i.e. $\hat{P}$ is a constant of motion; (i.e conserva quantity) it means that a symmetric state will always be symmetric and an anti-symmet will always be anti-symmetric. The symmetry and anti-symmetry of a state is a characteristic of particles. *

Symmetrization Postulates:

we know that wavefn of a system of identical particles is either symmetric or anti-symmetric under the interchange of any pair of particle. This shows ~~that~~ the very essence of the symmetrization postulate that, in nature, the states of system containing identical particles are either totally symmetric or totally anti-symmetric under the interchange of any pair of particles and that states with mixed symmetry do not exist.

Hence, we have two classes of identical Particles:

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Fermions:

Identical particles of half-odd integral spins ($1/2$, $3/2$ ...). They are described by anti-symmetric wave functions.

Example: electrons, protons, neutrons neutrinos (spin $1/2$).

Bosons:

Identical particles with integral spins. (0, 1, 2, ...). They are described by symmetric wave functions.

Examples: π mesons (spin 0), photon (spin 1) deutrons (spin 1), α - particles (spin 0) oxygen nuclei (spin 0).

$\Rightarrow$ Particles with half-odd-integral spins: Fermions obey Fermi-Dirac statistics.

$\Rightarrow$ Particles with integral spins: Bosons obey Bose-Einstein statistics.

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System of N-Identical Particles:

Let us consider N-identical particles with symmetric Hamiltonian

$$H = H(1, 2, \dots, N).$$

The wavefunction is

$$\psi = \psi(1, 2, \dots, N).$$

The exchange operator $\hat{P}_{ij}$ exchanges $i \longleftrightarrow j$.

$$\hat{P}_{ij}\, \psi(\dots, i, \dots, j, \dots) = \psi(\dots, j, \dots, i, \dots).$$

The symmetry of the Hamiltonian implies

$$[H, \hat{P}] = 0$$

The equation of motion of $\psi(1,2,\dots,N)$ is

$$i\hbar \frac{\partial}{\partial t} \psi(1,2,\dots,N) = H(1,2,\dots,N)\, \psi(1,2,\dots,N)$$

The normalization condition is

$$\int |\psi(1,2,\dots,N)|^2 \, d^3\underline{x}_1 \, d^3\underline{x}_2 \dots d^3\underline{x}_N = 1$$

The operators $\hat{P}_{ij}$ do not commute with one another e.g.

$$\hat{P}_{13}\hat{P}_{12}\,\psi(1,2,3) = \hat{P}_{13}\,\psi(2,1,3)$$

$$= \psi(2,3,1)$$

$$= \hat{P}_{23}\,\psi(3,2,1)$$

$$= \hat{P}_{23}\hat{P}_{13}\,\psi(1,2,3)$$

i.e.

$$\hat{P}_{13}\hat{P}_{12} = \hat{P}_{23}\hat{P}_{13} \neq \hat{P}_{12}\hat{P}_{13}.$$

The two eigenfunctions of $\hat{P}_{ij}$ are ψ_s and ψ_A with eigenvalues +1 and -1 respectively.

$$\hat{P}_{ij}\,\psi_s = \psi_s$$

$$\hat{P}_{ij}\,\psi_A = -\psi_A$$

The Hamiltonian of N non-interacting Particles is

$$H(1,2,\dots,N) = \sum_{i=1}^{N} H(i)$$

i.e. H is sum of N identical one particle Hamiltonians H(i). The one-particle Schrodinger equation is

$$H(i)\,\psi_i(i) = E_i\,\psi_i(i)$$

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The Product state is

$$\psi_1(1)\,\psi_2(2)\,\dots\,\psi_N(N).$$

with eigenenergy

$$E = E_1 + E_2 + \dots + E_N$$

Wave Functions of Two-Particle Systems:

For Fermions,

The two particle anti-symmetric wavefunction is

$$\psi_A = \frac{1}{\sqrt{2}}\left[\psi_1(1)\,\psi_2(2) - \psi_1(2)\,\psi_2(1)\right]$$

For N-particles, ~~one one~~ an anti-symmetric combination exists

$$\psi_A = \frac{1}{\sqrt{N!}} \sum_{P} (-1)^P \hat{P}\, \psi_1(1)\,\psi_2(2) \dots \psi_N(N)$$

For even permutations $(-1)^P = 1$

" odd " $(-1)^P = -1$

In determinant form..

$$\psi_A = \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_1(1) & \psi_1(2) \\ \psi_2(1) & \psi_2(2) \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[\psi_1(1)\,\psi_2(2) - \psi_1(2)\,\psi_2(1)\right]$$

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for N-Particle system, determinant is

$$= \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(1) & \psi_1(2) \cdots & \psi_1(N) \\ \psi_2(1) & \psi_2(2) \cdots & \psi_2(N) \\ \vdots & & \\ \psi_N(1) & \psi_N(2) \cdots & \psi_N(N) \end{vmatrix}$$

The determinant is called Slater determinant. The anti-symmetry of ψ_A is evident, since interchange of any two columns introduces a factor -1.

For Bosons,

The two particle symmetric wavefn is.

$$\psi_s = \frac{1}{\sqrt{2}} \left(\psi_1(1)\, \psi_2(2) + \psi_1(2)\, \psi_2(1) \right)$$

for N-Particles system.

$$\psi_s = \frac{1}{\sqrt{N!}} \sum_{P} P\, \psi_1(1)\, \psi_2(2) \cdots \psi_N(N)$$

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Since the wavefunctions with spin are expressed as a product of spatial part and a spin part i.e.

$$\Psi = \psi(x,t)\,\chi_{\pm}$$

The following possibilities exist for the total N-particle wave function:

Fermionic States:-

$$\Psi_A = \begin{cases} \text{(Symmetric N-particle spatial function X anti-symmetric N-particle spin state)} \\ \text{(Anti-symmetric N-particle spatial function X symmetric N-particle spin state.)} \end{cases}$$

Boson States:

$$\Psi_S = \begin{cases} \text{(Symmetric N-particle spatial fn X symmetric N-particle spin state.)} \\ \text{(Anti-symmetric N-particle spatial fn X anti-symmetric N-particle spin state.)} \end{cases}$$

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Let us consider two electron state.

The spatial and spin coordinates of two one-electron functions are.

$$\psi_1 = \psi_a \chi_\pm$$

$$\psi_2 = \psi_b \chi_\pm$$

we have two spatial fns.

$$\frac{1}{\sqrt{2}}\left[\psi_a(1)\,\psi_b(2) + \psi_b(1)\,\psi_a(2)\right]$$

(Symmetric)

$$\frac{1}{\sqrt{2}}\left[\psi_a(1)\,\psi_b(2) - \psi_b(1)\,\psi_a(2)\right]$$

(anti-symmetric)

The spin states are

$$\chi_+(1)\,\chi_+(2)$$

$$\chi_-(1)\,\chi_-(2)$$

$$\frac{1}{\sqrt{2}}\left[\chi_+(1)\,\chi_-(2) + \chi_-(1)\,\chi_+(2)\right]$$

(symmetric)

$$\frac{1}{\sqrt{2}}\left[\chi_+(1)\,\chi_-(2) - \chi_-(1)\,\chi_+(2)\right]$$

(anti-symmetric)

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The four anti-symmetric product states are.

$$\frac{1}{\sqrt{2}}\left[\psi_{ab}-\psi_{ba}\right]\cdot\begin{cases}\chi_{++}=\chi_{+}(1)\,\chi_{+}(2).\\ \frac{1}{\sqrt{2}}\left[\chi_{+-}+\chi_{-+}\right]\\ \chi_{--}=\chi_{-}(1)\,\chi_{-}(2)\end{cases}$$

$$\frac{1}{\sqrt{2}}\left[\psi_{ab}+\psi_{ba}\right]\cdot\frac{1}{\sqrt{2}}\left[\chi_{+-}-\chi_{-+}\right]$$

~~The~~ To summerize:

The complete wavefunction describing N-particle state is anti-symmetric under the exchange of any pair of identical fermions, and symmetric under the exchange of any pair of identical bosons.

PAULI'S Exclusion Principle:

It has been observed ~~that~~ experimently that the wavefunction for a system of electrons must be anti-symmetric in the exchange of the co-ordinates of any two particles ($\bar{e}$s). The two fermions wavefn is therefore,

$$\psi_A = \frac{1}{\sqrt{2}} \left(\psi_{12} - \psi_{21} \right)$$

If the two fermions are in the same spatial and spin quantum states i.e.

$$\psi_1 = \psi_2$$

then.

$$\psi_A = \frac{1}{\sqrt{2}} \left(\psi_{11} - \psi_{11} \right) = 0.$$

$$\psi_S = \frac{1}{\sqrt{2}} \left(\psi_{11} + \psi_{11} \right) = \sqrt{2}\, \psi_{11}.$$

i.e. ψ_A vanishes whereas ψ_S does not vanish. The vanishing ψ_A means, no such state exists. This is Pauli's exclusion principle which states:

" Two identical fermions can not exist in the same quantum state".

This principle also follows from the Slater determinant. If two or more columns or rows of the Slater determinant are identical, it vanishes.

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STATISTICS:

The property that particles either obey or do not obey exclusion principle has direct consequences on the distribution of energy in an assembly of particles.

Fermions (obeying exclusion principle). obey fermi-Dirac statistic. A collections of such non-interacting particles at temperature T has the energy distribution.

$$f_{FD} = \frac{1}{e^{(E_i - E_F)/k_B T} + 1}$$

This expression gives the average numbers of particles per state at energy E_i. The parameter E_F denotes the fermi-energy. At zero degree Kelvin, no states of energy greater than E_F are occupied.

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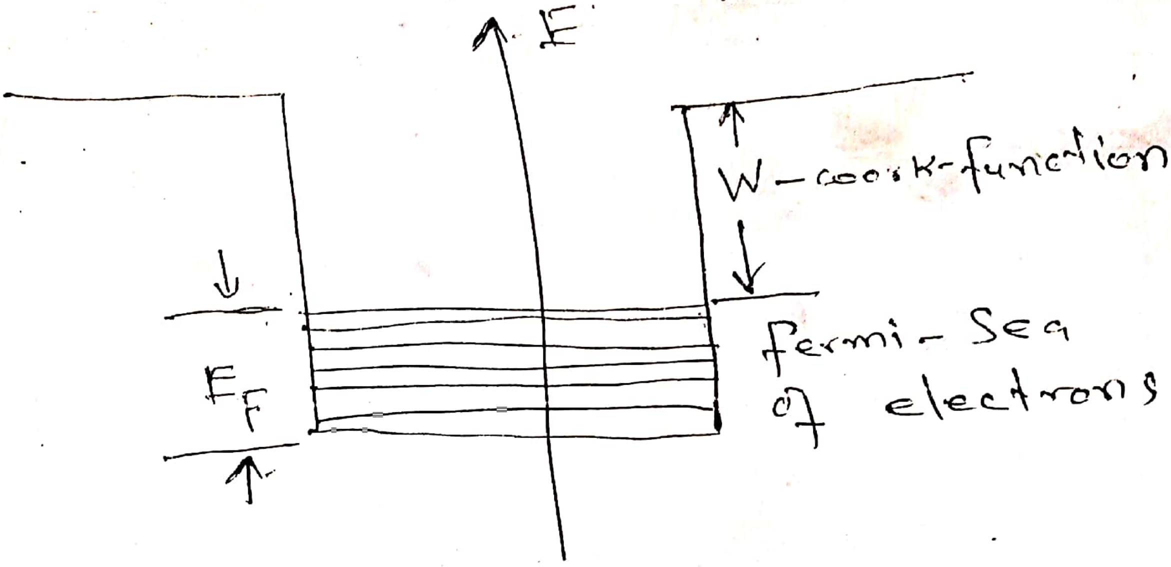

Bosons: (not obeying the exclusion principle)

obey Bose- Einstein statistics.

A collection of non-interacting bosons at temperature T has the energy distribution

$$f_{BE} = \frac{1}{e^{(E_i - \mu)/k_B T} - 1}$$

Here again f_{BE} represents the average number of particles per state at energy E_i and μ is the Chemical potential.

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