Chapter #08:- →Imp:-

Second Order circuit

↳ 2^{nd} ODE ↳ 2nd order/RLC.

↳ 1^{st} ODE (contains only single energy storage element either capacitor or inductor).

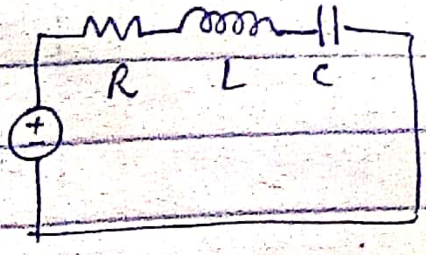
Introduction:- (RLC)

A second order circuit characterized by 2^{nd} order differential equation. It consist of resistor and equivlnt of two energy storage elements (L & C)

(pg# 314-315)

→ active elements produce gain while passive element not produces gain.

→ How to obtain the initial condition for circuit variable.



8.2 Finding initial & final values.

$v(0)$; $i(0)$; $\frac{dv(0)}{dt}$; $i(\infty)$; $v(\infty)$ → final value.

Steps:-

1. Carefully handle polarity of voltage $v(t)$ across the capacitor & direction of current $i(t)$ through the inductor.

→ Keep in mind that capacitor voltage is always continuous.

$$v(0^+) = v(0^-)$$
$$i(0^+) = i(0^-)$$

- $t<0$ / $t=0^-$ (before switching)
- $t>0$ / $t=0^+$ (after switching) → permanent switch is there.
- $t=0$ → $t=0$ (switching event take place)

→ capacitor's voltage can't change abruptly - slowly charge & discharge

→ Current in inductor can't change abruptly.

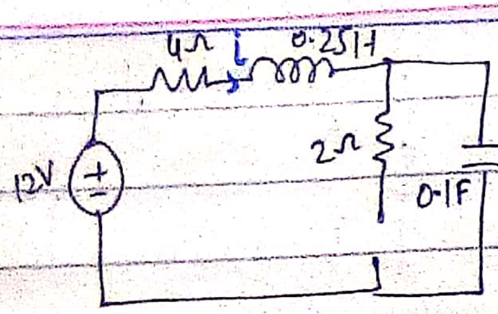
- At DC → capacitor ⇒ open cct. , Inductor ⇒ short cct.

Example #8.1:-

Switch in fig is closed for a long time. It is open at $t=0$. Find : (a) $i(0^+)$, $v(0^+)$,

(b) $di(0^+)/dt$, $dv(0^+)/dt$

(c) $i(\infty)$; $v(\infty)$



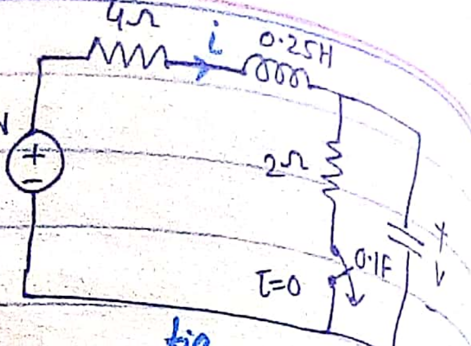
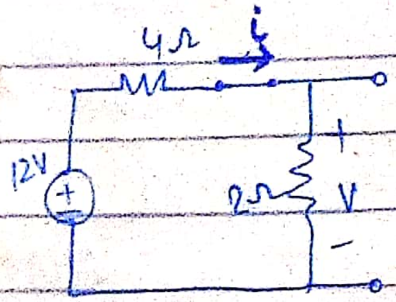
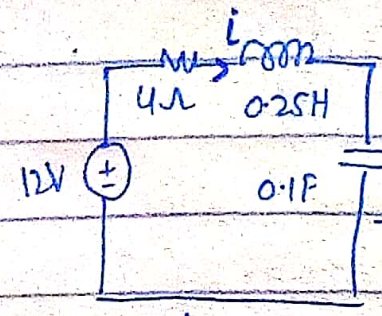


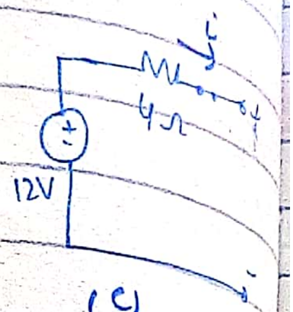
fig.



(a)



(b)



(c)

When switch is connected for a long time; Then C & L reaches to steady state. Hence Inductor is short & C → open. (a)

i) $\frac{V}{R} = i(0^-) = \frac{12}{4+2} = 2A$.

$$V(0^-) = 2 \cdot i(0^-) = 2 \times 2 = 4V$$

as: $i(0^+) = i(0^-) = 2A$;

$V(0^+) = V(0^-) = 4V$.

$t = 0^+$ (after-switching)

$$i_c(0^+) = i(0^+) = 2A.$$

$$i_c = C\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{i_c}{C} = \frac{2}{0.1} = \boxed{20\ V/s}$$

ii) $t=0$ (switch remove) → fig. (b)

$$V_L = L\frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{V_L(0^+)}{L}$$

$\because V_C(0^+) = 4$, $i(0^+) = 2$

KVL at (b):

$$-12 + 4i(0^+) + V_L(0^+) + V_C(0^+) = 0$$

$$-12 + 4(2) + V_L(0^+) + 4 = 0$$

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$$V_L(0^+) = 12-8-4 = 0.$$

$$\frac{di(0^+)}{dt} = \frac{V_L(0^+)}{L} = \frac{0}{0.25} = \boxed{0\,A/s}$$

iii) fig. C $t>0;\ t\to\infty$

$$i(\infty) = 0A.$$

$$V(\infty) = 12V$$

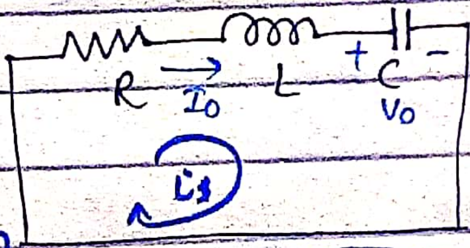
8.3 The Source-free Series RLC circuit:-

Natural response of RLC cct.:-

$\because V(0)=V_0$

$I(0)=I_0.$

$\because V_C = V_0$



KVL:-

$$V_R+V_L+V_C=0$$

$$\boxed{iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0} \quad (1)$$

$\because i = C\frac{dv}{dt}$

$\int i\,dt = \int C\,dv$

$\frac{1}{C}\int i\,dt = V_C.$

Taking derivative on both sides w.r.t 't' to eliminate integral.

$$\frac{di}{dt}\cdot R + \frac{d}{dt}\left(L\frac{di}{dt}\right) + \frac{d}{dt}\left(\frac{1}{C}\int i\,dt\right) = 0.$$

$$\frac{di}{dt}\cdot R + \frac{d^2i}{dt^2}\cdot L + \frac{1}{C}\cdot i = 0.$$

To remove L-coefficient; divide both sides with 'L' & re-arrange:-

$$\boxed{\frac{d^2i}{dt^2} + \frac{R}{L}\cdot\frac{di}{dt} + \frac{i}{LC} = 0} \rightarrow 2^{nd} \text{ order diff. eqn.} \quad (2) \ (2^{nd}\text{ ODE})$$

or from eq ①:-

$$iR + L\frac{di}{dt} + V_0 = 0.$$

$\because i(0) = I_0.$

$$R \cdot i(0) + L\frac{di(0)}{dt} + V_0 = 0.$$

$\because i(t) = I_0 e^{-t/\tau}$

$i = Ae^{st}$

$$L\frac{di(0)}{dt} = -V_0 - RI_0$$

$$\frac{di(0)}{dt} = \frac{-(V_0 + RI_0)}{L}$$

let: $\boxed{i = A \cdot e^{st}}$

from eq ② :- (2nd ODE)

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0. \quad ②$$

when : $i = Ae^{st}$

$$\frac{di}{dt} = Ae^{st}(S) = As.e^{st}$$

so. $\frac{d}{dt}\left(\frac{di}{dt}\right) = \frac{d}{dt}(As \cdot e^{st}).$

$$\boxed{\frac{d^2 i}{dt^2} = As^2 e^{st}}$$ put values in eq ②.

② →

$$\boxed{(As^2 e^{st}) + \frac{R}{L}(Ase^{st}) + \frac{(Ae^{st})}{LC} = 0}$$ → charac. eqn.

Common:-

charac. eqn.

$$Ae^{st}\underbrace{\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right]}_{} = 0.$$

$\because ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

$a = 1$; $b = \frac{R}{L}$; $c = \frac{1}{LC}$

put a, b, c in quadratic formula:-

$$S_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(R/L) \pm \sqrt{(R/L)^2 - 4(1)(1/LC)}}{2(L)}$$

$$S_1 = \frac{-\frac{R}{L} + \sqrt{(R/L)^2 - 4(1)(1/LC)}}{2(1)}$$

$$S_1 = \frac{-R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

$$S_1 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

or $S_1 = \frac{-R}{2L} + \frac{1}{\sqrt{4}}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$

$$S_1 = \frac{-R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{\cancel{4}}{\cancel{4}.LC}}$$

$$S_1 = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

let $\alpha = \frac{R}{2L}$

$\omega_0 = 1/\sqrt{LC}$

$\omega_0^2 = \frac{1}{LC}$

So, $\underline{S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}}$

$$S_2 = \frac{-\frac{R}{L} - \sqrt{(R/L)^2 - 4(1)(1/LC)}}{2(1)}$$

$$S_2 = \frac{-R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

$$S_2 = \frac{-R}{2L} - \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

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$$S_2 = \frac{-R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

let $\alpha = R/2L$

$\omega_0 = 1/\sqrt{LC}$

$\omega_0^2 = 1/LC$

So,

$\underline{S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$

So,

$$\boxed{S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}}$$

S_1 & $S_2 \rightarrow$ natural frequencies ; ω_0 = resonant freq.

$\alpha \rightarrow$ damping factor.

$\left\{\begin{array}{l} \text{i)}\ \alpha > \omega_0 \\ \text{ii)}\ \alpha < \omega_0 \\ \text{iii)}\ \alpha = 0 \end{array}\right\}$ → Over-damped (roots of eq. are real & unequal)
→ Under-damped
→ Critical damped.

As: $i = A e^{st}$

So, $i_1 = A_1 e^{s_1 t}$ & $i_2 = A_2 e^{s_2 t}$

Total response $i = i_1 + i_2$

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

case#1 (over-damped)
↳ $(\alpha > \omega_0)$

→ $b^2 - 4ac = 0$ $\quad \because \alpha = \frac{R}{2L}$; $\omega_0^2 = \frac{1}{LC}$

or $\alpha^2 - \omega_0^2 = 0$

$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0.$$

$$\frac{R^2}{4L^2} - \frac{1}{LC} = 0. \Rightarrow \frac{R^2}{4L^{\cancel{2}}} = \frac{1}{\cancel{L}C}$$

$$1 = \frac{4L}{R^2 C}.$$

So, $C = \frac{4L}{R^2}$ → $C > \frac{4L}{R^2}$

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

case#2:- (under-damped) $(\alpha < \omega_0)$

Monday
24-02-2020

→ What is Damping?

To cause a decrease in amplitude of oscillation.

RLC - series:-

over-damped case $(\alpha > \omega_0)$:-

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}.$$

ii) Critically-damped case:- $(\alpha = \omega_0)$

$$i(t) = (A_2 + A_1 t)\, e^{-\alpha t}.$$

iii) Underdamped case:- $(\alpha < \omega_0)$

$$i(t) = e^{-\alpha t}\,(B_1 \cos \omega_d t + B_2 \sin \omega_d t).$$

RLC-parallel:-

i) Over-damped case $(\alpha > \omega_0)$:-

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}.$$

ii) critically damped case $(\alpha = \omega_0)$:-

$$v(t) = (A_1 + A_2 t)\, e^{-\alpha t}$$

iii) Underdamped case $(\alpha < \omega_0)$:-

$$v(t) = e^{-\alpha t}\,(A_1 \cos \omega_d t + A_2 \sin \omega_d t).$$

Problem solving strategy:-

1- For $t<0$, we have to find the initial conditions $i(0)$ and $v(0)$.

2- For $t>0$ circuit, write KVL equation and then find V_L.

$$\frac{di}{dt} = \frac{V_L}{L}$$

Plug in values of $i(0)$ and $v(0)$ to get $\left.\frac{di}{dt}\right|_{t=0}$

3- For $t>0$; calculate α; ω_0 & roots $S_{1,2}$.

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

4- Based on value of α and ω_0, choose eqn for current $i(t)$ and plug it values in it.

5- Calculate value of A_1 & A_2 and plug it in to get final answer.

i) $i(0)$ & $v(0)$

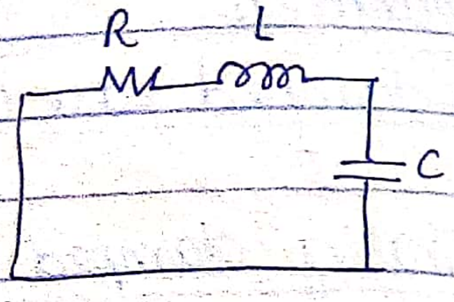
ii) $\left.\frac{di}{dt}\right|_{t=0}$ Through KVL.

iii) $\alpha, \omega_0, S_{1,2}$

iv) $i(t)$ eqn.

v) A_1 & A_2.

RLC - series:-



2nd order differential eqn.

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0.$$

$$Ae^{-st}\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right] = 0$$

roots after quadratic formula.

- $S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$
- $S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

$i(t) = Ae^{st}$; $i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$.

$\alpha = \omega_0$

$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$; $S_1 = S_2 = -\alpha$.

$$i(t) = (A_1 e^{-\alpha t} + A_2 e^{-\alpha t})$$

$$= (A_1 + A_2)e^{-\alpha t}$$

$$= \boxed{A_3 e^{-\alpha t}}$$

→ $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0.$ $\quad \because \alpha = \frac{R}{2L}$

$\frac{d^2i}{dt^2} + 2\left(\frac{R}{2L}\right)\frac{di}{dt} + \frac{i}{L\left(\frac{4L}{R^2}\right)}$ $\quad \because C = \frac{4L}{R^2}$

$$\frac{d^2i}{dt^2} + 2\alpha\frac{di}{dt} + \frac{iR^2}{4L^2} = 0.$$

$$\frac{d^2i}{dt^2} + 2\alpha\frac{di}{dt} + \left(\frac{R}{2L}\right)^2 i = 0$$

$$\frac{d^2i}{dt^2} + \underbrace{2\alpha\frac{di}{dt}}_{\text{splitting}} + \alpha^2 i = 0.$$

$$\left(\frac{d^2i}{dt^2} + \alpha\frac{di}{dt}\right) + \left(\alpha\frac{di}{dt} + \alpha^2 i\right) = 0$$

Taking $\frac{d}{dt}$ common:

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0.$$

Let: $f = \frac{di}{dt} + \alpha i$

So;

$$\frac{d}{dt}(f) + \alpha(f) = 0 \rightarrow \text{1st ODE}$$

$$\boxed{\because f = A_1 e^{-\alpha t}} \rightarrow \text{put } f \text{ in eq}$$

$$\frac{d}{dt} A_1 e^{-\alpha t} + \alpha A_1 e^{-\alpha t} = 0.$$

$$\sqrt{A_1 e^{-\alpha t}(-\alpha) + \alpha A_1 e^{-\alpha t} = 0}$$

$$f = \frac{d}{dt}(i) + \alpha i.$$

$$A_1 e^{-\alpha t} = \left(\frac{di}{dt} + \alpha i\right) e^{\alpha t}$$

$$A_1 = e^{\alpha t}\frac{di}{dt} + \alpha i e^{\alpha t}.$$

apply product formula:-

$$= \frac{d}{dt}(e^{\alpha t}.i)$$

$$= e^{\alpha t}\frac{di}{dt} + \alpha i \cdot e^{\alpha t} \quad \Rightarrow A_1 = \frac{d}{dt}(e^{\alpha t}.i)$$

Integrate on both sides to eliminate derivative

$$\int A_1 = \int \frac{d}{dt}\left(e^{\alpha t} \cdot i\right) dt$$

$$A_1 t + A_2 = e^{\alpha t} \cdot i$$

$$\boxed{i(t) = (A_2 + A_1 t)\, e^{-\alpha t}}$$

Example # 8.3 :-

$R = 40\,\Omega$; $L = 4H$; $C = \frac{1}{4}F$; $S_{1,2} = ?$

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times 1/4}} = 1$$

$\rightarrow$ $5 > 1$

$\alpha > \omega_0$

$\downarrow$ over-damped.

So, roots are:-

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -5 \pm \sqrt{25 - 1}$$

$$S_1 = -5 + \sqrt{24} \quad ; \quad S_2 = -5 - \sqrt{24}$$

$$\boxed{S_1 = -0.101 \quad ; \quad S_2 = -9.89}$$

P.P # 8.3 :-

$R = 10\,\Omega$; $L = 5H$; $C = 2mF$; $S, \alpha, \omega = ?$

$$\alpha = \frac{R}{2L} = \frac{10}{2(5)} = 1$$

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5 \times (2 \times 10^{-3})}} = 10$$

$\rightarrow$ $\omega > \alpha$

$\hookrightarrow$ under-damped.

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -1 \pm \sqrt{1 - 100}$$

$$\boxed{S_{1,2} = -1 \pm j9.95}$$

Example # 84:-

find $i(t)$; assume that circuit has reached steady state $t=0$.

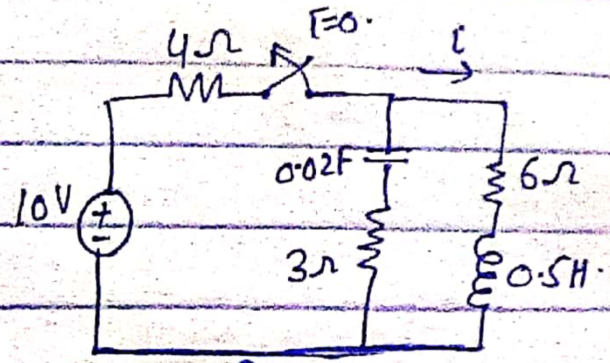


Fig-1.

capacitor → open
inductor → short

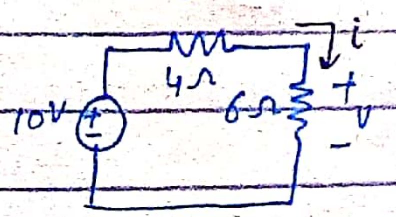


Fig-2.

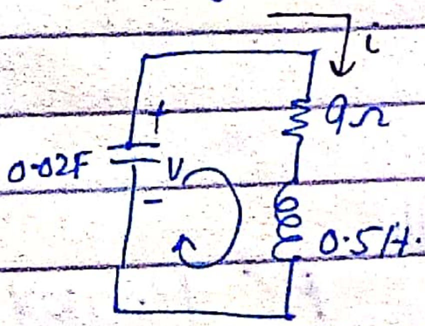


Fig-3.

Solution:-

steps 1- $i(0)$ and $V(0)$

2- $\frac{di}{dt}\big|_{t=0}$ through KVL

3- $\alpha, \omega_0, S_{1,2}$

4- $i(t)$ eqn.

5. A_1 & A_2.

Solution:-

step#1: Fig-2 ⇒ $i(0) = \frac{V}{R} = \frac{10}{10} = 1A$.

$$V(0) = iR_{6\Omega}$$

$$= (1)(6)$$

$$V(0) = 6V.$$

Fig#3 → step#2

$$\boxed{V_L = L\frac{di}{dt}}$$

$$\frac{di}{dt}\Big|_{t=0} = \frac{V_L}{L}$$

KVL on fig#3 to find V_L:-

$$-V_c + V_R + V_L = 0.$$

$$V_L(0) = V_c(0) - V_R(0).$$

$$V_L(0) = 6 - i(0)R.$$

$$= 6 - 1(9)$$

$$\boxed{V_L(0) = 6-9 = -3}$$ → put in eq:-

$$\left.\frac{di}{dt}\right|_{t=0} = \frac{-3}{0.5}$$

$$= -6\ A/s.$$

Step#3: $\alpha = \frac{R}{2L} = \frac{9}{2(0.5)} = 9.$

$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5(0.02)}} = 10$

$\rightarrow$ $\alpha < \omega$ under-damped.

So, $S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$= -9 \pm \sqrt{81-100}$

$= -9 \pm j4.359.$

Step#4 So, $i(t) = e^{-\alpha t}(A_1 \cos 4.359j + A_2 \sin 4.359j)$

$\boxed{\text{Ex. Eipp. } 8.1 \rightarrow 8.10, 8.16}$

$6 \rightarrow 10, 16 \rightarrow$ imp.

$\omega_\alpha = \sqrt{\omega_0^2 - \alpha^2}$

$\omega_\alpha = 4.359$