.8

critical regions from the central acceptance region are $-Z_{\alpha/2}$ and $Z_{\alpha/2}$. The critical value $-Z_{\alpha/2}$ has the area on its left equal to $\alpha/2$ and the critical value $+Z_{\alpha/2}$ has area on its right equal to $\alpha/2$. H_0 is rejected if the calculated value of Z lies in rejection region. The rejection region is $Z < -Z_{\alpha/2}$ and $Z > Z_{\alpha/2}$. When $\alpha = 0.05$, then $-Z_{\alpha/2} = -Z_{0.025} = -1.96$ and $Z_{0.025} = 1.96$.

(b) When H_1 is $\mu > \mu_0$, the rejection region equal to α is taken in the right end of the distribution as shown in Fig. 13.9. The test plan is called one-tailed to the right.

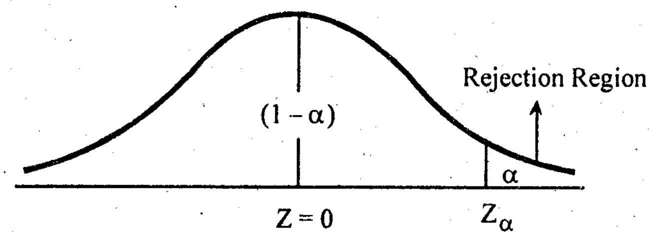

Figure 13.9

The hypothesis is rejected when the calculated value of Z is greater than Z_α, where Z_α is the critical point on the right of which the area is equal to α.

(c) When H_1 is $\mu < \mu_0$, the rejection region equal to α is taken in the left end of the distribution as shown in Fig. 13.10. The rejection plan is called one-tailed to the left.

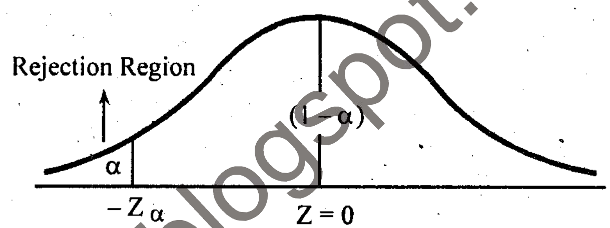

Figure 13.10

The hypothesis is rejected when the calculated value of Z is less than the critical value $-Z_\alpha$ where $-Z_\alpha$ is a critical point on the left of which the area is α. The rejection region is $Z < -Z_\alpha$. Corresponding to each null hypothesis, the alternate hypothesis and the rejection regions are given below:

Null hypothesis	Alternative hypothesis	Rejection region
(a) $H_0 : \mu = \mu_0$	$H_1 : \mu \neq \mu_0$ (two-sided)	$Z < -Z_{\alpha/2}$ and $Z > Z_{\alpha/2}$
(b) $H_0 : \mu \leq \mu_0$	$H_1 : \mu > \mu_0$ (one-sided)	$Z > Z_\alpha$
(c) $H_0 : \mu \geq \mu_0$	$H_1 : \mu < \mu_0$ (one-sided)	$Z < -Z_\alpha$

5. Computations:

The value of Z is calculated by using the formula: $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$

6. Conclusion:

If the value of Z lies in the acceptance region, the hypothesis is accepted. But acceptance is just an indication that the sample data has failed to provide evidence against the null hypothesis. If the value of Z lies in rejection region the hypothesis is rejected. When H_0 is rejected, there is only 100 α % chance that the null hypothesis is true.

Example 13.2.

Past records show that the average score of students in statistics is 57 with standard deviation 10. A new method of teaching is employed and a random sample of 70 students is selected. The sample average is 60. Can we conclude on the basis of these results, at 5 % level of significance, that the average score has increased?

Solution:

1. **Null hypothesis:** $H_0 : \mu = 57$ **Alternative hypothesis:** $H_1 : \mu > 57$
2. **Level of significance:** $\alpha = 0.05$
3. **Test – statistic:** $Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$
4. **Critical region:** $Z > 1.645$. Here we use one-sided test to the right. The hypothesis H_0: $\mu = 57$ will be rejected if Z lies in rejection region.

   (From the area table of normal distribution, we have $Z_\alpha = Z_{0.05} = 1.645$)
5. **Computations:** Here $n = 70$, $\bar{X} = 60$, $\sigma = 10$, and hence

   $$Z = \frac{60 - 57}{10/\sqrt{70}} = \frac{3}{10}\sqrt{70} = 2.51$$
6. **Conclusion:** Since the calculated value of Z = 2.51 falls in the critical region, so we reject our null hypothesis H_0: $\mu = 57$ at 5 % level of significance and we may conclude that the average score has increased.

Example 13.3.

An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed with a mean of 812 hours and a standard deviation of 40 hours. Test the hypothesis that $\mu = 812$ hours against the alternative $\mu \neq 812$ hours if a random sample of 36 bulbs has an average life of 800 hours. Use a 5 % level of significance.

Solution:

1. **Null hypothesis:** $H_0 : \mu = 812$ **Alternative hypothesis:** $H_1: \mu \neq 812$
2. **Level of significance:** $\alpha = 0.05$
3. **Test – statistic:** $Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$
4. **Critical region:** $|Z| > 1.96$ ($Z < -1.96$ and $Z > 1.96$)

   (From the area table of normal distribution, we have $Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$)

5. **Computations:** Here $n = 36$, $\bar{X} = 800$, $\sigma = 40$, and hence

$$Z = \frac{800 - 812}{40/\sqrt{36}} = -\frac{12}{40}(6) = -1.8$$

6. **Conclusion:** Since the calculated value of $Z = -1.8$ falls in the acceptance region. Thus $H_0: \mu = 812$ is not rejected.

13.8 HYPOTHESIS TESTING — POPULATION MEAN μ — σ NOT KNOWN (LARGE SAMPLE)

This is an important case in which σ is not known. When sample size n is large, the population may be normal or not, the sampling distribution of $\bar{X}$ has the normal distribution with mean μ and standard error $\sigma/\sqrt{n}$. But when σ is unknown, it is estimated by the sample standard deviation S and the estimated standard error is $S/\sqrt{n}$. The Z-statistic becomes $Z = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ where $S^2 = \frac{\Sigma(X - \bar{X})^2}{n}$. The remaining procedure is exactly the same as discussed earlier. The only difference is that S is used in place of σ in the calculation of Z.

Example 13.4.

A home heating oil delivery company would like to estimate the annual usage for its customers who live in single-family homes. A sample of 100 customers indicated an average annual usage of 1103 gallons and a sample standard deviation of 327.8 gallons. At the 1 % level of significance, is there evidence that the average annual usage exceeds 1000 gallons per year?

Solution:

1. **Null hypothesis:** $H_0: \mu \le 1000$ **Alternative hypothesis:** $H_1: \mu > 1000$
2. **Level of significance:** $\alpha = 0.01$
3. **Test – statistic:** $Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$
4. **Critical region:** $Z > 2.326$

   (From the area table of normal distribution, we have $Z_\alpha = Z_{0.01} = 2.326$)

5. **Computations:** Here $n = 100$, $\bar{X} = 1103$, $S = 327.8$, and hence

$$Z = \frac{1103 - 1000}{327.8/\sqrt{100}} = \frac{103}{327.8}(10) = 3.14$$

6. **Conclusion:** Since the calculated value of $Z = 3.14$ falls in the critical region, so we reject our null hypothesis $H_0: \mu \le 1000$ at 1 % level of significance and we may conclude that the average annual usage exceeds 1000 gallons per year.

Example 13.5.

A sample of 42 measurements was taken in order to test the null hypothesis that the population mean equals 8.5 against the alternative that it is different from 8.5. The sample mean and standard deviation were found to be 8.79 and 1.27, respectively. Perform the hypothesis test using 0.01 as the level of significance.

Solution:

1. **Null hypothesis:** $H_0 : \mu = 8.5$ **Alternative hypothesis:** $H_1 : \mu \neq 8.5$
2. **Level of significance:** $\alpha = 0.01$
3. **Test – statistic:** $Z = \dfrac{\bar{X} - \mu_0}{S / \sqrt{n}}$
4. **Critical region:** $|Z| > 2.575$ ($Z < -2.575$ and $Z > 2.575$)
   (From the area table of normal distribution, we have $Z_{\frac{\alpha}{2}} = Z_{0.005} = 2.575$)
5. **Computations:** Here $n = 42$, $\bar{X} = 8.79$, $S = 1.27$, and hence
   $$Z = \frac{8.79 - 8.5}{1.27 / \sqrt{42}} = \frac{0.29}{1.27}\sqrt{42} = 1.48$$
6. **Conclusion:** Since the calculated value of $Z = 1.48$ falls in the acceptance region, so we accept our null hypothesis $H_0 : \mu = 8.5$ at 1 % level of significance.

13.9 HYPOTHESIS TESTING – POPULATION MEAN μ, σ KNOWN – NORMAL POPULATION (SMALL SAMPLE)

Sometimes the hypothesis about the population which is normal and its standard deviation σ is known. In this case Z-test is used both for small and large sample size. Thus $Z = \dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}}$. The procedure for testing of population mean μ is the same as discussed earlier.

13.10 HYPOTHESIS TESTING – POPULATION MEAN μ, σ UNKNOWN – NORMAL POPULATION (SMALL SAMPLE)

When the standard deviation of the population is not known, it is estimated by the sample standard deviation 's' where $s = \sqrt{\dfrac{1}{n-1}\Sigma(X - \bar{X})^2}$. The procedure runs as follows:

The different forms of hypotheses are

1. (a) $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$
   (b) $H_0 : \mu \leq \mu_0$ and $H_1 : \mu > \mu_0$
   (c) $H_0 : \mu \geq \mu_0$ and $H_1 : \mu < \mu_0$
2. Level of significance α is decided.

3. Test – statistic:

When population is normal and sample size n is small, the sampling distribution of $\bar{X}$ has the t-distribution with (n – 1) degrees of freedom. The test-statistic is $t = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$.

4. Critical region:

The critical region is based on the alternative hypothesis.

(a) For the alternative hypothesis $H_1 : \mu \neq \mu_0$, the rejection region is two-sided as shown in Fig. 13.11. The two critical values $-t_{\alpha/2\,(n-1)}$ and $t_{\alpha/2\,(n-1)}$ are seen from the t-table below α/2 and against (n – 1) degrees of freedom. The critical region is $t > t_{\alpha/2(n-1)}$ or $t < -t_{\alpha/2(n-1)}$ as show in Fig. 13.11.

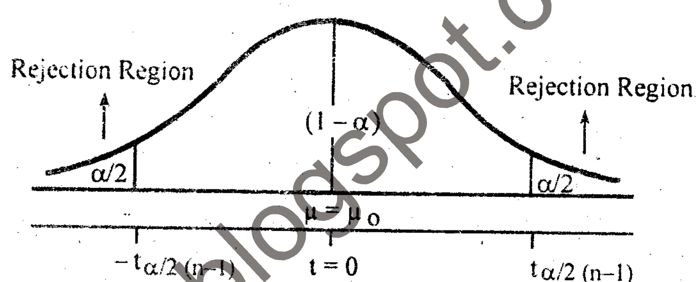

Figure 13.11

(b) When H_1 is $\mu > \mu_0$, the rejection region is taken on the extreme right side of the sampling distribution as shown in Fig. 13.12. The critical value $t_{\alpha\,(n-1)}$ is seen from the t-table below α and against (n – 1) degrees of freedom. The critical region is $t > t_{\alpha\,(n-1)}$.

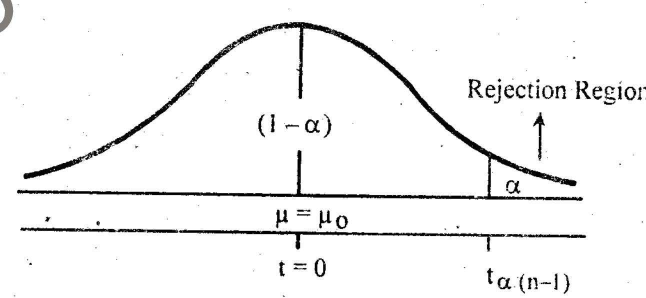

Figure 13.12

(c) When H_1 is $\mu < \mu_0$, the entire rejection region is taken on the left side of the sampling distribution as shown in Fig. 13.13. The critical value $t_{\alpha(n-1)}$ is seen from the t-table below α and against (n – 1) degrees of freedom. The critical region is $t < -t_{\alpha\,(n-1)}$.

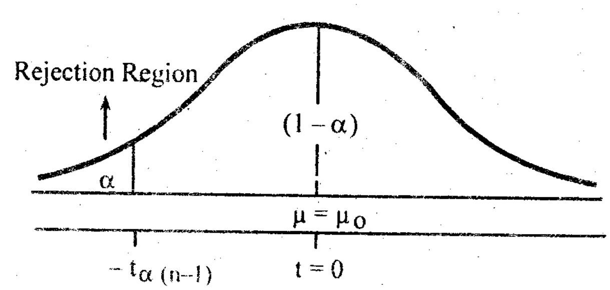

Figure 13.13

5. **Computations:**

The test-statistic 't' is calculated from the sample data where $t = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$

6. **Conclusion:**

The null hypothesis H_0 is rejected in favour of H_1 when the value of t lies in the rejection region. H_0 is accepted when the value of t lies in acceptance region.

Example 13.6.

A manufacturing company making automobile tires claims that the average life of its product is 35000 miles. A random sample of 16 tires was selected; and it was found that the mean life was 34000 miles with a standard deviation s = 2000 miles. Test hypothesis H_0: $\mu = 35000$ against the alternative H_1: $\mu < 35000$ at $\alpha = 0.05$.

Solution:

1. **Null hypothesis:** $H_0 : \mu = 35000$ **Alternative hypothesis:** H_1: $\mu < 35000$
2. **Level of significance:** $\alpha = 0.05$
3. **Test – statistic:** $t = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$
4. **Critical region:** $t < -1.753$
   (From the t–table, we have $-t_{\alpha(n-1)} = -t_{0.05(15)} = -1.753$)
5. **Computations:** Here n = 16, $\bar{X} = 34000$, s = 2000, and hence
   $$t = \frac{34000 - 35000}{2000/\sqrt{16}} = \frac{-1000}{2000}(4) = -2$$
6. **Conclusion:** Since the calculated value of t = –2 falls in the critical region, so we reject our null hypothesis H_0: $\mu = 35000$ at 5 % level of significance.

Example 13.7.

A random sample of 8 cigarettes of a certain brand has an average nicotine content of 4.2 milligrams and a standard deviation of 1.4 milligrams. Is this in line with the manufacturer's claim that the average nicotine content does not exceed 3.5 milligrams? Use 1 % level of significance and assume the distribution of nicotine contents to be normal.

Solution:

1. **Null hypothesis:** $H_0 : \mu \le 3.5$ **Alternative hypothesis:** $H_1 : \mu > 3.5$
2. **Level of significance:** $\alpha = 0.01$
3. **Test – statistic:** $t = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$

4. **Critical region:** $t > 2.998$

(From the t-table, we have $t_{\alpha(n-1)} = t_{0.01(7)} = 2.998$)

5. **Computations:** Here $n = 8$, $\bar{X} = 4.2$, $s = 1.4$, and hence

$$t = \frac{4.2 - 3.5}{1.4/\sqrt{8}} = \frac{0.7}{1.4}\sqrt{8} = 1.414$$

6. **Conclusion:** Since the calculated value of $t = 1.414$ falls in the acceptance region, so we accept our null hypothesis H_0: $\mu = 35$ at 1 % level of significance.

13.11 HYPOTHESIS TESTING – DIFFERENCE BETWEEN TWO POPULATION MEANS $\mu_1 - \mu_2$, σ_1^2 AND σ_2^2 KNOWN (LARGE SAMPLES)

Suppose there are two populations (normal or non-normal) with means μ_1 and μ_2 which are unknown and the variances σ_1^2 and σ_2^2 which are known. Two large random samples of sizes n_1 and n_2 are selected from the populations and the sample means $\bar{X}_1$ and $\bar{X}_2$ are calculated. The difference $(\bar{X}_1 - \bar{X}_2)$ is a random variable and its distribution is normal with mean $\mu_1 - \mu_2$ and standard error $\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$.

The procedure for testing the hypothesis $\mu_1 - \mu_2 = 0$ is explained below.

1. The null and the alternative hypotheses which are possible are

   (a) $H_0 : \mu_1 - \mu_2 = 0$ (or $\mu_1 = \mu_2$) and $H_1 : \mu_1 - \mu_2 \neq 0$ (or $\mu_1 \neq \mu_2$)

   (b) $H_0 : \mu_1 - \mu_2 \leq 0$ (or $\mu_1 \leq \mu_2$) and $H_1 : \mu_1 - \mu_2 > 0$ (or $\mu_1 > \mu_2$)

   (c) $H_0 : \mu_1 - \mu_2 \geq 0$ (or $\mu_1 \geq \mu_2$) and $H_1 : \mu_1 - \mu_2 < 0$ (or $\mu_1 < \mu_2$)

2. Level of significance α is decided.

3. **Test – statistic:**

   The distribution of $(\bar{X}_1 - \bar{X}_2)$ is normal, therefore the test-statistic to be used is Z,

   where $$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

4. **Critical region:**

   For each alternate hypothesis H_1, there is a rejection plan as explained earlier.

5. **Computations:**

The Z-statistic is calculated using the sample data where,

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Sometimes the null hypothesis states some difference between μ_1 and μ_2 and the difference is denoted by Δ. In that case H_o is $\mu_1 - \mu_2 = \Delta$ (say) and

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - \Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

6. **Conclusion:**

The hypothesis is rejected if the calculated value of Z lies in rejection region. If Z lies in acceptance region, the hypothesis is accepted.

Example 13.8.

Suppose you wish to estimate the effects of a certain sleeping pill on men and women. Two samples are independently taken, and the relevant data are shown below:

	Men	Women
Sample size	$n_1 = 36$	$n_2 = 64$
Sample mean	$\bar{X}_1 = 8.75$	$\bar{X}_2 = 7.25$
Population variance	$\sigma_1^2 = 9$	$\sigma_2^2 = 4$

Test the null hypothesis H_o: $\mu_1 = \mu_2$ against the alternative hypothesis H_1: $\mu_1 > \mu_2$ at $\alpha = 0.05$.

Solution:

1. **Null hypothesis:** $H_o : \mu_1 = \mu_2$ **Alternative hypothesis:** $H_1 : \mu_1 > \mu_2$
2. **Level of significance:** $\alpha = 0.05$
3. **Test – statistic:** $$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$
4. **Critical region:** $Z > 1.645$
   (From the area table of normal distribution, we have $Z_\alpha = Z_{0.05} = 1.645$)
5. **Computations:** Here $n_1 = 36$, $\bar{X}_1 = 8.75$, $\sigma_1^2 = 9$, $n_2 = 64$, $\bar{X}_2 = 7.25$, $\sigma_2^2 = 4$,

   and hence $$Z = \frac{(8.75 - 7.25) - 0}{\sqrt{\frac{9}{36} + \frac{4}{64}}} = \frac{1.5}{0.5590} = 2.683$$

6. **Conclusion:** Since the calculated value of Z = 2.683 falls in the critical region, so we reject our null hypothesis H_0: $\mu_1 = \mu_2$ at 5 % level of significance.

Example 13.9.

Two astronomers recorded observations on a certain star. The mean of 30 observations obtained by first astronomer is 8.85 and mean of 40 observations made by second astronomer is 8.20. Past experience shows that each astronomer obtained readings with variance of 1.2. Using $\alpha = 0.01$, can we say that the difference between two results is significant.

Solution:

1. **Null hypothesis:** $H_0 : \mu_1 = \mu_2$ **Alternative hypothesis:** $H_1 : \mu_1 \neq \mu_2$
2. **Level of significance:** $\alpha = 0.01$
3. **Test – statistic:**

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$(\because \ \sigma_1^2 = \sigma_2^2 = \sigma^2)$$

4. **Critical region:** $|Z| > 2.575$ ($Z < -2.575$ and $Z > 2.575$)
   (From the area table of normal distribution, we have $Z_{\frac{\alpha}{2}} = Z_{0.005} = 2.575$)
5. **Computations:** Here $n_1 = 30$, $\bar{X}_1 = 8.85$, $n_2 = 40$, $\bar{X}_2 = 8.20$, $\sigma^2 = 1.2$, $\sigma = 1.10$

and hence $$Z = \frac{(8.85 - 8.20) - 0}{1.10\sqrt{\frac{1}{30} + \frac{1}{40}}} = \frac{0.65}{0.27} = 2.407$$

6. **Conclusion:** Since the calculated value of Z = 2.407 falls in the acceptance region, so we accept our null hypothesis H_0: $\mu_1 = \mu_2$ at 1 % level of significance. We may conclude that the difference between two results is insignificant.

13.12 HYPOTHESIS TESTING – DIFFERENCE BETWEEN TWO POPULATION MEANS $\mu_1 - \mu_2$, σ_1^2 AND σ_2^2 UNKNOWN (LARGE SAMPLES)

When the population variances σ_1^2 and σ_2^2 are unknown, they are estimated by their sample variances S_1^2 and S_2^2 and the test-statistic to be used becomes,

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

This formula is used only for large sample sizes but the populations may or may not be normal. The procedure for testing H_0 is the same as explained earlier.

Example 13.10.

Suppose that two randomly selected samples yield the following information:

	Sample I	Sample II
Size	$n_1 = 82$	$n_2 = 41$
Mean	$\bar{X}_1 = 50$	$\bar{X}_2 = 55$
Variance	$S_1^2 = 405$	$S_2^2 = 324$

Test the null hypothesis that the two population means are equal that is, $H_0: \mu_1 = \mu_2$ against the alternative hypothesis $H_1: \mu_1 < \mu_2$ at $\alpha = 0.01$.

Solution:

1. **Null hypothesis:** $H_0 : \mu_1 = \mu_2$ **Alternative hypothesis:** $H_1 : \mu_1 < \mu_2$
2. **Level of significance:** $\alpha = 0.01$
3. **Test – statistic:** $Z = \dfrac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$
4. **Critical region:** $Z < -2.326$
   (From the area table of normal distribution, we have $-Z_{\alpha} = -Z_{0.01} = -2.326$)
5. **Computations:** Here $n_1 = 82$, $\bar{X}_1 = 50$, $S_1^2 = 405$, $n_2 = 41$, $\bar{X}_2 = 55$, $S_2^2 = 324$,
   and hence $Z = \dfrac{(50 - 55) - 0}{\sqrt{\dfrac{405}{82} + \dfrac{324}{41}}} = \dfrac{-5}{3.58} = -1.40$
6. **Conclusion:** Since the calculated value of $Z = -1.40$ falls in the acceptance region, so we accept our null hypothesis $H_0: \mu_1 = \mu_2$ at 1 % level of significance.

13.13 TEST ABOUT $\mu_1 - \mu_2$, σ_1^2 AND σ_2^2 KNOWN, POPULATIONS NORMAL (SMALL SAMPLES)

In case of small sample sizes, we can use Z-test for testing the difference between μ_1 and μ_2 when σ_1^2 and σ_2^2 are known and the populations are necessarily normal. The Z-test used is $Z = \dfrac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$

13.14 TEST ABOUT $\mu_1 - \mu_2$, σ_1^2 AND σ_2^2 NOT KNOWN, POPULATIONS NORMAL (SMALL SAMPLES)

This is a case which is different from the previous three cases. Here the conditions are that:

(i) the populations are normal

(ii) σ_1^2 and σ_2^2 are unknown but assumed to be equal.

(iii) the sample sizes n_1 and n_2 are small and are selected independently.

The variances σ_1^2 and σ_2^2 are unknown but $\sigma_1^2 = \sigma_2^2 = \sigma^2$. The parameter σ^2 is estimated by the sample variances. The sample estimator of σ^2 is s_p^2, where

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = \frac{\Sigma(X_1-\bar{X}_1)^2 + \Sigma(X_2-\bar{X}_2)^2}{n_1+n_2-2}$$

and $$s_p = \sqrt{\frac{\Sigma(X_1-\bar{X}_1)^2 + \Sigma(X_2-\bar{X}_2)^2}{n_1+n_2-2}}$$

s_p^2 is called pooled estimator of the common population variance σ^2. The difference $(\bar{X}_1 - \bar{X}_2)$ has the t – distribution with $(n_1 + n_2 - 2)$ degrees of freedom where

$$t = \frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{\sqrt{\frac{s_p^2}{n_1}+\frac{s_p^2}{n_2}}} = \frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

The tabulated value of 't' for $n_1 + n_2 - 2$ degrees of freedom is seen from the t-table.

For $H_1 : \mu_1 \neq \mu_2$ the critical values are $-t_{\alpha/2\,(n_1+n_2-2)}$ and $t_{\alpha/2\,(n_1+n_2-2)}$

For $H_1 : \mu_1 > \mu_2$ the critical value is $t_{\alpha\,(n_1+n_2-2)}$

and For $H_1 : \mu_1 < \mu_2$ the critical value is $-t_{\alpha\,(n_1+n_2-2)}$

The null hypothesis H_0 is rejected when the calculated value of t lies in rejection region.

Example 13.11.

Two samples are randomly selected from two classes of students who have been taught by different methods. An examination is given and the results are shown as follows:

	Class I	Class II
Sample Size	$n_1 = 8$	$n_2 = 10$
Mean	$\bar{X}_1 = 95$	$\bar{X}_2 = 97$
Variance	$s_1^2 = 47$	$s_2^2 = 30$

On the assumption that the test scores of the two classes of students have identical variances, determine whether the two different methods of teaching are equally effective at $\alpha = 0.01$.

Solution:

1. **Null hypothesis:** $H_0 : \mu_1 = \mu_2$ **Alternative hypothesis:** $H_1 : \mu_1 \neq \mu_2$

2. **Level of significance:** $\alpha = 0.01$

3. **Test – statistic:**
$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

4. **Critical region:** $|t| > 2.921$ $(t < -2.921 \text{ and } t > 2.921)$
(From the t–table, we have $t_{\frac{\alpha}{2}(n_1 + n_2 - 2)} = t_{0.005(16)} = 2.921$)

5. **Computations:** Here $n_1 = 8$, $\bar{X}_1 = 95$, $s_1^2 = 47$, $n_2 = 10$, $\bar{X}_2 = 97$, $s_2^2 = 30$,
$$s_p^2 = \frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1 + n_2 - 2} = \frac{(8-1)\,47 + (10-1)\,30}{8 + 10 - 2} = \frac{599}{16} = 37.4375,$$
$$s_p = \sqrt{37.4375} = 6.12, \text{ and hence } t = \frac{(95 - 97) - 0}{6.12\sqrt{\frac{1}{8} + \frac{1}{10}}} = \frac{-2}{2.9030} = -0.689$$

6. **Conclusion:** Since the calculated value of $t = -0.689$ falls in the acceptance region, so we accept our null hypothesis H_0: $\mu_1 = \mu_2$ at 1 % level of significance. On the basis of the evidence, we may conclude that the two different methods of teaching are equally effective.

13.15 TEST ABOUT $\mu_1 - \mu_2$, DEPENDENT SAMPLES, POPULATIONS NORMAL

Suppose there are two populations with mean μ_1 and μ_2 which are unknown. Two random samples of sizes n_1 and n_2 are selected. It is further assumed that the samples are dependent. Suppose we record blood pressures of a sample of some patients. The patients are given a treatment for some period and again their blood pressures are recorded. These two sets of observations are called dependent samples. The first set of observations is called 'before' and the second set of observations is called 'after' observations. These observations are in pairs. If $X_1, X_2, X_3, \cdots, X_n$ are the 'before' observations and $Y_1, Y_2, Y_3, \cdots, Y_n$ are the 'after' observations, then the paired observations are $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), \cdots, (X_n, Y_n)$. Let us find the difference between the paired values. Let difference $d_1 = X_1 - Y_1$, $d_2 = X_2 - Y_2$, $d_3 = X_3 - Y_3, \cdots, d_n = X_n - Y_n$

The mean of the sample 'd' values is denoted by $\bar{d}$. Suppose the corresponding parameter of the difference between paired observations in the populations is denoted by μ_D. The various steps of the procedure are:

1. Three different forms of null and alternative hypotheses are

   (a) $H_0 : \mu_D = 0$ (or $\mu_1 = \mu_2$) and $H_1 : \mu_D \neq 0$ (or $\mu_1 \neq \mu_2$)

   (b) $H_0 : \mu_D \leq 0$ (or $\mu_1 \leq \mu_2$) and $H_1 : \mu_D > 0$ (or $\mu_1 > \mu_2$)

   (c) $H_0 : \mu_D \geq 0$ (or $\mu_1 \geq \mu_2$) and $H_1 : \mu_D < 0$ (or $\mu_1 < \mu_2$)

   Sometimes we have to examine that the differences of the paired observations in the population have some specified value say Δ. In that case $\mu_D = \Delta$.

2. Level of significance α is decided.

3. **Test-statistic:**

   $\bar{d}$ has the t-distribution with $(n - 1)$ degrees of freedom.

$$t = \frac{\bar{d} - d_0}{s_d / \sqrt{n}} \quad \text{where } s_d = \sqrt{\frac{\Sigma(d - \bar{d})^2}{n - 1}} = \sqrt{\frac{1}{n-1}\left[\Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n}\right]}$$

4. **Critical region:**

   Corresponding to each H_1, there is a critical region.

5. **Computations:** The test-statistic t is calculated where $t = \dfrac{\bar{d} - d_0}{s_d / \sqrt{n}}$.

   When H_0 is $\mu_D = 0$, then $t = \dfrac{\bar{d} - 0}{s_d / \sqrt{n}} = \dfrac{\bar{d}\sqrt{n}}{s_d}$

6. **Conclusion:**

   The hypothesis $\mu_D = 0$ is rejected if the calculated value of 't' lies in the rejection region.

Example 13.12.

Suppose that a shoe company wanted to test material for the sales of shoes. For each pair of shoes the new material was placed on one shoe and the old material was placed on the other shoe. After a given period of time a random sample of ten pairs of shoes was selected and the wear was measured on a ten-point scale with the following results:

Pair number	1	2	3	4	5	6	7	8	9	10
New material	2	4	5	7	7	5	9	8	8	7
Old material	4	5	3	8	9	4	7	8	5	6
Differences	−2	−1	+2	−1	−2	+1	+2	0	+3	+1

At the 0.05 level of significance, is there evidence that the average wear is higher for the new material than the old material?

Solution:

1. **Null hypothesis:** $H_0 : \mu_{new} \le \mu_{old}$ or $\mu_D = \mu_{new} - \mu_{old} \le 0$

   Alternative hypothesis: $H_1 : \mu_{new} > \mu_{old}$ or $\mu_D = \mu_{new} - \mu_{old} > 0$

2. **Level of significance:** $\alpha = 0.05$

3. **Test – statistic:** $t = \dfrac{\bar{d} - d_0}{s_d / \sqrt{n}}$

4. **Critical region:** $t > 1.833$

   (From the t-table, we have $t_{\alpha(n-1)} = t_{0.05(9)} = 1.833$)

5. **Computations:** Let X_1 = new material and X_2 = old material.

   The necessary calculations are given below:

X_1	2	4	5	7	7	5	9	8	8	7
X_2	4	5	3	8	9	4	7	8	5	6
$d = X_1 - X_2$	−2	−1	+2	−1	−2	+1	+2	0	+3	+1
d^2	4	1	4	1	4	1	4	0	9	1

Here $n = 10$, $\Sigma d = 3$, $\Sigma d^2 = 29$, $\bar{d} = \dfrac{\Sigma d}{n} = \dfrac{3}{10} = 0.3$,

$$s_d^2 = \frac{1}{n-1}\left[\Sigma d^2 - \frac{(\Sigma d)^2}{n}\right] = \frac{1}{10-1}\left[29 - \frac{(3)^2}{10}\right]$$

$= 3.1222$, $s_d = 1.77$, and hence

$$t = \frac{0.3 - 0}{1.77/\sqrt{10}} = \frac{0.3}{1.77}\sqrt{10} = 0.536$$

6. **Conclusion:** Since the calculated value of $t = 0.536$ falls in the acceptance region, so we accept our null hypothesis $H_0: \mu_{new} \le \mu_{old}$ at 5 % level of significance. On the basis of the evidence, we may conclude that the average wear is not higher for the new material than the old material.

Example 13.13.

Two varieties of wheat are each planted in ten localities with differences in yield as follows: 2, 4, 2, 2, 3, 6, 2, 2, 4, 3. Test the hypothesis that the population mean difference is zero, using $\alpha = 0.01$.

Solution:

1. **Null hypothesis:** $H_0 : \mu_1 = \mu_2$ or $\mu_D = \mu_1 - \mu_2 = 0$

   Alternative hypothesis: $H_1 : \mu_1 \ne \mu_2$ or $\mu_D = \mu_1 - \mu_2 \ne 0$

2. **Level of significance:** $\alpha = 0.01$

3. **Test – statistic:** $t = \dfrac{\bar{d} - d_o}{s_d / \sqrt{n}}$

4. **Critical region:** $|t| > 3.250$ ($t < -3.250$ and $t > 3.250$)
(From the t–table, we have $t_{\frac{\alpha}{2}(n-1)} = t_{0.005(9)} = 3.250$)

5. **Computations:** Here $n = 10, \Sigma d = 30, \Sigma d^2 = 106, \bar{d} = \dfrac{\Sigma d}{n} = \dfrac{30}{10} = 3,$

$$s_d^2 = \frac{1}{n-1}\left[\Sigma d^2 - \frac{(\Sigma d)^2}{n}\right] = \frac{1}{10-1}\left[106 - \frac{(30)^2}{10}\right]$$

$= 1.7778,$ $s_d = 1.33$, and hence

$$t = \frac{3-0}{1.33/\sqrt{10}} = \frac{3}{1.33}\sqrt{10} = 7.133$$

6. **Conclusion:** Since the calculated value of $t = 7.133$ falls in the critical region, so we reject our null hypothesis $H_0: \mu_1 = \mu_2$ at 1 % level of significance.

13.16 TEST OF POPULATION PROPORTION p (LARGE SAMPLE)

Let us consider a binomial population with a proportion p which is unknown and we have to test a hypothesis about the unknown population parameter. A random sample of size n (n > 30) is selected from the population and the sample proportion $\hat{p}$ is calculated. When sample size is large, the distribution of $\hat{p}$ is normal with mean p and standard error $\sqrt{\dfrac{p\,q}{n}}$. The random variable Z can be calculated from $\hat{p}$. Thus $Z = \dfrac{\hat{p} - p}{\sqrt{\dfrac{p\,q}{n}}}$.

The random variable Z is used as test statistic and the value of Z makes a base for the acceptance or rejection of the null hypothesis about the population proportion. The procedure for testing p runs as below:

1. We frame a hypothesis about the population proportion p. Let us specify a value p_0 for the population parameter p. The null hypothesis H_0 and the alternative hypothesis H_1 can take any one of the following three forms:

   (a) $H_0 : p = p_0$ and $H_1 : p \neq p_0$ (b) $H_0 : p \leq p_0$ and $H_1 : p > p_0$

   (c) $H_0 : p \geq p_0$ and $H_1 : p < p_0$

2. Level of significance is decided. It is denoted by α.

3. **Test-statistic:** Used in this case is $Z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0\, q_0}{n}}}$ where $q_0 = 1 - p_0$

The sample proportion $\hat{p}$ can also be written as $\hat{p} = \frac{X}{n}$, where 'X' is the number of successes in the sample of size n. Putting $\hat{p} = \frac{X}{n}$ in the above formula for Z,

we get
$$Z = \frac{\frac{X}{n} - p_o}{\sqrt{\frac{p_o q_o}{n}}} = \frac{\frac{X - n p_o}{n}}{\sqrt{\frac{p_o q_o}{n}}} = \frac{X - n p_o}{n\sqrt{\frac{p_o q_o}{n}}} = \frac{X - n p_o}{\sqrt{n p_o q_o}}$$

Thus $Z = \frac{X - n p_o}{\sqrt{n p_o q_o}}$ can also be used as *test-statistic* for testing population proportion p.

4. Critical region:

The critical region depends upon the alternative hypothesis H_1. The three forms of H_1 are:

(a) H_1 is $p \neq p_o$. In this case the rejection region is taken in both ends of the sampling distribution. The rejection region on each side is equal to $\alpha/2$. The two critical values $-Z_{\alpha/2}$ and $Z_{\alpha/2}$ separate the critical region from the acceptance region as shown in Fig. 13.14. H_o is rejected when the calculated value of Z lies in rejection region. H_o is rejected when $Z < -Z_{\alpha/2}$ or $Z > Z_{\alpha/2}$. The values between $-Z_{\alpha/2}$ and $Z_{\alpha/2}$ form the acceptance region. The test is called two – sided.

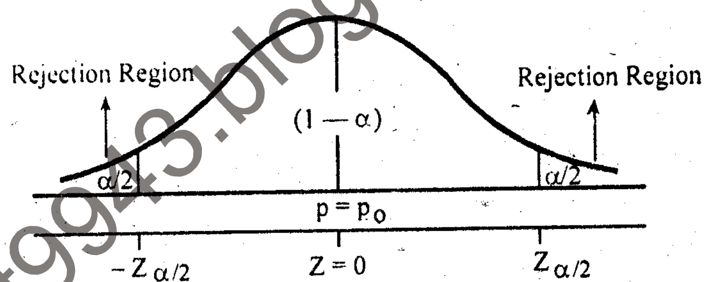

Figure 13.14

(b) $H_1 : p > p_o$. In this case the rejection region is taken only in the right side of the sampling distribution. The test is called one – sided to the right. The critical value between the acceptance region and the rejection region is Z_α as shown in Fig. 13.15. The values above Z_α form the critical region and the values less than Z_α form the acceptance region where as Z_α is the critical value and should not be used for acceptance or rejection of H_o.

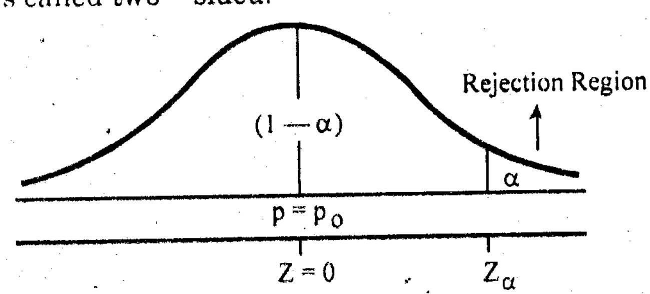

Figure 13.15

(c) When H_1 is $p < p_o$, the entire rejection region falls in the left side of the sampling distribution. The test is called one-sided to the left. The critical value $-Z_\alpha$ is a point between the critical region and the acceptance region as

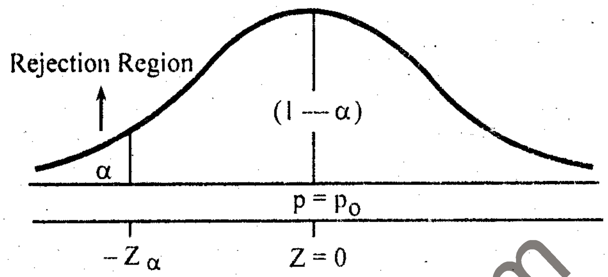

Figure 13.16

shown in Fig. 13.16. The value less than $-Z_\alpha$ form the critical region. H_0 is rejected when the Z value calculated from the sample data falls in the rejection region otherwise the null hypothesis H_0 is accepted with the usual meaning of the term 'acceptance'. The rejection region is $Z < -Z_\alpha$.

5. Computation   6. Conclusion

Example 13.14.

In a poll of 1000 voters selected at random from all the voters in a certain district, it is found that 518 voters are in favour of a particular candidate. Test the null hypothesis that the proportion of all the voters in the district who favour the candidate is equal to or less than 50 percent against the alternative that it is greater than 50 percent at $\alpha = 0.05$.

Solution:

1. **Null hypothesis:** $H_0 : p \leq 0.50$ **Alternative hypothesis:** $H_1 : p > 0.50$
2. **Level of significance:** $\alpha = 0.05$
3. **Test – statistic:** $Z = \dfrac{\hat{p} - p_o}{\sqrt{\dfrac{p_o q_o}{n}}}$
4. **Critical region:** $Z > 1.645$

   (From the area table of normal distribution, we have $Z_\alpha = Z_{0.05} = 1.645$)
5. **Computations:** Here $n = 1000$, $X = 518$, $\hat{p} = \dfrac{X}{n} = \dfrac{518}{1000} = 0.518$

   $p_o = 0.50$, $q_o = 1 - p_o = 0.50$, and hence

   $$Z = \frac{(0.518 - 0.50)}{\sqrt{\dfrac{(0.50)(0.50)}{1000}}} = \frac{0.018}{0.016} = 1.125$$
6. **Conclusion:** Since the calculated value of $Z = 1.125$ falls in the acceptance region, so we accept our null hypothesis H_0: $p \leq 0.50$ at 5 % level of significance.

Example 13.15.

At a certain college it is estimated that at most 25 % of the students ride bicycles to class. Does this seem to be a valid estimate, if in a random sample of 90 college students, 28 are found to ride bicycles to class? Use a 5 % level of significance.

Solution:

1. **Null hypothesis:** $H_0 : p \le 0.25$ **Alternative hypothesis:** $H_1 : p > 0.25$
2. **Level of significance:** $\alpha = 0.05$
3. **Test – statistic:** $$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}}$$
4. **Critical region:** $Z > 1.645$
   (From the area table of normal distribution, we have $Z_\alpha = Z_{0.05} = 1.645$)
5. **Computations:** Here $n = 90$, $X = 28$, $\hat{p} = \frac{X}{n} = \frac{28}{90} = 0.31$,
   $p_0 = 0.25$, $q_0 = 1 - p_0 = 0.75$, and hence
   $$Z = \frac{0.31 - 0.25}{\sqrt{\frac{(0.25)(0.75)}{90}}} = \frac{0.06}{0.0456} = 1.32$$
6. **Conclusion:** Since the calculated value of $Z = 1.32$ falls in the acceptance region, so we accept our null hypothesis H_0: $p \le 0.25$ at 5 % level of significance. On the basis of the evidence, we may conclude that at most 25 % of the students ride bicycles to class.

13.17 TEST OF DIFFERENCE BETWEEN TWO POPULATION PROPORTIONS, $p_1 - p_2$ (LARGE SAMPLES)

Suppose there are two binomial populations with proportions p_1 and p_2 which are unknown. Two independent large random samples of sizes n_1 and n_2 are selected from the populations and sample proportion $\hat{p}_1$ and $\hat{p}_2$ are calculated. The difference $(\hat{p}_1 - \hat{p}_2)$ is a random variable and has the normal distribution with mean $p_1 - p_2$ and standard error $\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$

The procedure for testing of the difference between p_1 and p_2 is given below:

1. Three forms of the hypotheses are as below:
   (a) $H_0 : p_1 - p_2 = 0$ (or $p_1 = p_2$) and $H_1 : p_1 - p_2 \ne 0$ (or $p_1 \ne p_2$)
   (b) $H_0 : p_1 - p_2 \le 0$ (or $p_1 \le p_2$) and $H_1 : p_1 - p_2 > 0$ (or $p_1 > p_2$)
   (c) $H_0 : p_1 - p_2 \ge 0$ (or $p_1 \ge p_2$) and $H_1 : p_1 - p_2 < 0$ (or $p_1 < p_2$)
2. Level of significance is decided and is denoted by α.
3. **Test-statistic:**
   The random variable Z is used as test statistic where
   $$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}}$$

but Z as defined above is only in theory. In actual practice when H_o is $p_1 - p_2 = 0$ (or $p_1 = p_2$), the values of p_1, q_1, p_2 and q_2 are not known because these are all unknown parameters. When H_o is $p_1 = p_2$, then we assume that the common population proportion for both populations is p_c. This proportion p_c is estimated by $\hat{p}_c$ by pooling the data from both samples. Thus $\hat{p}_c = \dfrac{X_1 + X_2}{n_1 + n_2} = \dfrac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}$.

Thus the test – statistic used in actual practice is

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\dfrac{\hat{p}_c\,\hat{q}_c}{n_1} + \dfrac{\hat{p}_c\,\hat{q}_c}{n_2}}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}_c\,\hat{q}_c\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$$

When H_o is $p_1 - p_2 = \Delta$ (say), then the test statistic used is

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - \Delta}{\sqrt{\dfrac{\hat{p}_1\hat{q}_1}{n_1} + \dfrac{\hat{p}_2\hat{q}_2}{n_2}}}$$

4. **Critical region:**

The critical region depends upon the alternative hypothesis H_1. For three forms of H_1, the rejection regions are:

(a) When H_1 is $p_1 - p_2 \neq 0$ or $p_1 \neq p_2$, the rejection region is taken in both ends of the sampling distribution. The critical values are $-Z_{\alpha/2}$ and $Z_{\alpha/2}$. The values greater than $Z_{\alpha/2}$ and less than $-Z_{\alpha/2}$ form the rejection region. The values which lie between $-Z_{\alpha/2}$ and $Z_{\alpha/2}$ form the acceptance region. H_o is rejected if $Z < -Z_{\alpha/2}$ or $Z > Z_{\alpha/2}$. When H_o is $p_1 - p_2 = 0$, then it does not make any difference whether we take $(\hat{p}_1 - \hat{p}_2)$ or $(\hat{p}_2 - \hat{p}_1)$ in the test-statistic.

(b) When H_1 is $p_1 - p_2 > 0$ or $p_1 > p_2$, the entire rejection region is taken in the right side of the curve. It is called one – tailed test to the right. The critical value is Z_α and if Z lies in rejection region the hypothesis $(p_1 - p_2) \leq 0$ or $(p_1 \leq p_2)$ is rejected and $H_1 : p_1 > p_2$ is accepted. It is important to note that if H_1 is $p_2 > p_1$, then the difference $(\hat{p}_2 - \hat{p}_1)$ is used in the test – statistic.

Thus $Z = \dfrac{(\hat{p}_2 - \hat{p}_1)}{\sqrt{\hat{p}_c\,\hat{q}_c\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$. The rejection region is $Z > Z_\alpha$.

(c) When H_1 is $(p_1 - p_2) < 0$ (or $p_1 < p_2$), the rejection region equal to α is taken in the extreme left side. The critical value is $-Z_\alpha$ and the hypothesis $H_o : (p_1 - p_2) \geq 0$ is rejected and $H_1 : (p_1 - p_2) < 0$ is accepted. The critical region is $Z < -Z_\alpha$.

5. Computation 6. Conclusion

Example 13.16.

The cigarette-manufacturing firm distributes two brands of cigarettes. It is found that 56 of 200 smokers prefer brand 'A' and that 30 of 150 smokers prefer brand 'B'. Test the hypothesis at 0.05 level of significance that brand 'A' outsells brand 'B' by 10% against the alternative hypothesis that the difference is less than 10 %.

Solution:

1. **Null hypothesis:** $H_0 : p_1 - p_2 \geq 0.10$

   Alternative hypothesis: $H_1 : p_1 - p_2 < 0.10$

2. **Level of significance:** $\alpha = 0.05$

3. **Test – statistic:**
$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}}$$

4. **Critical region:** $Z < -1.645$

   (From the area table of normal distribution, we have $-Z_{\alpha} = -Z_{0.05} = -1.645$)

5. **Computations:** Here $n_1 = 200$, $X_1 = 56$ (No. of smokers who prefer brand A), $n_2 = 150$, $X_2 = 30$ (No. of smokers who prefer brand B),
$$\hat{p}_1 = \frac{X_1}{n_1} = \frac{56}{200} = 0.28, \quad \hat{q}_1 = 1 - \hat{p}_1 = 0.72,$$
$$\hat{p}_2 = \frac{X_2}{n_2} = \frac{30}{150} = 0.2, \quad \hat{q}_2 = 1 - \hat{p}_2 = 0.8, \text{ and hence}$$
$$Z = \frac{(0.28 - 0.2) - 0.10}{\sqrt{\frac{(0.28)(0.72)}{200} + \frac{(0.2)(0.8)}{150}}} = \frac{-0.02}{0.0455} = -0.44$$

6. **Conclusion:** Since the calculated value of $Z = -0.44$ falls in the acceptance region, so we accept our null hypothesis H_0: $p_1 - p_2 \geq 0.10$ at 5 % level of significance and we may conclude that the brand 'A' outsells brand 'B'.

Example 13.17.

A random sample of 150 high school students was asked whether they would turn to their fathers or their mothers for help with a home work assignment in Mathematics and another random sample of 150 high school students was asked the same question with regard to a homework assignment in English. Use the result shown in the following table at the 0.01 level of significance to test whether or not there is a difference between the true proportions of high school students who turn to their fathers rather than their mothers for help in these two subjects:

	Mathematics	English
Mother	59	85
Father	91	65

Solution:

1. **Null hypothesis:** $H_0 : p_1 = p_2$ or $p_1 - p_2 = 0$

   Alternative hypothesis: $H_1 : p_1 \neq p_2$ or $p_1 - p_2 \neq 0$

2. **Level of significance:** $\alpha = 0.01$

3. **Test – statistic:** $$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\hat{p}_c \hat{q}_c \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

4. **Critical region:** $|Z| > 2.575$ $(Z < -2.575$ and $Z > 2.575)$

   (From the area table of normal distribution, we have $Z_{\frac{\alpha}{2}} = Z_{0.005} = 2.575$

5. **Computations:** Here $n_1 = 150$, $X_1 = 91$, $n_2 = 150$, $X_2 = 65$,

$$\hat{p}_1 = \frac{X_1}{n_1} = \frac{91}{150}, \quad \hat{p}_2 = \frac{X_2}{n_2} = \frac{65}{150},$$

$$\hat{p}_c = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \frac{150\left(\frac{91}{150}\right) + 150\left(\frac{65}{150}\right)}{150 + 150} = \frac{91 + 65}{300} = \frac{156}{300} = 0.52,$$

$$\hat{q}_c = 1 - \hat{p}_c = 1 - 0.52 = 0.48, \quad \text{and hence}$$

$$Z = \frac{\left(\frac{91}{150} - \frac{65}{150}\right) - 0}{\sqrt{(0.52)(0.48)\left(\frac{1}{150} + \frac{1}{150}\right)}} = \frac{\left(\frac{26}{150}\right)}{\sqrt{0.003328}} = \frac{0.1733}{0.0577} = 3.003;$$

6. **Conclusion:** Since the calculated value of $Z = 3.003$ falls in the critical region, so we reject our null hypothesis $H_0: p_1 = p_2$ at 1 % level of significance and we may conclude that there is a difference between the true proportions of high school students.

13.18 CHOICE OF PROPER TEST – STATISTIC

In a certain given situation, we have to choose the proper test-statistic. For example the population mean μ can be tested with the help of Z-test and t-test. The testing of hypotheses along with other things, mainly depends upon the sample size. The sample size plays a major role in the testing of hypothesis. The following table can be used for guidance in choosing the proper test-statistic.

	n – Large	n – Small
σ – Known	Z – test	Z – test
σ – Unknown	Z – test	t – test

SHORT DEFINITIONS

Hypothesis
A statement about a population parameter developed for the purpose of testing.

or

Hypothesis is a statement which may or may not appears be true after conclusion.

Hypothesis Testing
The objective of hypothesis testing is to check the validity of a statement about a population parameter.

or

A procedure based on sample evidence and probability theory to determine whether the hypothesis is a reasonable statement or not is called hypothesis testing.

Statistical Hypothesis
A statistical hypothesis is a statement about the numerical value of a population parameter.

or

A statistical hypothesis is a quantitative statement about a population.

Null Hypothesis
A null hypothesis is any hypothesis which is tested for possible rejection or acceptance under the assumption that it is true.

or

The null hypothesis is a statement about the value of a population parameter.

Alternative Hypothesis or Research Hypothesis
The alternative hypothesis is usually the hypothesis for which the researcher wants to gather supporting evidence.

or

A statement specifying that the population parameter is some value other than the ~~one specified under~~ the null hypothesis.

Simple Hypothesis
A hypothesis which specify all values of parameters of a distribution is called simple hypothesis.

or

A hypothesis is said to be a simple hypothesis if the hypothesis uniquely specifies the distribution from which the sample is taken.

Composite Hypothesis
A hypothesis is said to be a composite hypothesis if it does not completely specify the probability distribution.

or

A hypothesis which does not specify all values of parameters of a distribution is called composite hypothesis.

Significance Level or Level of Significance
The probability of rejecting a true null hypothesis is called the significance level α.

or

The probability of making a type I error is called the significance level of the hypothesis test and is denoted by α (alpha).

Tests of Significance

A significance test is a statistical test laying down the procedure for deciding whether to accept or reject a statistical hypothesis.

Test Statistic

A statistic used as a basis for deciding whether the null hypothesis should be rejected is called test statistic.

or

The sample quantity on which the decision to support H_0 or H_1 is based is called the test statistic.

Rejection Region

The rejection region is the set of possible computed values of the test statistic for which the null hypothesis will be rejected.

or

The set of values for the test statistic that lead to rejection of the null hypothesis H_0 is called rejection region.

Acceptance Region

The set of values for the test statistic that lead to accept the null hypothesis is called acceptance region.

or

The portion of the area under a curve that includes those values of a statistic that lead to acceptance of the null hypothesis.

One-Tailed Test

A statistical test in which the critical region is at one end of sampling distribution is called as one-tailed test.

or

A one-tailed test of hypothesis is one in which the alternative hypothesis is directional, and includes either the symbol " < " or " > ".

Two-Tailed Test

A two-tailed test of hypothesis is one in which the alternative hypothesis does not specify departure from H_0 in a particular direction; such an alternative is written with the symbol " ≠ ".

or

A statistical test in which the critical region is located at both ends of sampling distribution is known as two-tailed test.

Critical Value

The value which separates the rejection and acceptance regions is called the critical value of the test statistic.

or

The dividing point between the region where the null hypothesis is rejected and the region where it is accepted is said to be critical value.