FUNCTION

[(UOS:2006,2010,2015/II)(UAJK:2014)(UOH:2009)(GCUF:2013/I,2014/II)(UOPR:2012,2016)]

Simply the relationship between the variables is known as function. In other words, the dependence of one variable upon the other is called a **functional relationship**. For example, if in a country, the high birth rate depends upon poverty — –then we will say that the birth rate is a function of poverty. In such case the birth rate is a dependent variable while poverty is an independent variable. With this background we define a function as " As y is a dependent variable and x is an independent variable, if y depends upon x —such relationship between y and x is called a function. Moreover, in case of a function, corresponding to one value of x there is only one value of y and any perpendicular drawn on x–axis cuts the graph of the function only at one point ". It is explained with an example .

EXAMPLE. If $y = 2x + 3$

Putting $x = 0, 1, 2, 3$ we get

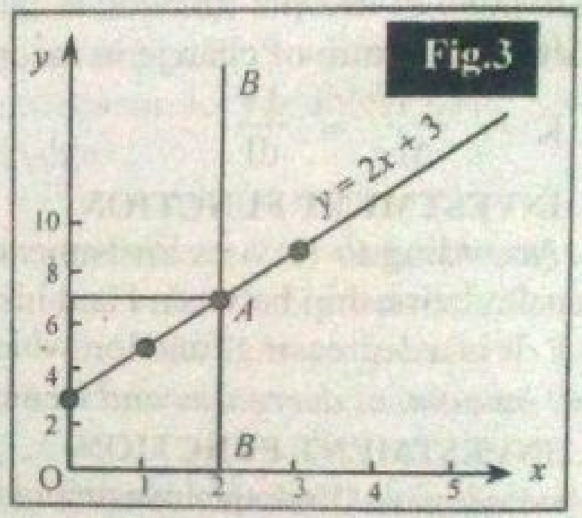
when $x = 0$, $y = 2(0) + 3 = 3$

$x = 1$ $y = 2(1) + 3 = 5$

$x = 2$, $y = 2(2) + 3 = 7$

$x = 3$, $y = 2(3) + 3 = 9$

$x =$	0	1	2	3
$y =$	3	5	7	9



With the help of values of x and y we constructed Fig. 3. We explain it with the help of properties of function presented in its definition.

1. The value of y which is 7 has been determined by the value of x which is 2. Thus it is obvious that the value of y depends upon the value of x.
2. Corresponding to one value of x, i.e., 2, there is only one value of y which is 7.
3. The perpendicular BB drawn on x–axis intersects the graph of the function $y = 2x + 3$ only at one point which is A.

Thus we conclude that $y = 2x + 3$ is a function or there exists a functional relationship between y and x.

TYPES OF FUNCTIONS (GCUF:2013/I)

1. INCREASING FUNCTION **[(Isamia University Bahawalpur (IUBWR):2014/I, 2015./3rd year, 2015./II) (UOH:2004,2009) (UOPR:2014)]**

If alongwith increase in the values of x, the values of y also increase; or alongwith decrease in the values of x the values of y also decrease —— the relationship between y and x will be given the name of an increasing function. It is as: $y = f(x)$

$x\uparrow, y\uparrow$ and $x\downarrow y\downarrow$

EXAMPLE. If $y = 2x + 2$. Supposing the values of x and putting them in the function.

Putting $x = 1, 2, 3$ we get

when $x = 1, y = 2(1) + 2 =$

$x = 2, y = 2(2) + 2 = 6$

$x = 3, y = 2(3) + 2 = 8$

$x =$	1	2	3
$y =$	4	6	8

The values of y are increasing alongwith increase in the values of x — increasing function. As in economics we take a specific consumption function. $C = 40 + 0.6\,Y$

Putting $Y = 100, 200, 300$ we get

$C = 40 + 0.6(100) = 100$

$C = 40 + 0.6(200) = 160$

$C = 40 + 0.6(300) = 220$

Y =	100	200	300
C =	100	160	220

The values of C are increasing alongwith increase in the values of Y — increasing function.

2. DECREASING FUNCTION **[(GCUF:2013IA)(IUBWR:2014/I,2015/3RD YEAR, 2015/II) (UOPR:2014)(UOH:2004,2009)]**

If alongwith increase in the values of x, the values of y decrease; or alongwith decrease in the values of x the values of y increase —— the relationship between y and x will be given the name of a decreasing function. It is as : $y = f(x)$

$x\uparrow, y\downarrow$ and $x\downarrow y\uparrow$

(i): If $y = 3x - 10$
(ii): $y = 4x^2 - 2x + 5$
(iii): $y = 3^x$, draw the graphs (UOH:2005)

EXAMPLE. If $y = 16 - 2x$. Supposing the values of x and putting them in the function

Putting $x = 1, 2, 3$ we get

when $x = 1, y = 16 - 2(1) = 14$
$x = 2, y = 16 - 2(2) = 12$
$x = 3, y = 16 - 2(3) = 10$

$x =$	1	2	3
$y =$	14	12	10

It is clear that values of y are decreasing alongwith increase in the values of x — decreasing function. As in economics we take a specific demand function :

EXAMPLE. If $Q = 10 - 2P$.

Supposing the values of P and putting them in the function

Putting P = 1, 2, 3 we get

when P = 1, Q = 10 − 2 (1) = 8
P = 2, Q = 10 − 2 (2) = 6
P = 3, Q = 10 − 2 (3) = 4

P =	1	2	3
Q =	8	6	4

The values of Q are decreasing alongwith increase in the values of P — decreasing function.

3. SINGLE–VALUED FUNCTION (GCUF:2013/I) (UOS:2016)

When the dependent variable (y) depends upon the independent variable (x) and corresponding to one value of x there is only one value of y such relationship is known as single valued function.

In case of single valued function the perpendicular drawn on the x–axis cuts the graph just at one point. The concept of a single valued function is similar to that of a function. Therefore, to explain a single valued function the examples presented in function can be used.

4. MULTI–VALUED FUNCTION (GCUF:2013/I) (UOS:2016)

When the dependent variable (y) depends upon the independent variable (x), but corresponding to one value of x there are more than one value of y such relationship is termed as multi-valued function in the old terminology. Moreover, in such case the perpendicular drawn on x–axis cuts the graph of multi-valued function at more than one point. The concept of multi-valued function is similar to that of a **relation** which we are going to explain in the coming pages.

5. EXPLICIT FUNCTION [(GCUF:2013/I)(UOP:2013)(UOS:2006)(UOG:2014) (UOH:2003,2014)(UOPR:2011,2013,2016)]

The variable y will be explicit function of x if y — the dependent variable entirely depends upon x — the independent variable. This concept is also similar to the concept of a function. In case of an explicit function by supposing the values of x the values of y can be found. As if $y = f(x) = 2x + 3$, then supposing x we can find y. This shows that the function where the values of y have clearly (explicitly) been expressed in terms of x is called an **explicit function**. As in economics, most of the functions are explicit functions , i.e.,

$$U = f(Q), \quad C = f(Q), \quad R = f(Q) \text{ etc.}$$

6. IMPLICIT FUNCTION [(GCUF:2013/I)(UOP:2012)(UOG:2014)(UOH:2003,2004)(UOPR:2011,2013,2016)(UOS:2016)]

In case of explicit function the value of y entirely, completely and explicitly depends upon the value of x. As $y = f(x)$. But in case of implicit function the values of both the variables x and y depend upon each other. As $F(x, y) = 0$. Thus the implicit function shows the functional relationship between both the variables. In case of implicit function the variable y is attached with variable x on the one side of the equation or the y is mixed up with x. All is shown as :

$y = 5x$	Explicit function
$5x - y = 0$	Implicit function
$xy = 5$	Implicit function

As in economics $Y = f(I)$ and $I = f(Y)$ are the examples of explicit functions. These functions show that both Y and I depend upon each other. With this we formulate the implicit function $F(Y,I) = 0$. Again on the basis of explicit functions $Q = f(P)$ and $P = f(Q)$, we construct implicit function $F(P,Q) = 0$. On such pattern a specific demand function is as : $PQ = 1200$.

7. INVERSE FUNCTION [(GCUF:2013/I)(UOP:2013)(UOH:2004)(UOPR:2013)(UOS:2016)]

Certain functions can also be represented in their inverse form. As if the function is $y = f(x)$, its inverse function will be $x = f(y)$. The inverse function can also be represented as $x = f^{-1}(y)$. In case of inverse function, there also exists one value of y corresponding to one value of x.

Now we give the economic example of inverse function

$Q = f(P)$ Direct function, $P = f(Q)$ Inverse function

The standard demand function is $Q = a - bP$

Solving for P : $bP = a - Q \Rightarrow P = \frac{a - Q}{b} \Rightarrow P = \frac{a}{b} - \frac{1}{b} Q$

If $\frac{a}{b} = a_0$ and $\frac{1}{b} = a_1$, then $P = a_0 - a_1 Q$

is a standard inverse demand function.

Again we take the investment function $I = f(Y)$ and its standard function is $I = Io + eY$. The inverse standard function of investment is derived as :

$I = Io + eY \Rightarrow I - Io = eY \Rightarrow eY = I - Io \Rightarrow Y = \frac{I - Io}{e}$

is a standard form of inverse investment function $Y = f(I)$.

FURTHER TYPES OF FUNCTIONS

The functions are also divided into **Algebraic** and **non-Algebraic** functions. The algebraic functions are further classified into (1) Polynomial functions (2) Rational functions. The non-algebraic functions are classified into (1) Exponential Functions (2) Logarithmic Functions (3) Trigonometric Functions.

POLYNOMIAL FUNCTION (UOPR:2013)

A polynomial function is such an algebraic function which has many an algebraic sentences. The standard form of a polynomial function which has one independent variable x is as: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$

where $a_0, a_1, a_2, a_3, \ldots, a_n$ represent coefficients, while x represents a single independent variable. The numbers which lie above on x like 1 , 2 , 3 , , n represent the powers of the variable x. The polynomial function has the following types :

1. CONSTANT FUNCTION (UOPR:2013)

The algebraic function whose range consists of just one element is called a constant function. Its standard form is as: $y = a_0$, where a_0 = constant. Hence the value of y remains the same whatsoever are the values of x. Accordingly in the cartesian plane the graph of a constant function is a horizontal line. As we take a specific example of constant function $y = f(x) = 7$. Plotting it we get Fig. 5.

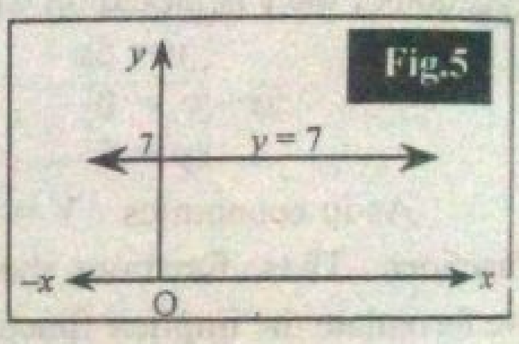


Fig.5

As in economics all the autonomous or exogenous variables represent constant quantities. As I = Io (autonomous investment) and G = Go (autonomous govt. expenditure). Plotting G = Go = 50,we get Fig.6.

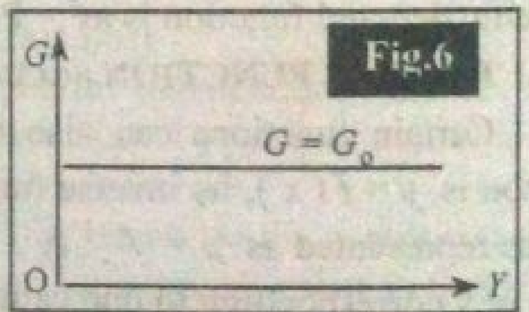


Fig.6

3. LINEAR FUNCTION [(GCUF:2014/I)(Quaid-I-Azam University Islamabad(QAU:2014) (University of Hazara (UOH:2003)]

It is a function where the highest power of independent variable is 1. The graphical representation of a linear function is a straight line. The standard linear function is as : $y = a + bx$ or $y = a_o + a_1x$ where a_o represents y–intercept. It is told that y–intercept represents the point where the graph of the function intersects the y–axis. This occurs when the value of x is kept equal to zero. As $x = 0$, then $y = a_0 + a_1x = a_o + a_1(0) = a_o$. Thus y–intercept consists of the ordered pair $(0, a_o)$. The b (or a_1) represents the slope of the curve. If $a_1 > 0$, then the slope of the linear curve will be positive. If $a_1 < 0$, then the slope of the linear curve will be negative.

Now we demonstrate the linear function with a mathematical example.

If $y = 2x + 3$, then supposing x we can find the values of y. With these values we construct Fig.7.

EXAMPLE. If $y = 2x + 3$

Putting $x = -3, -2, -1, 0, 1, 2, 3.$

$x = -3,\ y = 2(-3) + 3 = -3$

$x = -2,\ y = 2(-2) + 3 = -1$

$x = -1,\ y = 2(-1) + 3 = 1$

$x = 0,\ y = 2(0) + 3 = 3$

$x = 1,\ y = 2(1) + 3 = 5$

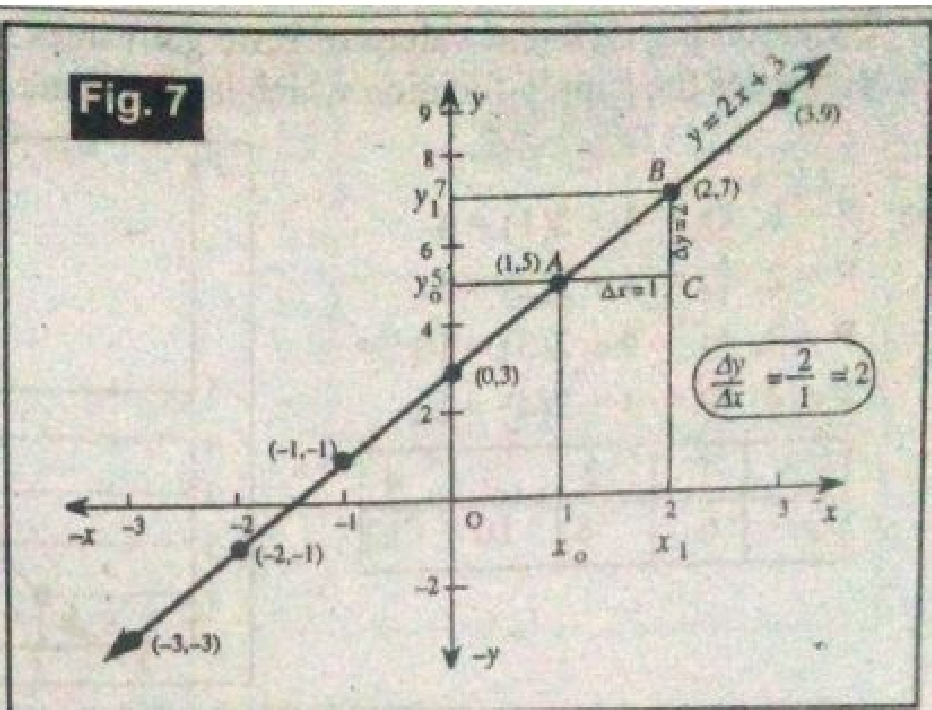
$x = 2,\ y = 2(2) + 3 = 7$

$x = 3,\ y = 2(3) + 3 = 9$

x =	−3	−2	−1	
y =	−3	−1	1	

x =	0	1	2	3
y =	3	5	7	9

Fig. 7



The y–intercept of the graph is 3, while the slope of the graph is 2. It is proved as:

$$\text{Slope} = \frac{\text{Perpendicular}}{\text{Base}} = \frac{\Delta y}{\Delta x} = \frac{Oy_o - Oy_1}{Ox_o - Ox_1} = \frac{5-7}{1-2} = \frac{-2}{-1} = 2$$

As the slope of the curve is positive — the function is increasing one.

4. QUADRATIC OR NON-LINEAR FUNCTION

[(Govt College University Faisalabad (GCUF):2014/I,II)(QAU:2014)(UOH:2003)]

A quadratic function is such a function where the highest power of unknown or the independent variable is 2. The graphical representation of a quadratic function is a parabola.

The standard quadratic function : $y = ax^2 + bx + c$

A specific quadratic function : $y = x^2 + 2x - 1$

By assuming different value of x and putting them in the specific function we get values of y. Then with these values of x and y we get Fig.12 which is a parabola.

EXAMPLE. $y = x^2 + 2x - 1$

Putting $x = -3, -2, -1, 0, 1, 2, 3.$

when $x = -3,\ y = (-3)^2 + 2(-3) - 1 = 2$

$x = -2,\ y = (-2)^2 + 2(-2) - 1 = -1$

$x = -1,\ y = (-1)^2 + 2(-1) - 1 = -2$

$x = 0,\ y = (0)^2 + 2(0) - 1 = -1$

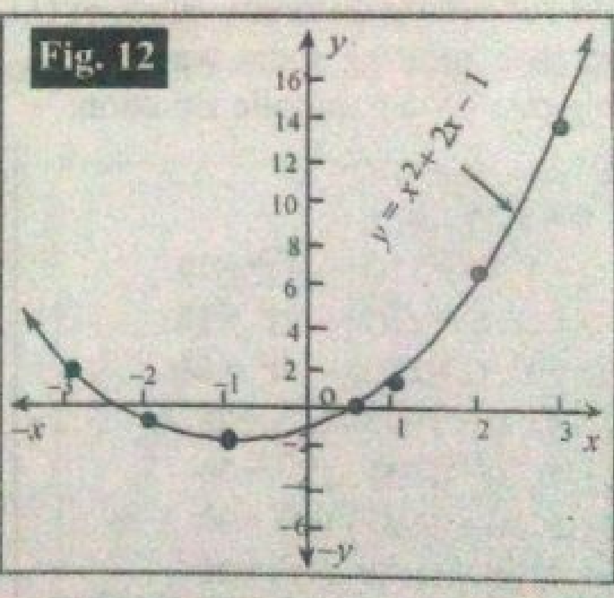
$x = 1,\ y = (1)^2 + 2(1) - 1 = 2$

$x = 2,\ y = (2)^2 + 2(2) - 1 = 7$

$x = 3,\ y = (3)^2 + 2(3) - 1 = 14$

x =	-3	-2	-1	
y =	2	-1	-2	
x =	0	1	2	3
y =	-1	2	7	14

Fig. 12



4. **CUBIC FUNCTION**

The algebraic function in which the highest power of the independent variable x is 3 is known as a cubic function. Its standard form is: (GCUF:2013/I)

$$y = ax^3 + bx^2 + cx + d$$

We take the following specific function : $y = 2x^3 + 4x^2 + 2x - 3$

By supposing and putting the values of x, we can find the corresponding values of y. Then with these values of x and y we construct the Fig. 17 —the cubic graph.

EXAMPLE. $y = 2x^3 + 4x^2 + 2x - 3$

Putting $x = -3, -2, -1, 0, 1, 2, 3.$

$y = 2(-3)^3 + 4(-3)^2 + 2(-3) - 3 = -27$

$y = 2(-2)^3 + 4(-2)^2 + 2(-2) - 3 = -7$

$y = 2(-1)^3 + 4(-1)^2 + 2(-1) - 3 = -3$

$y = 2(0)^3 + 4(0)^2 + 2(0) - 3 = -3$

$y = 2(1)^3 + 4(1)^2 + 2(1) - 3 = 5$

$y = 2(2)^3 + 4(2)^2 + 2(2) - 3 = 33$

$y = 2(3)^3 + 4(3)^2 + 2(3) - 3 = 93$

x =	–3	–2	–1	
y =	–27	–7	–3	
x =	0	1	2	3
y =	–3	5	33	93

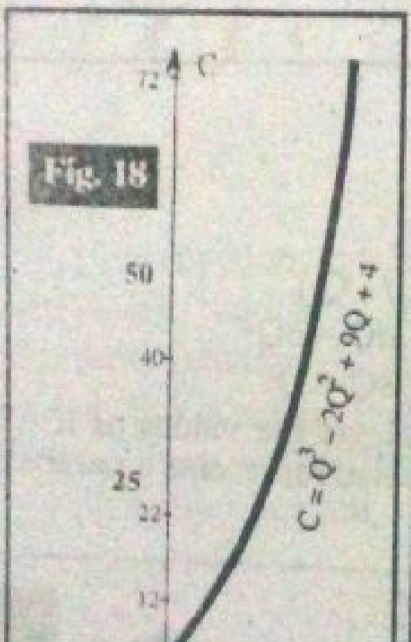


Fig. 18

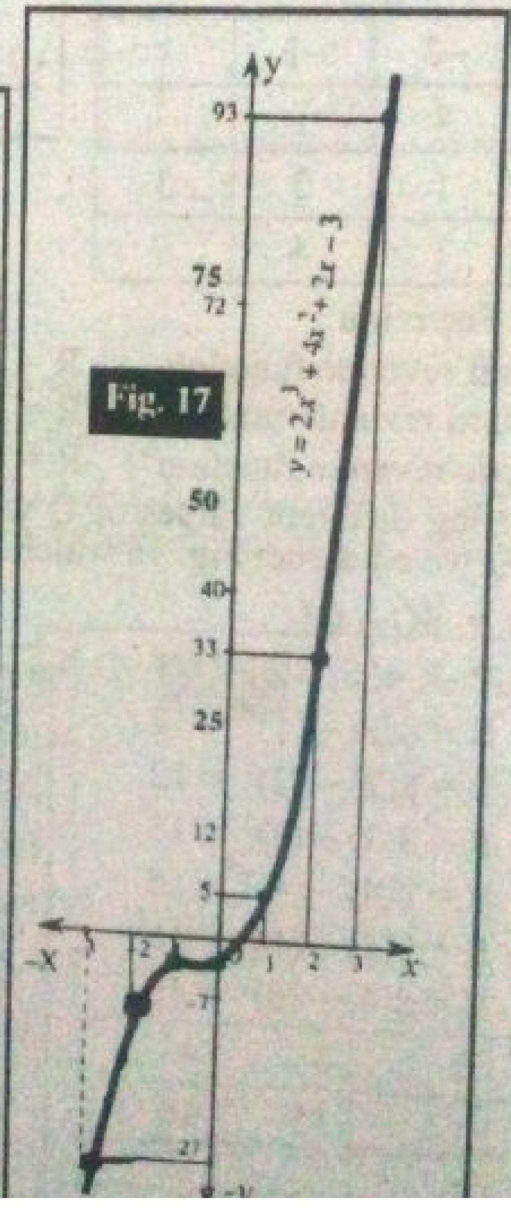


Fig. 17

$C = (5) - 2(5) + 9(5) + 4 = 124$

5. RATIONAL FUNCTION (GCUF:2013/I)

The function where the dependent variable y has been expressed in the ratio of two polynomial functions is called a **rational function**. Actually each polynomial function is rational function because it can be expressed in the ratio of 1 which is a constant function. The standard form of a rational function is : $y = \frac{a}{x}$ or $xy = a$. On this pattern we take a specific function as : $y = \frac{12}{x}$.

By supposing values of x (= 1, 2, 3, 4, say) we can find corresponding values of y and then plotting them we can get the graph of the rational function.

A specific demand function is : $PQ = 1200$

Converting it into standard rational function : $Q = \frac{1200}{P}$

Supposing the values of P we can find the corresponding values of Q. Then by plotting these pairs of values we get Fig. 19. The graphical representation of a rational function is called **Rectangular Hyperbola**.

EXAMPLE. $Q = \frac{1200}{P}$

Putting P = 100, 200 , 300 , 400,

when P = 100, $Q = \frac{1200}{100} = 12$

P = 200, $Q = \frac{1200}{200} = 6$

P = 300, $Q = \frac{1200}{300} = 4$

P = 400, $Q = \frac{1200}{400} = 3$

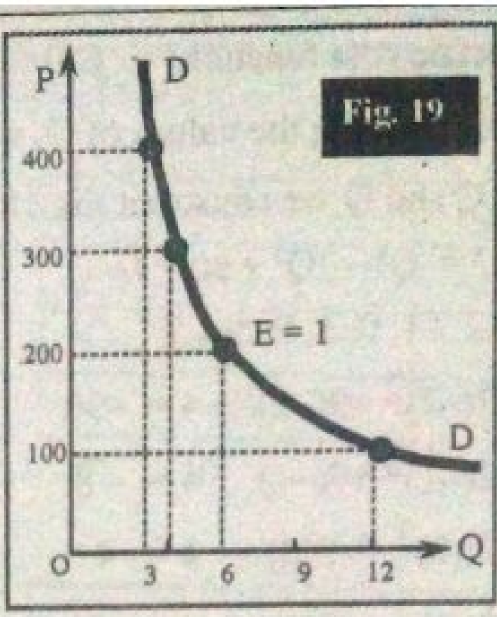


Fig. 19

P =	100	200	300	400
Q =	12	6	4	3

The demand curve which we derived in Fig. 19 is of such a type which shows that the elasticity of demand is equal to one (E = 1).

In addition to demand function we have AFC function. As TFC (Total Fixed Costs) remain the same while quantity produced (Q) goes on to increase. Consequently, AFC (Average Fixed Costs) go on to fall. In this way, AFC runs parallel to *x*–axis or it is asymptote to *x*–axis —as shown in Fig. 20.

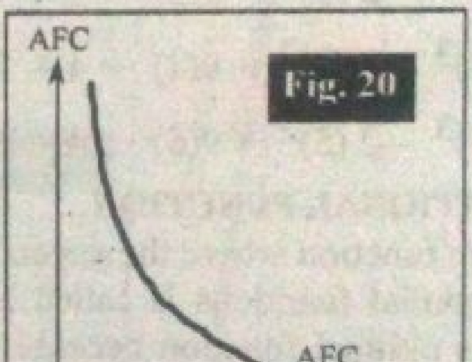


Fig. 20

NON–ALGEBRAIC FUNCTIONS (UOPR:2013)

1. EXPONENTIAL FUNCTION [(UOP:2011,2014)(UOS:2010,2015/II)(UOPR:2011,2013)]

The function where the independent variable x has been represented in terms of power is called **exponential function.** In other words, the function whose base is a constant like a , b , etc. and the power is x, is called exponential function.

The standard form of an exponential function : $y = a^x$

We take a specific exponential function : $y = 2^x$

Supposing the values of x and putting them in y , we get corresponding values of y. In this way we get Fig.22 which is known as exponential curve.

EXAMPLE. $y = 2^x$

Putting $x = -3, -2, -1, 0, 1, 2, 3.$

when $x = -3,\ y = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$x = -2,\ y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

$x = -1,\ y = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

$x = 0,\ y = 2^0 = 1$

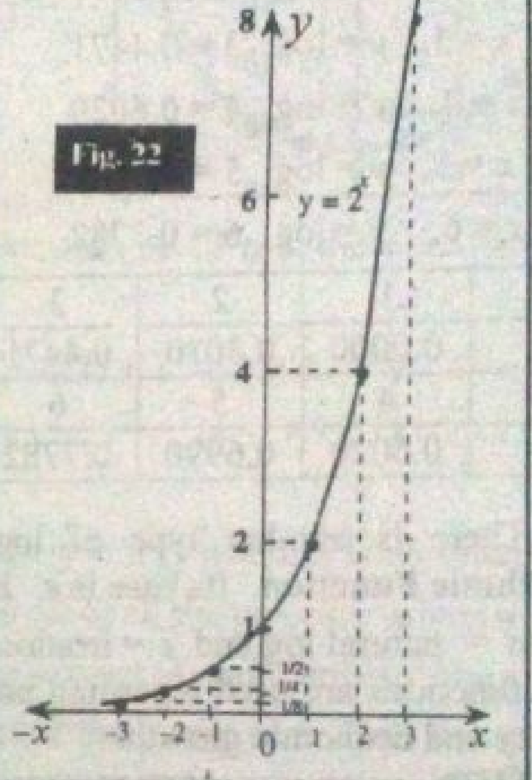
$x = 1,\ y = 2^1 = 2$

$x = 2,\ y = 2^2 = 4$

$x = 3,\ y = 2^3 = 8$

x =	–3	–2	–1	
y =	1/8	1/4	1/2	
x =	0	1	2	3
y =	1	2	4	8

Fig. 22



There is another type of exponential function which is called **Natural Exponential Function.** The base of such function is e , rather a . Its standard form is as : $y = e^x$

It is told that e is an irrational number whose approximate value is 2.718. The exponential function is applicable in the case of production function, returns to scale and effects of interest on principal amount etc. in economics.

2. LOGARITHMIC FUNCTION [(UOP: 2011 (UOS:2010,2015/II)]

Before defining logarithmic function we present the concept of logarithm.

The log. of a number is a power which is given to any base to get the original number. As $10^2 = 100 \Rightarrow Log_{10}100 = 2$

$10^3 = 1000 \Rightarrow Log_{10}1000 = 3$

By log. function we mean a function where the dependent variable y has been expressed in terms of log of independent variable x.

The log function whose base is a is known as **Common Logarithmic function.** The base of a common log function is 10. The standard common log function is: $y = \log_a x$

By supposing values of x we can find the corresponding values of y. To find the values of y we will have to use the calculator. With these values of x and y we construct Fig. 23.

EXAMPLE. $y = \log_{10} x$

Putting $x = 1, 2, 3, 4, 5, 6.$

when $x = 1,\ y = \log_{10} 1 = 0.0000$

$x = 2,\ y = \log_{10} 2 = 0.3010$

$x = 3,\ y = \log_{10} 3 = 0.4471$

$x = 4,\ y = \log_{10} 4 = 0.6020$

$x = 5,\ y = \log_{10} 5 = 0.6990$

$x = 6,\ y = \log_{10} 6 = 0.7782$

x =	1	2	3
y =	0.0000	0.3010	0.4471
x =	4	5	6
y =	0.6020	0.6990	0.7782

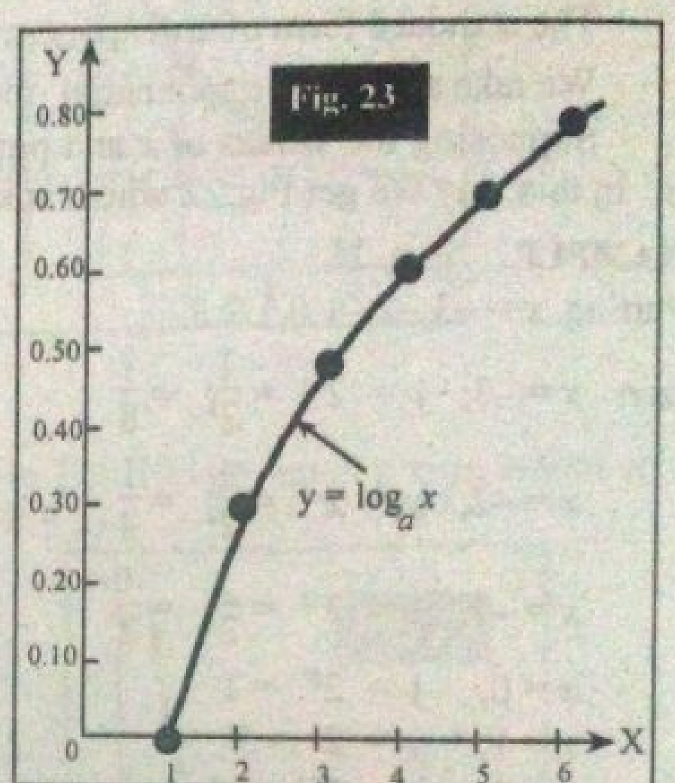


Fig. 23

There is another type of logarithmic function which is called **Natural Logarithmic Function**. Its base is e. Its standard form is : $y = \log_e x = \ln x$ where ln = natural log and e = irrational number whose value is 2.718. In economics the log functions are used regarding population growth, production function, quantity of money and economic growth.

RELATION [(UOP:2013)(GCUF:2014/II)(UOS:2006,2015/II)(UAJK:2014)(UOH:2009)]

As y is a dependent variable and x is an independent variable, therefore, if y depends on x and corresponding to one value of x there are more than one values of y — such situation is called a **relation**. Moreover, in case of a relation, the perpendicular drawn on x–axis cuts the graph of a relation at the points more than one. It is explained with an example.

EXAMPLE. $x^2 + y^2 = 16 \Rightarrow y^2 = 16 - x^2 \Rightarrow y = \pm\sqrt{16 - x^2}$

Putting $x = -4, -3, -2, -1, 0, 1, 2, 3, 4,$ we get

when $x = -4,\ y = \pm\sqrt{16-(-4)^2} = \pm\sqrt{16-16} = 0$

$x = -3,\ y = \pm\sqrt{16-(-3)^2} = \pm\sqrt{16-9} = \pm\sqrt{7} = \pm 2.6$

$x = -2,\ y = \pm\sqrt{16-(-2)^2} = \pm\sqrt{16-4} = \pm\sqrt{12} = \pm 3.4$

$x = -1,\ y = \pm\sqrt{16-(-1)^2} = \pm\sqrt{16-1} = \pm\sqrt{15} = \pm 3.8$

$x = 0,\ y = \pm\sqrt{16-(0)^2} = \pm\sqrt{16-0} = \pm\sqrt{14} = \pm 4$

$x = 1,\ y = \pm\sqrt{16-(1)^2} = \pm\sqrt{16-1} = \pm\sqrt{15} = \pm 3.8$

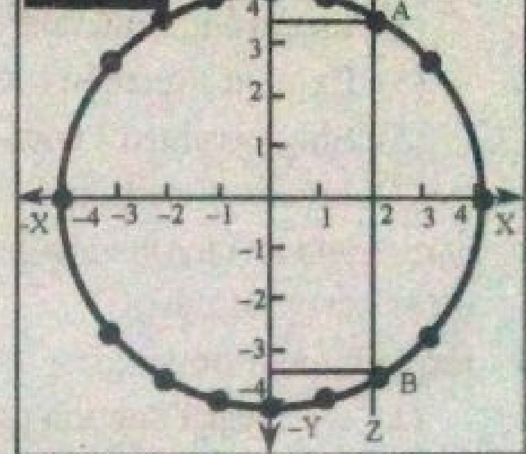
$x = 2,\ y = \pm\sqrt{16-(2)^2} = \pm\sqrt{16-4} = \pm\sqrt{12} = \pm 3.4$

$x = 3, \quad y = \pm\sqrt{16-(3)^2} = \pm\sqrt{16-9} = \pm\sqrt{7} = \pm 2.6$

$x =$	−4	−3	−2	−1	0	1	2	3	4
$y =$	0	± 2.6	± 3.4	± 3.8	± 4	± 3.8	± 3.4	± 2.6	0

With values of x and y the Fig.24 has been constructed. The perpendicular ZZ has been drawn on x–axis and it is cutting the graph of the above relation at two points namely A and B. These points show that corresponding to one value of x, i.e. $x = 2$, there are two values of y, i.e., $y = 3.4$ and $y = -3.4$.

Fig. 24



ECONOMIC EXAMPLE OF RELATION

A specific supply function is : $Q^2 = 4P + 1$

This shows that corresponding to one value of P, i.e. , P = 2, there are two values of Q, i.e., Q = 3 and Q = − 3.

EXAMPLE: $Q^2 = 4P + 1$ When P = 2, $Q^2 = 4(2) + 1 = 8 + 1 = 9$

$Q^2 = 9 \Rightarrow \quad Q = \pm\sqrt{9} = \pm 3$, i.e., $\quad Q = +3,\ Q = -3.$

Domain and Range [(IUBWR:2014/I)(UOH:2003)]

We know that the general form of the function is: $y = f(x)$ where x is an independent variable while y is a dependent variable. Thus the set of all the possible values which 'x' can assume is called Domain. Such set will be the sub-set of real numbers. The value of y in which the value of x has been mapped is called the image of x. Thus the set of all the values which are adopted by 'y' in a situation is called range. It means that domain is concerned with values of x, while range is concerned with values of y.

Economic Example of Domain and Range

As the cost functions is: $C = f(Q)$, we take a specific cost function:

$C = 150 + 7Q$. If the firm's daily capacity to produce is 100 units, we find the domain and range. The value of daily production is between zero and hundred. Then the domain will be as: $0 < Q < 100$. It is shown as: Domain = $\{Q \mid 0 \leq Q \leq 100\}$. Putting them in cost function: $C = 150 + 7Q$ $\quad C = 150 + 7(0) = 150$,

$$C = 150 + 7(100) = 850$$

This shows that costs of production (C) may assume the values between 150 and 850. Thus the range will be as: Range = $\{C \mid 150 \leq C \leq 850\}$.